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CONTRACTIVE CURVES

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We discuss the dynamics of the correspondences associated to those plane curves whose local sections contract the Poincaré metric in a hyperbolic planar domain.

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1. Introduction. We consider certain 1-dimensional, holomorphic correspondences of hyperbolic type, which we call "contractive curves." These are curves whose local sections contract the Poincaré metric of a hyperbolic planar domain. The model for our analysis is given by the hyperbolic Julia sets.

This work has been motivated by the work in Ochs' recent paper [5]. In the first part, we adapt to our purposes the Schwarz lemma for correspondences that appears in Minda's paper [3]. In the second part, we discuss the dynamics of a contractive curve, and the properties of the associated attractor. Such curves usually do not have global sections that contract the hyperbolic metric. Nevertheless, the associated dynamical systems have much in common with the iterated function systems of hyperbolic type. The basis for our discussion is the paper [2] by Barnsley and Demko.

2. Preliminaries

2.1. Hyperbolic metric. The hyperbolic metric (infinitesimal length-element) on the unit disk $\Delta = \{|z| < 1\} \subset \mathbb{C}$ is $ds_\Delta(z) = (2/(1-|z|^2))|dz|$.

Let $U \subset \mathbb{C}$ be a planar domain (open and connected subset of $\mathbb{C}$), and assume that the complement $\mathbb{C}\setminus U$ contains at least two points, so that U is a hyperbolic Riemann surface; the universal covering of U is (biholomorphic to) Δ. The density ρ_U of the hyperbolic metric in U, $ds_U(z) = \rho_U(z)|dz|$, is defined as follows. Fix an unramified covering $\Delta \xrightarrow{\phi} U$. Given $z \in U$, choose any $t \in \phi^{-1}(z)$, and define $\rho_U(z) = 2/(1 - |t|^2)|\phi'(t)|$. The definition of $\rho_U(z)$ does not depend on the choice of $\Delta \xrightarrow{\phi} U$, nor on the choice of $t \in \phi^{-1}(z)$. Note that ρ_U is positive and real-analytic.

The hyperbolic distance s_U in U is obtained by integrating the infinitesimal length-element ds_U. The metric space (U, s_U) is complete.

The (Gaussian) curvature of a metric $\rho(z)|dz|$ is $K_\rho = -(4/\rho^2)(\partial^2(\log\rho)/\partial z\partial\overline{z})$. The hyperbolic metric ds_U has constant negative curvature, $K_{\rho_U} \equiv -1$.

Holomorphic maps do not expand the hyperbolic metric; given any holomorphic map $U \xrightarrow{f} V$ of hyperbolic domains, $f^*ds_V \le ds_U$, that is, $\rho_V(f)|f'| \le \rho_U$. Equality at some point of U implies that $U \xrightarrow{f} V$ is an unramified covering; in this case, equality holds everywhere in U.

2.2. Hausdorff distance. Given two subsets $K \neq \emptyset \neq H$ of a metric space (X,d), let $d(K,H) := \inf_{(k,h)\in K\times H} d(k,h)$, and define the Hausdorff distance

$$\mathcal{H}_d(K,H) := \max\left(\sup_{k\in K} d(k,H), \sup_{h\in H} d(K,h)\right). \tag{2.1}$$

When $K_1 \subset K$, $d(K_1,H) \le \mathcal{H}_d(K,H)$.

Denote by $\mathcal{H}(X,d) := (\mathcal{H}_X,\mathcal{H}_d)$ the metric space of nonempty compact subsets of X. Clearly, (X,d) is isometrically embedded in $\mathcal{H}(X,d)$, $\mathcal{H}_d(x,y) = d(x,y)$.

If (X,d) is a complete metric space, then $\mathcal{H}(X,d)$ is a complete metric space. Therefore, when U is a hyperbolic domain, $\mathcal{H}(U,s_U)$ is a complete metric space.

Given $(X,d_X) \xrightarrow{f} (Y,d_Y)$ continuous, let $\mathcal{H}(X,d_X) \xrightarrow{\mathcal{H}_f} \mathcal{H}(Y,d_Y)$ denote the continuous map $\mathcal{H}_f(K) := f(K)$. Let $\operatorname{Lip}(f)$ denote the Lipschitz "norm" of f,

$$\operatorname{Lip}(f) = \sup\left\{\frac{d_Y(f(x),f(u))}{d_X(x,u)} : (x,u) \in X\times X,\ x \neq u\right\}. \tag{2.2}$$

We have $\operatorname{Lip}(\mathcal{H}_f) = \operatorname{Lip}(f)$.

