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# EXTENSIONS OF THE HEISENBERG-WEYL INEQUALITY

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ABSTRACT. In this paper a number of generalizations of the classical Heisenberg-Weyl uncertainty inequality are given. We prove the n-dimensional Hirschman entropy inequality (Theorem 2.1) from the optimal form of the Hausdorff-Young theorem and deduce a higher dimensional uncertainty inequality (Theorem 2.2). From a general weighted form of the Hausdorff-Young theorem, a one-dimensional weighted entropy inequality is proved and some weighted forms of the Heisenberg-Weyl inequalities are given.

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## 1. INTRODUCTION.

Let $\hat{f}$ be the Fourier transform of f defined by

$$\hat{f}(x) = \int e^{-2\pi ixy} f(y)dy, \qquad x \in \mathbb{R}.$$

If $f \in L^2(\mathbb{R})$ with $L^2$ -norm $||f||_2 = 1$, then by Plancherel's theorem $||\hat{f}||_2 = 1$, so that $|f(x)|^2$ and $|\hat{f}(y)|^2$ are probability frequency functions. The variance of a probability frequency function g is defined by

$$V[g] = \int_{\mathbb{R}} (x-m)^2 g(x)dx \qquad \text{where} \qquad m = \int_{\mathbb{R}} xg(x)dx$$

is the mean. With these notations, the Heisenberg uncertainty principle of quantum mechanics can be stated in terms of the Fourier transform by the inequality

$$V[|f|^2]V[|\hat{f}|^2] \geq (16\pi^2)^{-1}. \tag{1.1}$$

In the sequel, we assume without loss of generality that the mean $m = 0$. If g is a probability frequency function, then the entropy of g is defined by

$$E[g] = \int_{\mathbb{R}} g(x)\log g(x)dx.$$

With f as above, Hirschman [1] proved that

$$E[|f|^2] + E[|\hat{f}|^2] \leq E_H \tag{1.2}$$

with $E_H = 0$, and suggested that (1.2) holds with $E_H = \log 2-1$. If $E_H$ has that form, then by an inequality of Shannon and Weaver [2] it follows that (1.2) implies (1.1). Using the Babenko-Beckner optimal form of the Hausdorff-Young inequality ([3])

$$||\hat{f}||_{p'} \leq A(p)||f||_p, \; 1 < p < 2, \; A(p) = [p^{1/p}(p')^{-1/p'}]^{1/2}, \tag{1.3}$$

in Hirschman's proof of (1.2), then as Beckner [3] noted, (1.2) holds with $E_H = \log 2-1$.

A modest extension of (1.1) is obtained as follows: Let $f$ on $\mathbb{R}$ be differentiable, such that $f(0) = 0$. Then Hölder's and Hardy's inequality [4, Theorem 3.27] yield with $1 < p \leq 2$

$$\int_0^\infty |f(x)|^2dx \leq (\int_0^\infty |x\ f(x)|^p dx)^{1/p}(\int_0^\infty |f(x)/x|^{p'}dx)^{1/p'}$$
$$\leq p(\int_0^\infty |x\ f(x)|^p dx)^{1/p}(\int_0^\infty |f'(x)|^{p'}dx)^{1/p'}.$$

Applying this estimate also to $f(-x)$, then

$$||f||_2^2 = \int_0^\infty |f(x)|^2dx + \int_0^\infty |f(-x)|^2dx$$
$$\leq p[(\int_0^\infty |x\ f(x)|^p dx)^{1/p}(\int_0^\infty |f'(x)|^{p'}dx)^{1/p'}$$
$$+ (\int_0^\infty |x\ f(-x)|^p dx)^{1/p}(\int_0^\infty |f'(-x)|^{p'}dx)^{1/p'}]$$
$$\leq p(\int_{-\infty}^\infty |x\ f(x)|^p dx)^{1/p}(\int_{-\infty}^\infty |f'(x)|^{p'}dx)^{1/p'},$$

where the last inequality follows from Hölder's inequality. Now by (1.3) and the fact that $\hat{f}'(y) = 2\pi iy\ \hat{f}(y)$ we obtain

THEOREM 1.1. If $f \in S(\mathbb{R})$ and $f(0) = 0$, then for $1 < p \leq 2$

$$||f||_2^2 \leq 2\pi\ p\ A(p)\ ||xf||_p ||y\hat{f}||_p. \tag{1.4}$$

Note that the constant in (1.4) is slightly better than that in [4,§1.4] but unlikely best possible.