2.3. Stolz subdomains. Given a subdomain V of a hyperbolic planar domain U, $\rho_U(z) \le \rho_V(z)$ for all $z \in V$. Equality at some point $z \in V$ implies that $U = V$.

Definition 2.1. A subdomain V of a hyperbolic planar domain U is called *Stolz* if and only if there exists a constant $k > 1$ so that $\rho_V(z)/\rho_U(z) \ge k > 1$ for all $z \in V$.

Equivalently, V is Stolz in U if and only if the inclusion $(V,s_V) \xrightarrow{i} (U,s_U)$ is a strict contraction, that is, $\operatorname{Lip}(i) < 1$.

If $\overline{V} \subset U$, then V is Stolz in U. Roughly speaking, if $V \subset U$ are planar domains with piecewise C^1-smooth boundaries, then V is Stolz in U if and only if the boundaries ∂U and ∂V have no tangency at any common boundary point.

Example 2.2. The disk $\{|z-R| < R\}$ is not Stolz in the half-plane $\{\Re(z) > 0\}$. For $a > b \ge 1$, the angle $A_a = \{|\arg(z)| < \pi/2a\}$ is Stolz in A_b.

3. Schwarz lemma for correspondences

3.1. Ahlfors' lemma. Recall the definition of ultra-hyperbolic metrics $\rho(z)|dz|$ in planar domains. An upper semicontinuous function $V \xrightarrow{\rho} [0,\infty)$ is an *ultra-hyperbolic density function* if and only if, for all $z \in V$ with $\rho(z) > 0$, there exists a positive C^2-smooth function $V_z \xrightarrow{\rho_z} (0,\infty)$, defined in a neighborhood V_z of z, so that $K_{\rho_z} \le -1$, $\rho_z \le \rho$ and $\rho_z(z) = \rho(z)$. Recall Ahlfors' lemma (see [1]).

Proposition 3.1. *If $U \xrightarrow{f} V$ is a holomorphic map from a hyperbolic planar domain U to an ultra-hyperbolic domain (V,ρ), then $\rho(f)|f'| \le \rho_U$. In particular, $\rho \le \rho_V$.*

3.2. Proper curves. By a *curve* in a given holomorphic manifold, we always mean a *closed* analytic subset of pure dimension one.

Definition 3.2. Let U and V be planar domains. A curve $C \subset U \times V$ is called *proper* if and only if the following two conditions are satisfied:

(i) the projection $C \xrightarrow{p} U$ is proper;

(ii) no local branch of C has vertical tangent.

Definition 3.3. A *local section* of a curve $C \subset U \times V$ is a local section of the projection $C \xrightarrow{p} U$, that is, a holomorphic map $U_0 \xrightarrow{f} V$ defined on a subdomain $U_0 \subset U$ so that $\mathrm{Graph}(f) \subset C$.

3.3. Schwarz lemma of Nehari and Minda. Using Ahlfors' lemma, Nehari [4] and then Minda [3] prove a Schwarz-Pick lemma for multivalent functions. We formulate and prove their result, in a more convenient form.

Proposition 3.4. *Let U and V be hyperbolic planar domains. If $C \subset U \times V$ is a proper curve, then $\rho_V(f)|f'| \le \rho_U$, for every local section f of C.*

Proof. Given a local branch $B = (C_0, c)$ of C, let $p(B) := p(c) \in U$ and $q(B) := q(c) \in V$. Let $(\mathbb{C}, 0) \xrightarrow{\nu_B} B$ be a normalization, $\nu_B(t) = (x(t), y(t))$. The slope of B is $s(B) = \lim_{t\to 0}(y'(t)/x'(t))$.

Let $\tilde{C} \xrightarrow{\nu} C$ be a normalization map of the (possibly reducible) plane curve C. Recall that $\tilde{C}$ is the curve of local branches of C. Let $\tilde{C} \xrightarrow{\check{\nu}} \check{C}$ be the dual curve of C, that is, the curve of tangents of local branches of C. Then $\check{C} \subset \mathcal{L}$, where $\mathcal{L} \simeq \mathbb{C}^2$ is the space of nonvertical lines in $\mathbb{C}^2$. Let $\mathcal{L} \xrightarrow{\sigma} \mathbb{C}$ be the holomorphic map that associates to a line its slope. The map $\tilde{C} \xrightarrow{s} \mathbb{C}$, $B \mapsto s(B)$, is holomorphic since it factorizes as $\tilde{C} \xrightarrow{\check{\nu}} \check{C} \subset \mathcal{L} \xrightarrow{\sigma} \mathbb{C}$.

Given $x \in U$, let $C(x)$ be the set of local branches B of C with $p(B) = x$. Since the projection $C \xrightarrow{p} U$ is proper, $C(x)$ is finite. Define the function $U \xrightarrow{\rho} [0,\infty)$, $\rho(x) := \max_{B\in C(x)} \rho_V(q(B)) \cdot |s(B)|$. By definition, $\rho_V(f)|f'| \le \rho$ for every local section f of C. Clearly, $\rho \in C(U)$. We show that ρ is ultra-hyperbolic.

Let B be a local branch of C with $s(B) \ne 0$. Let $m(B)$ denote the multiplicity of B. Since B has nonvertical tangent, $m(B)$ equals the local degree of $B \xrightarrow{p} U$. Since B has nonhorizontal tangent, $m(B)$ equals the local degree of $q : B \to V$. We can define, in a punctured neighborhood of $p(B)$, the function

$$\rho_B(x) := \left[\prod_{\beta\in B(x)} \rho_V(q(\beta)) \cdot |s(\beta)|\right]^{1/m(B)}. \tag{3.1}$$

Clearly, ρ_B is real-analytic, and $0 < \rho_B \le \rho$ in suitable local coordinates in $\tilde{C}$, U, and V, we see that ρ_B is C^2-smooth at $\rho(B)$ with $\rho_B(p(B)) = \rho_V(q(B)) \cdot |s(B)|$. We prove now that $K_{\rho_B} \le -1$.

Let $W \xrightarrow{\rho_i} (0,\infty)$, $1 \le i \le m$, be C^2-smooth positive functions in a planar domain, and let $W \xrightarrow{g} (0,\infty)$ denote their geometric mean, $g = [\prod_i \rho_i]^{1/m}$. The inequality between the arithmetic mean and the geometric mean of finitely many positive numbers implies that *if* $K := \max_i(K_{\rho_i}) < 0$, *then* $K_g \le K$.

If $x \ne p(B)$ is in a small neighborhood of $p(B)$, then B is given over a small neighborhood W of x by the union of graphs of $m(B)$ biholomorphisms $W \xrightarrow{f_i} V_i \subset V$,

$1 \le i \le m(B)$. Let $W \xrightarrow{\rho_i} (0,\infty)$, $\rho_i := \rho_V(f_i)|f_i'|$. The restriction to W of ρ_B is the geometric mean of the functions ρ_i. The conformal invariance of the curvature implies that $K_{\rho_i} = K_{\rho_V} = -1$. Therefore, $K_{\rho_B} \le -1$ in W. It follows that $K_{\rho_B} \le -1$ in a punctured neighborhood of $q(B)$. By continuity, $K_{\rho_B}(q(B)) \le -1$.

Now, fix $x \in U$ with $\rho(x) > 0$, and choose $B \in C(x)$ with $\rho(x) = \rho_V(q(B)) \cdot |s(B)|$. Then ρ_B is a support function of ρ of x. In conclusion, ρ is ultra-hyperbolic in U. Ahlfors' lemma implies that $\rho \le \rho_U$, and the proof is finished. □

Remark 3.5. Let $C \subset U \times V \ni (x,y)$ be a curve, with projections $C \xrightarrow{p} U$ and $C \xrightarrow{q} V$. There exists $F \in \mathbb{O}(U \times V)$ so that $C = \{F = 0\}$. Let $\operatorname{Reg}(C)$ denote the smooth locus of C. Define $\|C\| = \sup_{\operatorname{Reg}(C)}[(\rho_V(q)/\rho_U(p)) \cdot ((\partial F/\partial x)/(\partial F/\partial y))]$. The right-hand side is independent of F. For every local section f of C, $f^* ds_V \le \|C\| ds_U$. When p is proper, Proposition 3.4 says that either $\|C\| = \infty$ (when C has vertical tangents), or $\|C\| \le 1$ (when C has no vertical tangent).