The purpose of this paper is to give extensions of the Heisenberg-Weyl inequality (1.1). In the next section a new proof of the entropy inequality (1.2) for functions on $\mathbb{R}^n$ is given and an n-dimensional Heisenberg-Weyl inequality is deduced. The n-dimensional generalization of inequality (1.4) is also given in the next section. The two inequalities are quite different, even in the case $p = 2$, but depend strongly on the sharp Hausdorff-Young inequality. In the third section a weighted form of the Heisenberg-Weyl inequality in one dimension is obtained from a weighted form of the Hausdorff-Young inequality ([5][6][7][8]). Unlike the constant $A(p)$ in (1.3) the constant of the weighted Hausdorff-Young inequality (3.3) of (Theorem 3.1) is far from sharp. If the constant is not too large, then a weighted form of Hirschman's entropy inequality can also be given, from which another uncertainty inequality is deduced.

Throughout, $p' = p/(p-1)$, with $p' = \infty$ if $p = 1$, is the conjugate index of $p$, and similarly for other letters. $S(\mathbb{R}^n)$ is the Schwartz class of slowly increasing functions on $\mathbb{R}^n$. We say $g$ is in the weighted $L_w^r$-space with weight $w$, if $wg \in L^r$ and norm $||g||_{r,w} = ||wg||_r$. If $x \in \mathbb{R}^n$, then $x = (x_1,x_2,\ldots,x_n)$ and $dx = dx_1\ldots dx_n$ the n-dimensional Lebesgue measure. $f_i(x)$, $x \in \mathbb{R}^n$ denotes the partial derivative of $f$ with respect to the $i^{th}$ component and $f_{ij} = (f_i)_j$. The letter C denotes a constant which may be different at different occurrences, but is independent of f.

## 2. THE HIRSCHMAN INEQUALITY.

The Fourier transform of $f$ on $\mathbb{R}^n$ is given by

$$\hat{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi ix\cdot y}f(y)dy, \quad x \in \mathbb{R}^n, \quad x\cdot y = x_1y_1 +\ldots+ x_ny_n;$$

and the entropy of a function on $\mathbb{R}^n$ is defined as before with $\mathbb{R}$ replaced by $\mathbb{R}^n$. We shall need the following well known result (c.f. [9; §13.32 ii]):

If $\int_X d\mu = 1$, then

$$\lim_{p\to 0+} \left(\int_X |f|^p d\mu\right)^{1/p} = \exp \int_X \log |f| d\mu. \tag{2.1}$$

Using this fact we obtain easily the n-dimensional form of Hirschman's inequality (1.2)

THEOREM 2.1. If $f \in L^2(\mathbb{R}^n)$ such that $||f||_2 = ||\hat{f}||_2 = 1$, then

$$E[|f|^2] + E[|\hat{f}|^2] \leqslant n[\log 2-1], \tag{2.2}$$

whenever the left side has meaning.

PROOF. Let $f \in (L^1 \cap L^2)(\mathbb{R}^n)$, then $f \in L^p(\mathbb{R}^n)$, $1 < p < 2$, and by the n-dimensional form of the sharp Hausdorff-Young inequality [3] (that is, (1.3) with A(p) replaced by $[A(p)]^n$) we obtain with $p = 2-r$, $r > 0$ and $p' = 2-r'$, $r' < 0$

$$\left(\int_{\mathbb{R}^n} |\hat{f}(y)|^{2-r'} dy\right)^{-1/r'} \leqslant \left[\frac{(2-r)^{1/2r)}}{(2-r')^{-1/(2r')}}\right]^n \left(\int_{\mathbb{R}^n} |f(x)|^{2-r} dx\right)^{1/r}.$$

Now let $d\hat{\mu} = |\hat{f}(y)|^2 dy$ and $d\mu = |f(x)|^2 dx$, then $\int_{\mathbb{R}^n} d\hat{\mu} = \int_{\mathbb{R}^n} d\mu = 1$, so that the inequality becomes

$$\left(\int_{\mathbb{R}^n} |\hat{f}(y)|^{-r'} d\hat{\mu}\right)^{-1/r'} \Big/ \left(\int_{\mathbb{R}^n} (1/|f(x)|)^r d\mu\right)^{1/r} \leq \left[(2-r)^{-1/(2r)} \Big/ (2-r')^{-1/(2r')}\right]^n.$$

But as $r \to 0+$, $-r' \to 0+$, so that by (2.1)

$$\exp\left(\int_{\mathbb{R}^n} \log |\hat{f}(y)| d\hat{\mu}\right) \Big/ \exp\left(\int_{\mathbb{R}^n} \log(|f(x)|^{-1}) d\mu\right)$$

$$= \exp\left(\int_{\mathbb{R}^n} |\hat{f}(y)|^2 \log|\hat{f}(y)| dy + \int_{\mathbb{R}^n} |f(x)|^2 \log|f(x)| dx\right) \leq \lim_{r\to 0} \frac{(2-r)^{\frac{n}{(2r)}}}{(2-r')^{-n/(2r')}}$$

$$= 2^{n/2} e^{-n/2}.$$

Taking logarithms on both sides we get

$$\int_{\mathbb{R}^n} |\hat{f}(y)|^2 \log|\hat{f}(y)| dy + \int_{\mathbb{R}^n} |f(x)|^2 \log|f(x)| dx \leqslant \frac{n}{2} [\log 2-1]$$

and this implies (2.2) in the case $f \in (L^1 \cap L^2)(\mathbb{R}^n)$.

If $f \in L^2$ the result is obtained as in [1] only now one takes for $\omega_T$, $\omega_\varepsilon(x) = e^{-\pi\varepsilon|x|^2}$ and for $\Omega_T$, $\hat{\omega}_\varepsilon(y) = \varepsilon^{-n/2} e^{-\pi|y|^2/\varepsilon}$. We omit the details.

If $|g| \in L^2(\mathbb{R})$ is a probability frequency function, then the relation between entropy and variance is expressed by $E[|g|^2] \geqslant -\frac{1}{2} - \frac{1}{2}\log(2\pi V[|g|^2])$([2; p. 55-56]). The n-dimensional form of this inequality is given in the following lemma:

LEMMA 2.1. ([2; p. 56-57]). Let $g \in L^2(\mathbb{R}^n)$ with $||g||_2 = 1$. If $B = (b_{ij})$ is the matrix with entries

$$b_{ij} = V[|g|^2] = \int_{\mathbb{R}^n} x_i x_j |g(x)|^2 dx, \qquad i,j = 1,2,\ldots,n;$$

then

$$E[|g|^2] \geqslant \frac{n}{2} \log (2\pi |b_{ij}|^{1/n}) - n/2$$

where $|b_{ij}| = \det B$.

Using the lemma and Theorem 2.1, we easily establish an n-dimensional extension of the Heisenberg-Weyl inequality.

THEOREM 2.2. Let $f \in L^2(\mathbb{R}^n)$ with $||f||_2 = ||\hat{f}||_2 = 1$ and

$$b_{ij} = \int_{\mathbb{R}^n} x_i x_j |f(x)|^2 dx, \qquad \hat{b}_{ij} = \int_{\mathbb{R}} y_i y_j |\hat{f}(y)|^2 dy,$$

$i,j = 1,2,\ldots n$; be the entries of the matrices $B$ and $\hat{B}$ respectively, then

$$(\det B)(\det \hat{B}) \geq (16\,\pi^2)^{-n}.$$

PROOF. By (2.2) and Lemma 2.1,

$$n[\log 2-1] \geq E[|f|^2] + E[|\hat{f}|^2]$$

$$\geq -\frac{n}{2}\log(2\pi\,|b_{ij}|^{1/n}) - \frac{n}{2}\log(2\pi\,|\hat{b}_{ij}|^{1/n}) - n,$$

so that

$$\log 2 \geq -\frac{1}{2}\log(4\pi^2|b_{ij}|^{1/n}|\hat{b}_{ij}|^{1/n}).$$

But then

$$4 \geq 1/[(\det B)^{1/n}(\det \hat{B})^{1/n}4\pi^2],$$

which implies the result.