Remark 3.6. Given an integer $n \ge 1$ and a holomorphic $\Delta \xrightarrow{g} \Delta$, consider the curve $C = \{y^n = g(x)\} \subset \Delta \times \Delta$. Let $m(g)$ denote the minimum of the orders of the zeros of g. Clearly $\|C\| = \infty$ if and only if $m(g) < n$. Assume that $m(g) \ge n$. Then $\|C\| \le 1$, and the Schwarz-Pick lemma implies, for all $z \in \Delta$,

$$|g'(z)| \le \frac{n|g(z)|^{1-1/n}(1-|g(z)|^{2/n})}{1-|z|^2}. \tag{3.2}$$

It is amusing to see this directly, as follows. Rewrite the inequality as $(1-|z|^2)|g'(z)| \le |g(z)| \cdot u_n(|g(z)|)$, with $(0,1] \xrightarrow{u_n} (0,\infty)$, $u_n(t) = n(t^{-1/n} - t^{1/n})$. Using the conformal invariance of the quantity $(1-|z|^2)|g'(z)|$, we may assume that $z = 0$ and $g(0) \ne 0$. We need to show that $|g'/g|(0) \le u_n(|g(0)|)$. Note two properties of u_n:

(i) $u_n(s) + u_n(t) \le u_n(st)$ for s, t in $(0,1]$;

(ii) $u_n \searrow u_\infty$, with $u_\infty(t) = -2\log(t)$.

Write $g = Bh$, where $B(z) = \prod_j((z-z_j)/(1-\bar{z}_j z))^{n_j}$ is the Blaschke product associated to g, $n_j \ge n$. Note that $\Delta \xrightarrow{h} \Delta$ is nonvanishing, hence, by the classical Schwarz lemma, $|h'/h|(0) \le u_\infty(|h(0)|)$. Also,

$$\begin{aligned} \left|\frac{B'}{B}\right|(0) &= \sum_j n_j \frac{1-|a_j|^2}{|a_j|} = \sum_j u_{n_j}(|a_j|^{n_j}) \\ &\le \sum_j u_n(|a_j|^{n_j}) \le u_n\Bigl(\prod_j |a_j|^{n_j}\Bigr) \\ &= u_n(|B(0)|), \end{aligned} \tag{3.3}$$

that is,

$$\left|\frac{B'}{B}\right|(0) \le u_n(|B(0)|). \tag{3.4}$$

Therefore,

$$\begin{aligned}\left|\frac{g'}{g}\right|(0) &\le \left|\frac{B'}{B}\right|(0)+\left|\frac{h'}{h}\right|(0)\\ &\le u_n(|B(0)|)+u_\infty(|h(0)|)\\ &\le u_n(|B(0)|)+u_n(|h(0)|)\\ &\le u_n(|g(0)|),\end{aligned} \tag{3.5}$$

and the inequality (3.2) is shown.

Let U, V be planar domains, and $C \subset U \times V$ a curve, with projections $C \xrightarrow{p} U$ and $C \xrightarrow{q} V$. Assume that p is proper. If K is compact in U, then $p^{-1}(K)$ is compact in C, hence $q(p^{-1}(K))$ is compact in V. We have therefore a map $\mathcal{K}_U \xrightarrow{\mathcal{K}_C} \mathcal{K}_V$, $\mathcal{K}_C(K) = q(p^{-1}(K))$.

Corollary 3.7. *Let U, V be hyperbolic planar domains. If $C \subset U \times V$ is a proper curve, then* $\mathrm{Lip}(\mathcal{K}_C) \le 1$.

Proof. Given K, H in $\mathcal{K}_U$, fix $y_0 \in \mathcal{K}_C(K)$, $x_0 \in K$ with $(x_0, y_0) \in C$, and $u_0 \in H$ with $s_U(x_0,u_0) = s_U(x_0,H)$. It suffices to find $v_0 \in \mathcal{K}_C(u_0)$ with $s_V(y_0,v_0) \le s_U(x_0,u_0)$. To show that such v_0 exists, we use an analytic continuation argument.