Clearly, if $n = 1$ we obtain at once (1.1). If $n = 2$ then

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \int_{\mathbb{R}^2} x_1^2|f|^2dx, & \int_{\mathbb{R}^2} x_1x_2|f|^2dx \\ \int_{\mathbb{R}^2} x_2x_1|f|^2dx, & \int_{\mathbb{R}^2} x_2^2|f|^2dx \end{pmatrix},$$

with a similar expression for $\hat{B}$. Applying Theorem 2.2 we obtain

$$(\det B)(\det \hat{B}) = [(\int_{\mathbb{R}^2} x_1^2|f|^2dx)(\int_{\mathbb{R}^2} x_2^2|f|^2dx) - (\int_{\mathbb{R}^2} x_1x_2|f|^2dx)^2]$$

$$\cdot[(\int_{\mathbb{R}^2} y_1^2|\hat{f}|^2dy) - (\int_{\mathbb{R}^2} y_1y_2|\hat{f}|^2dy)^2] \geq (16\,\pi^2)^{-2}.$$

If we denote the bracketed terms above by $D[|f|^2]$ and $D[|\hat{f}|^2]$, the discrepancy of Schwarz's inequality, or the difference between variance and covariance of $|f|^2$ and $|\hat{f}|^2$, then the two dimensional Heisenberg-Weyl inequality shows that the discrepancies of $|\hat{f}|^2$ and $|\hat{f}|^2$ cannot both be small; $D[|f|^2]\; D[|\hat{f}|^2] \geq (16\,\pi^2)^{-2}$.

A different generalization of (1.1) may be obtained along the lines of Theorem 1.1.

THEOREM 2.3. Let $f \in S(\mathbb{R}^n)$, such that $f(x_1,x_2,\ldots,x_n) = 0$, whenever $x_i = 0$ for some $i$. If $1 < p \leq 2$ and $A(p)$ is the constant of (1.3), then

$$||f||_2^2 \leq [2\pi p\; A(p)]^n||x_1\ldots x_nf||_p||y_1\ldots y_n\hat{f}||_p.$$

PROOF. We only give the proof for $n = 2$ since the general case follows in exactly the same way. Let $f_{21}(x,y) = g(x,y)$, then

$$f(x,y) = \int_0^x \int_0^y g(s,t)dtds$$

and by Hölder's and the two dimensional Hardy inequality, with $\mathbb{R}_+^2 = (0,\infty)x(0,\infty)$,

$$\int_{\mathbb{R}_+^2}|f(x,y)|^2dxdy \leq (\int_{\mathbb{R}_+^2}|xy\; f(x,y)|^pdxdy)^{1/p}(\int_{\mathbb{R}_+^2}|f(x,y)/xy|^{p'}dxdy)^{1/p'}$$

$$\leq p^2(\int_{\mathbb{R}_+^2}|xy\; f(x,y)|^pdxdy)^{1/p}(\int_{\mathbb{R}_+^2}|f_{21}(x,y)|^{p'}dxdy)^{1/p'}.$$

On applying this estimate four times we obtain with $d\mu = dxdy$

$$||f||_2^2 = \int_{\mathbb{R}_+^2}(|f(x,y)|^2 + |f(x,-y)|^2 + |f(-x,-y)|^2 + |f(-x,y)|^2)d\mu$$

$$\leq p^2\{(\int_{\mathbb{R}^2_+}|xy\ f(x,y)|^p d\mu)^{1/p}(\int_{\mathbb{R}^2_+}|f_{21}(x,y)|^{p'}d\mu)^{1/p'}$$

$$+ (\int_{\mathbb{R}^2_+}|xyf(x,-y)|^p d\mu)^{1/p}(\int_{\mathbb{R}^2_+}|f_{21}(x,-y)|^{p'}d\mu)^{1/p'}$$

$$+ (\int_{\mathbb{R}^2_+}|xyf(-x,-y)|^p d\mu)^{1/p}(\int_{\mathbb{R}^2_+}|f_{21}(-x,-y)|^{p'}d\mu)^{1/p'}$$

$$+ (\int_{\mathbb{R}^2_+}|xyf(-x,y)|^p d\mu)^{1/p}(\int_{\mathbb{R}^2_+}|f_{21}(-x,y)|^{p'}d\mu)^{1/p'}\}$$

$$\leq p^2\{(\int_{\mathbb{R}^2_+}|xy|^p[|f(x,y)|^p + |f(x,-y)|^p + |f(-x,-y)|^p + |f(-x,y))^p]d\mu)^{1/p}$$

$$x(\int_{\mathbb{R}^2_+}|f_{21}(x,y)|^{p'} + |f_{21}(x,-y)|^{p'}$$

$$+|f_{21}(-x,-y)|^{p'} + |f_{21}(-x,y)|^{p'}]d\mu)^{1/p'}\}$$

$$= p^2(\int_{\mathbb{R}^2}|xyf(x,y)|^p d\mu)^{1/p}(\int_{\mathbb{R}^2}|f_{21}(x,y)|^{p'}d\mu)^{1/p'},$$

where the last inequality follows from Hölder's inequality. But by the sharp form of the Hausdorff-Young inequality with $n = 2$ we obtain $||f||_2^2 \leq [p\ A(p)]^2||xyf||_p||\hat{f}_{21}||_{p'}$. Since $(\hat{f}_{21})(s,t) = 4\pi^2 st\ \hat{f}(s,t)$ the result follows.

## 3. WEIGHTED HIRSCHMAN ENTROPY INEQUALITY AND WEIGHTED HEISENBERG-WEYL INEQUALITY.

The results of the last section show that the Heisenberg-Weyl inequality is a consequence of the Hausdorff-Young theorem. Recently a number of weighted Hausdorff-Young inequalities have been obtained [5], [6], [7] and [8]. We shall use these results in this section to obtain a weighted Hirschman entropy inequality as well as weighted form of the Heisenberg-Weyl inequality. Here we consider weighted extensions in $\mathbb{R}^1$ only.

Recall that if $g$ is a Lebesgue measurable function on $\mathbb{R}$, then the equi-measurable decreasing rearrangement of $g$ is defined by $g^*(t) = \inf\{y > 0: |\{x \in \mathbb{R}: |g(x)| > y\}| \leq t\}$, where $y > 0$ and $|E|$ denotes Lebesgue measure of the set $E$. Clearly, if $g$ is an even function on $\mathbb{R}$, decreasing on $(0,\infty)$, then for $t > 0$, $g^*(t) = g(t/2)$. We shall use this fact below.

DEFINITION 3.1. Let $u$ and $v$ be locally integrable functions of $\mathbb{R}$. We write $(u,v) \in F^*_{p,q}$, $1 \leq p \leq q < \infty$, if

$$\sup (\int_0^s[u^*(t)]^q dt)^{1/q}(\int_0^{1/s}[(1/v)^*(t)]^{p'}dt)^{1/p'} < \infty, \tag{3.1}$$

where in the case $p = 1$ the second integral is replaced by the essential supremum of $(1/v)^*(t)$ over $(0, 1/s)$.

If $u$ and $1/v$ are even and decreasing on $(0, \infty)$ then (3.1) is equivalent to

$$\sup_{s>0} (\int_0^{s/2}[u(x)]^q dx)^{1/q}(\int_0^{1/(2s)}v(x)^{-p'}dx)^{1/p'} < \infty \tag{3.2}$$

and in this case we write $(u, v) \in F_{p,q}$.

The weighted Hausdorff-Young inequality is given in the following theorem:

THEOREM 3.1. ([5; Theorem 1.1]). Suppose $(u,v) \in F^*_{p,q}$, $1 \leq p \leq q < \infty$ and $f \in L^p_v$.

(i) If $\lim_{n\to\infty} ||f_n - f||_{p,v} = 0$ for a sequence of simple functions, then $\{\hat{f}_n\}$ converges in $L^q_u$ to a function $\hat{f} \in L^q_u$. $\hat{f}$ is independent of the sequence $\{\hat{f}_n\}$ and is called the Fourier transform of f.

(ii) there is a constant B > 0 such that for all $f \in L^p_v$

$$||\hat{f}||_{q,u} < B||f_{p,v}. \tag{3.3}$$

(iii) If $g \in L^q_{1/u}$, $q > 1$, then Parseval's formula

$$\int_{\mathbb{R}} \hat{f}(y)g(y)dy = \int_{\mathbb{R}} f(t)\hat{g}(t)dt$$

holds.