Any local branch of C at (x_0,y_0) has a normalization of form $t \mapsto (x,y) = (x_0 + t^m, \psi(t))$. Fix a local section $D_0 \xrightarrow{f_0} V$ of C that extends continuously to a section $\overline{D_0} \xrightarrow{\overline{f_0}} V$, with $x_0 \in \partial D_0$ and $\overline{f_0}(x_0) = y_0$.

Let $B(p) \subset U$ denote the discrete set of branch points of $C \xrightarrow{p} U$. Choose $x \in D_0 \setminus B(p)$ and $u \in U \setminus B(p)$. Let $y = f_0(x)$. Let $D \subset U \setminus B(p)$ be a simply connected domain with $\{x,u\} \subset D$. The section-germ (f_0,x) can be analytically extended to a section $D \xrightarrow{f} U$. If we put $v = s(u)$, we have $s_V(y,v) = s_V(f(x), f(u)) \le s_U(x,u)$.

Now, let $D_0 \setminus B(p) \ni x_n \to x_0$ and $U \setminus B(p) \ni u_n \to u_0$. Let $D_n \subset U \setminus B(p)$ be a simply connected domain with $\{x_n,u_n\} \subset D_n$, and let $D_n \xrightarrow{f_n} U$ be the analytic extension of the section-germ (f_0,x_n). Then

$$y_n := f_n(x_n) = f_0(x_n) = \overline{f_0}(x_n) \to \overline{f_0}(x_0) = f_0(x_0) = y_0. \tag{3.6}$$

Let $v_n = f_n(u_n)$. Since p is proper, we may assume (extracting a subsequence), that $v_n \to v_0$, for some $v_0 \in V$. Then $(u_0,v_0) \in C$, that is, $v_0 \in \mathcal{K}_C(u_0)$. Therefore,

$$\begin{aligned}s_V(y_0,v_0) &\le s_V(y_0,y_n)+s_V(y_n,v_n)+s_V(v_n,v_0)\\ &\le s_V(y_0,y_n)+s_U(x_n,u_n)+s_V(v_n,v_0) \longrightarrow s_U(x_0,u_0).\end{aligned} \tag{3.7}$$

The proof is complete. □

4. Attractor associated to a contractive curve

4.1. Contractive curves. Given $V \subset U$, let $\mathcal{K}_V \xrightarrow{\mathcal{K}_i} \mathcal{K}_U$ denote the inclusion map. Given a proper curve $C \subset U \times V$, $\mathcal{K}_U \xrightarrow{\tilde{\mathcal{K}}_C} \mathcal{K}_U$ denotes the self-map $\tilde{\mathcal{K}}_C = \mathcal{K}_i \mathcal{K}_C$.

Definition 4.1. A *contractive curve* is a proper curve $C \subset U \times V$, where U is a hyperbolic planar domain and V is a Stolz subdomain of U.

Remark 4.2. The proof of Corollary 3.7 shows that contractive curves have the following *coherence* property: *there is a constant* $k = k(U,V) > 1$ *such that, for all* $((x,y),u) \in C \times U$, *there exists* $v \in V$ *with* $(u,v) \in C$ *and* $ks_U(y,v) \leq s_U(x,u)$.

Theorem 4.3. *If* $C \subset U \times V$ *is a contractive curve, then* $\mathcal{K}(U,s_U) \xrightarrow{\tilde{\mathcal{K}}_C} \mathcal{K}(U,s_U)$ *is a strict contraction. Consequently,* $\tilde{\mathcal{K}}_C$ *has a unique fixed point* A_C. *For all* $K \in \mathcal{K}_U$, $\lim_{r\to\infty} \tilde{\mathcal{K}}_C^r(K) = A_C$ *in* $\mathcal{K}(U,s_U)$.

Proof. Indeed, $\text{Lip}(\tilde{\mathcal{K}}_C) \leq \text{Lip}(\mathcal{K}_i)\,\text{Lip}(\mathcal{K}_C) \leq \text{Lip}(\mathcal{K}_i) < 1$. Since $\mathcal{K}(U,s_U)$ is complete, Banach's principle finishes the proof. □

The hyperbolic metric is locally equivalent to the Euclidean metric $de_U = |dz|$, hence $\lim_{r\to\infty} \tilde{\mathcal{K}}_C^r(K) = A_C$ in $\mathcal{K}(U,e_U)$, for all $K \in \mathcal{K}_U$. We call A_C the *attractor* associated to the contractive curve C.

4.2. Orbits and limit sets. Let U be a planar domain, $V \subset U$ a subdomain, $C \subset U \times V$ a curve with $C \xrightarrow{p} U$ proper. A point $x \in U$ is *fixed* if and only if $(x,x) \in C$. The point x is *periodic* if and only if $x \in \tilde{\mathcal{K}}_C^r(x)$ for some $r \geq 1$. We denote by $\text{Fix}(C)$ the set of fixed points of C, and by $\text{Per}(C)$ the set of periodic points of C.

A *weak orbit* of a point $x \in U$ is a sequence of points $x_r \in \tilde{\mathcal{K}}_C^r(x)$, $r \geq 0$. An *orbit* of x is a sequence of points $x_r \in U$, $r \geq 0$, with $x_0 = x$ and $(x_r, x_{r+1}) \in C$ for all $r \geq 0$. A (weak) *suborbit* of x is a subsequence of a (weak) orbit.

The limit set $A^{\text{wo}}(x)$ is the set of points $u \in U$ such that some weak orbit of x converges to u; $A^{\text{so}}(x)$ is the set of points $u \in U$ such that some suborbit of x converges to u. Similarly, define the limit sets $A^{\text{wso}}(x)$ and $A^{\text{o}}(x)$.

The *total orbit* X^+ of $X \subset U$ is the union of orbits of the points of X.

Corollary 4.4. *If* $C \subset U \times V$ *is a contractive curve, then, for all points* $x \in U$,

$$A^{\text{wo}}(x) = A^{\text{wso}}(x) = A^{\text{so}}(x) = A_C, \qquad A^{\text{o}}(x) = \text{Fix}(C). \tag{4.1}$$

Proof. Given arbitrary points $x \in U$ and $a \in A_C$, $\lim_r \mathcal{K}_{s_U}(\tilde{\mathcal{K}}_C^r(x), A_C) = 0$, hence $\lim_r s_U(\tilde{\mathcal{K}}_C^r(x), a) = 0$. We get a sequence $x_r \in \tilde{\mathcal{K}}_C^r(x)$ with $\lim_r x_r = a$, so that $a \in A^{\text{wo}}(x)$. Therefore, $A_C \subset A^{\text{wo}}(x)$. Fix $u \in A^{\text{wso}}(x)$, and let $(x_{r_j})_j$ be a weak suborbit of x that converges to u. Since $\lim_j \mathcal{K}_{s_U}(\tilde{\mathcal{K}}_C^{r_j}(x), A_C) = 0$, we get $\lim_j s_U(x_{r_j}, A_C) = 0$, and then $s_U(u, A_C) = 0$. Since A_C is compact, $u \in A_C$. Therefore, $A^{\text{wso}}(x) \subset A_C$.

Clearly, $A^{\text{wo}}(x) \subset A^{\text{wso}}(x)$, and we obtain $A_C = A^{\text{wo}}(x) = A^{\text{wso}}(x)$.

Let $(\epsilon_j)_j$ be a sequence that decreases to 0. Since $a \in A^{\text{wo}}(x)$, there exists $x_1 \in \tilde{\mathcal{K}}_C^{r_1}(x)$ with $s_U(a,x_1) \leq \epsilon_1$. Since $a \in A^{\text{wo}}(x_1)$, there exists $x_2 \in \tilde{\mathcal{K}}_C^{r_2}(x_1)$ with $s_U(a,x_2) \leq \epsilon_2$. Repeating this procedure, we get a suborbit x_j of x with $s_U(a,x_j) \leq \epsilon_j$, hence $\lim_j x_j = a$. It follows that $A_C \subset A^{\text{so}}(x)$. Since $A^{\text{so}}(x) \subset A^{\text{wso}}(x) = A_C$, we get $A^{\text{so}}(x) = A_C$.

Fix an orbit $(x_r)_r$ of x that converges to u. Since C is closed, $(x_r, x_{r+1}) \in C$, and $\lim_r (x_r, x_{r+1}) = (u,u)$, we get $(u,u) \in C$. Therefore, $A^{\text{o}}(x) \subset \text{Fix}(C)$.