We note ([5], [6], [8]) that Theorem 3.1 is sharp in the sense that if u and v are even and satisfy (3.3), then (u, v) satisfies (3.2). The constant B in (3.3) is not sharp, however it is of the form B = k.C where k = k(p,q) is independent of u and v and C is the supremum of (3.1), and in the case u, 1/v decreasing and even the supremum (3.2).

A special case of Theorem 3.1 is the following:

COROLLARY 3.1. Suppose $f \in L^p_{v}1/2/p$, $(u^{1-2/p'}, v^{1-2/p}) \in F^*_{p,p'}$, $1 < p < 2$, where u and v are even, decreasing as $(0,\infty)$ then

$$(\int_{\mathbb{R}} u(y)^{p'-2}|\hat{f}(y)|^{p'} < k.C_p(\int_{\mathbb{R}} v(x)^{p-2}|f(x)|^p dx)^{1/p} \tag{3.4}$$

where $C_p = \sup_{s>o} (\int_0^{s/2} u(x)^{p'-2}dx)^{1/p'}(\int_0^{1/(2s)} v(x)^{(2-p)p'/p}dx)^{1/p'}$.

Utilizing the last result we now give a weighted form of Hirschman's entropy inequality.

PROPOSITION 3.1. Suppose $f \in L^2 \cap L^1_{1/v}$, where u and v satisfy the conditions of Corollary 3.1. If $||f||_2 = 1$ and (3.4) holds with $0 < k \leqslant 2$ and $C_p$ remains bounded as $p \to 2$, then

$$\int_{\mathbb{R}} |\hat{f}(y)|^2\log|u(y)\hat{f}(y)|^2dy + \int_{\mathbb{R}} |f(x)|^2\log |v(x)f(x)|^2dx$$

$$\leqslant 2 \log k + 8 \sup_{s>o} (\int_0^{s/2}\int_0^{1/(2s)}\log |u(x)v(y)|dxdy).$$

PROOF. Since $f \in L^p_v\ 1-2/p$, $1 < p < 2$, we apply Corollary 3.1 with $p = 2-r$, $r > 0$, $p' = 2-r'$, $r' < 0$ and $d\hat{\mu}(y) = |\hat{f}(y)|^2dy$, $d\mu(x) = |f(x)|^2dx$. Then (3.4) has the form

$$(\int_{\mathbb{R}} |u(y)\hat{f}(y)|^{-r'}d\mu)^{1/(2-r')} \leqslant k \sup_{s>o} [\int_0^{s/2} u(x)^{-r'}dx \int_0^{1/(2s)} v(x)^{-r'}dx]^{1/(2-r')}$$

$$\bullet (\int_{\mathbb{R}} |v(x)f(x)|^{-r}d\mu)^{1/(2-r)}$$

or, on raising the inequality ot the power (2-r')(-1/r'), equivalently

$$(\int_{\mathbb{R}} |u(y)\hat{f}(y)|^{-r'})^{-1/r'}/(\int_{\mathbb{R}} |v(x)f(x)|^{-r}d\mu)^{1/r} \leqslant k(\tfrac{k}{2})^{-2/r'}M_r,$$

where

$$M_r = \sup_{s>o} [\int_0^{s/2}\int_0^{1/(2s)} [u(x)v(y)]^{-r'}4dxdy]^{-1/r'}.$$

Given $\varepsilon > 0$ there is an $s_o > 0$ such that

$$M_r \leqslant [\int_0^{s_o/2}\int^{1/(2s)} [u(x)v(y)]^{-r'}rdxdy]^{-1/r'} + \varepsilon$$

so that

$$(\int_{\mathbb{R}} |u(y)\hat{f}(y)|^{-r'}d\hat{\mu})^{-1/r'} / (\int_{\mathbb{R}} |v(x)f(x)|^{-r}d\mu)^{1/r}$$

$$\leqslant k([\int_0^{s_o/2}\int_0^{1/(2s_o)}[u(x)v(y)]^{-r'}4dxdy]^{-1/r'} + \varepsilon), \qquad (3.5)$$

where we used the fact that $k/2 \leqslant 1$. Now as $r \to o+$, $r' \to o-$, then on applying (2.1) to both sides of (3.5) we obtain

$$\exp(\int_{\mathbb{R}} \log|u(y)\hat{f}(y)|d\hat{\mu})/\exp(\int_{\mathbb{R}} \log|1/[v(x)f(x)]|d\mu)$$

$$\leqslant k[\exp \int_0^{s_o/2}\int_0^{1/(2s_o)} \log[v(y)u(x)]4dxdy + \varepsilon]$$

$$\leqslant k[\exp \sup_{s>o} \int_0^{s/2}\int_0^{1/(2s)} \log[u(x)v(y)]4dxdy + \varepsilon].$$