Fix $u \in \text{Fix}(C)$. Since $(u,u) \in C$, there exists $x_1 \in U$ with $(x,x_1) \in C$ and $s_U(u,x_1) \leq (1/k)s_U(u,x)$. Since $(u,u) \in C$, there exists $x_2 \in U$ with $(x_1,x_2) \in C$ and $s_U(u,x_2) \leq (1/k)s_U(u,x_1)$. Repeating the procedure, we get an orbit x_r of x with $s_U(u,x_r) \leq (1/k)^r s_U(u,x)$, hence $\lim_r x_r = u$. Therefore, $\text{Fix}(C) \subset A^{\text{o}}(x)$. □

4.3. Discrete points. Let $K^\star$ denote the set of isolated points of a topological space K. Given a curve $C \subset U \times V$, let $L := \{v \in V : U \times \{v\} \subset C\}$.

Corollary 4.5. *$A_C^\star \subset \mathrm{Per}(C) \subset A_C$ for all contractive curves $C \subset U \times V$. When A_C is infinite, $A_C^\star \subset L^+ \subset A_C$. When $L = \varnothing$, A_C is either finite, or perfect.*

Proof. Clearly, $L \subset \mathrm{Fix}(C) \subset \mathrm{Per}(C) \subset A_C$.

Fix $u \in A_C$ and $v \in A_C^\star$. There is a weak orbit $(u_r)_r$ of u that converges to v. Since $\mathcal{K}_C(A_C) = A_C$, $u_r \in A_C$. Since $v \in A_C^\star$, $u_r = v$ for all $r \geq r_0$. Therefore, every $u \in A_C$ is pre-periodic to any $v \in A_C^\star$. In particular, $A_C^\star \subset \mathrm{Per}(C)$.

If $(u,v) \in C$, $u \in A_C \setminus A_C^\star$, and $v \in A_C^\star$, then $v \in L$. Indeed, choose in A_C distinct points u_n with $\lim_n u_n = u$. Construct $v_n \in A_C$ with $(u_n, v_n) \in C$ and $s_U(v_n, v) \leq s_U(u_n, u)$. Then $\lim_n v_n = v$. Since $v \in A_C^\star$, $v_n = v$ for all $n \geq n_0$. Therefore, $(u_n, v) \in C$ for all $n \geq n_0$. Since C is closed in $U \times V$, $C \cap (U \times \{v\})$ is analytic in this line. Since $\lim_n (u_n, v) = (u, v)$ and U is connected, we get $v \in L$.

Assume that $v \in A_C^\star \setminus L^+$. If $(u,v) \in C$ and $u \in A_C$, then $u \in A_C^\star \setminus L^+$. Since every $u \in A_C$ is pre-periodic to v, we get $A_C \subset A_C^\star$, hence $A_C = A_C^\star$. Since A_C is compact, it is finite. □

4.4. Periodic points. We prove the density in A_C of the periodic points of a contractive curve C without singular branches.

Corollary 4.6. *If a contractive curve $C \subset U \times V$ has no singular branches, then $A_C = \overline{\mathrm{Per}(C)}$.*

Proof. Clearly, $\overline{\mathrm{Per}(C)} \subset A_C$. To prove the other inclusion, note that, under the assumption that C has no singular branches, every local branch B of C has a section defined in a hyperbolic disk $\Delta_U(p(c), \epsilon)$, for some $\epsilon > 0$. Let $\epsilon(B)$ be the supremum of such ϵ. The function $\tilde{C} \ni B \mapsto \epsilon(B) \in (0, \infty]$ is lower semicontinuous. Let $\epsilon_C := \min\{\epsilon(B) : B \in \nu^{-1} p^{-1} A_C\} > 0$. Then, for all $x \in A_C$ and all branches B of C at x, B has a section defined on $\Delta_U(x, \epsilon_C)$. In particular, for all $x \in A_C$ and all $y \in \mathcal{K}_C(x)$, there exists a section $\Delta_U(x, \epsilon_C) \xrightarrow{f} V$ of C with $f(x) = y$.