But $\varepsilon > 0$ is arbitrary so that on taking logarithms we have

$$\frac{1}{2}\int_{-\infty}^{\infty} |\hat{f}(y)|^2\log|u(y)\hat{f}(y)|^2dy + \frac{1}{2}\int_{-\infty}^{\infty} |f(x)|^2\log|v(x)f(x)|^2dx$$

$$\leqslant \log k + 4 \sup_{s>o}(\int_0^{s/2}\log u(x)dx + \int_0^{1/(2s)}\log v(y)dy)$$

which yields the result.

Note that if $u = v \equiv 1$ and if $k \leqslant \sqrt{2/e}$ we obtain (2.2) with $n = 1$.

We can write the conclusion of Proposition 3.1 in the form

$$E[|f|^2] + E[|\hat{f}|^2] \leqslant 2 \log k + 8 \sup_{s>o} (\int_0^{s/2}\int_0^{1/(2s)}\log|u(x)v(y)|dxdy)$$

$$- \int_{\mathbb{R}}|\hat{f}(y)|^2\log|u(y)|^2dy - \int_{\mathbb{R}}|f(x)|^2\log|v(x)|^2dx.$$

But since ([2]) $E[|f|^2] \geqslant -\frac{1}{2} - \frac{1}{2}\log(2\pi\, V[|f|^2])$ and also with $f$ replaced by $\hat{f}$ we obtain another uncertainty inequality

$$V[|f|^2]\, V[|\hat{f}|^2] \geqslant \frac{k^{-4}}{4\pi^2e^2} \exp[-16 \sup_{s>o} \int_0^{s/2}\int_0^{1/(2s)}\log|uv|dxdy]$$

$$\times \exp(2 \int_{\mathbb{R}} |\hat{f}|^2\log|u|^2dy)\, \exp(2 \int_{\mathbb{R}}|f|^2\log|v|^2dx).$$

If $u = v \equiv 1$ and $k = \sqrt{2/e}$ in this estimate we obtain (1.1).

THEOREM 3.2. (Heisenberg-Weyl inequality). If $(1/u, v) \in F^*_{p,q}$, $1 \leqslant p \leqslant q < \infty$ and $f \in S(\mathbb{R})$, then

$$||f||_2^2 \leqslant C(\int_{\mathbb{R}}|u(x)xf(x)|^{q'}dx)^{1/q'}(\int_{\mathbb{R}}|v(y)y\hat{f}(y)|^pdy)^{1/p}. \qquad (3.6)$$

PROOF. Integration by parts and Hölder's inequality show that for $1 \leqslant q < \infty$

$$||f||_2^2 \leqslant 2 \int_{\mathbb{R}}|x||f(x)||f'(x)|dx$$

$$\leqslant 2(\int_{\mathbb{R}}|xu(x)f(x)|^{q'}dx)^{1/q'}(\int_{\mathbb{R}}|f'(x)/u(x)|^qdx)^{1/q}$$

$$\leqslant 2C(\int_{\mathbb{R}}|xu(x)f(x)|^{q'}dx)^{1/q'}(\int_{\mathbb{R}}|v(y)\hat{f}'(y)|^pdx)^{1/p},$$

where the last inequality follows from (3.3). Since $\hat{f}'(y) = 2\pi iy\, \hat{f}(y)$ the result follows.

Note that the case $p = 1$ also holds, provided the second integral in the $F^*_{p,q}$ condition is interpreted as the essential supremum of $(1/v)^*$ over $(0, 1/s)$.

The same result holds also if we take $(1/u, v) \in F_{p,q}$.

Observe also that the case $u = v \equiv 1$ and $q = p'$, $1 < p \leqslant 2$ reduces to (1.4), but with a different constant.

Weighted inequalities of the form (3.6) were also obtained by Cowling and Price [3] but by quite different methods.

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