Fix $x \in A_C$. Let $O = (x = x_0, x_1, \ldots, x_{r-1}, x_r = u)$ be a finite orbit. Since $x_{j-1} \in A_C$ and $(x_{j-1}, x_j) \in C$ for all $1 \leq j \leq r$, there is a section $\Delta_U(x, \epsilon_C) \xrightarrow{f_j} V$ of C with $f_j(x_{j-1}) = x_j$. Since $\mathrm{Lip}(f_j) \leq 1/k < 1$, $f_j(\Delta_U(x_{j-1}, \epsilon_C)) \subset \Delta_U(x_j, \epsilon_C)$, so that $f = f_r \circ f_{r-1} \circ \cdots \circ f_1$ is well defined, $\Delta_U(x, \epsilon_C) \xrightarrow{f} V$, with $f(x) = u$ and $\mathrm{Lip}(f) \leq 1/k^r$.

Fix $0 < \epsilon < \epsilon_C$. Since $x \in A^{\mathrm{wo}}(x)$, we can choose the orbit O so that $r \geq 1$ and $s_U(u, x) \leq \epsilon((k-1)/2k)$. Then

$$f(\Delta_U(x, \epsilon)) \subset \Delta_U\left(u, \frac{\epsilon}{k^r}\right) \subset \Delta_U\left(x, \epsilon\left(\frac{1}{k^r} + \frac{k-1}{2k}\right)\right) \subset \Delta_U\left(x, \epsilon\frac{k+1}{2k}\right). \tag{4.2}$$

Therefore, $\overline{f(\Delta_U(x, \epsilon))} \subset \Delta_U(x, \epsilon)$, and f must have a fixed point $z \in \Delta_U(x, \epsilon)$. Then $z \in \mathcal{K}_C^r(z)$, so that $z \in \mathrm{Per}(C)$. □

4.5. Continuity. Let $V \overset{i}{\hookrightarrow} U$ be a Stolz subdomain of a hyperbolic domain $U \subset \mathbb{C}$, $k(U, V) = 1/\mathrm{Lip}(i) > 1$. Define on $U \times V$ the distance

$$s((x,y),(u,v)) = \max(s_U(x,u), s_V(y,v)). \tag{4.3}$$

Corollary 4.7. *Let $(\mathscr{C}(U,V),\mathscr{H}_s)$ be the space of contractive curves in $U\times V$, and denote by*

$$(\mathscr{C}(U,V),\mathscr{H}_s)\xrightarrow{\mathscr{A}}\mathscr{K}(U,s_U) \tag{4.4}$$

the map $C\mapsto A_C$. Then $\mathrm{Lip}(\mathscr{A})\le 2/(k(U,V)-1)$.

Proof. Fix $C,D\in\mathscr{C}(U,V)$, with $\delta:=\mathscr{H}_s(C,D)$. Given $(x,y)\in C$ and $u\in U$, there exists $v\in V$ with $(u,v)\in D$ and $s_U(y,v)\le 2\delta/k+s_U(x,u)/k$. Indeed, there exists $(a,b)\in D$ with $s_U(x,a)\le\delta$ and $s_U(y,b)\le s_V(y,b)/k\le\delta/k$. Also, there exists $v\in\tilde{\mathscr{K}}_V(u)$ with $s_U(b,v)\le s_U(a,u)/k$. We get

$$\begin{aligned} s_U(y,v)\le s_U(y,b)+s_U(b,v)&\le\frac{\delta}{k}+\frac{s_U(a,u)}{k}\\ &\le\frac{\delta}{k}+\frac{s_U(x,a)+s_U(x,u)}{k}\le\frac{2\delta}{k}+\frac{s_U(x,u)}{k}. \end{aligned}\tag{4.5}$$

Fix $x_0\in U$. Since $\lim_r\tilde{\mathscr{K}}_C^r(x_0)=A_C$ and $\lim_r\tilde{\mathscr{K}}_D^r(x_0)=A_D$, it suffices to show that $\mathscr{H}_{s_U}(\tilde{\mathscr{K}}_C^r(x_0),\tilde{\mathscr{K}}_D^r(x_0))\le 2\delta/(k-1)$. Given $y\in\tilde{\mathscr{K}}_C^r(x_0)$, we have to find $v\in\tilde{\mathscr{K}}_D^r(x_0)$ with $s_U(y,v)\le 2\delta/(k-1)$. Fix a C-orbit $(x_0,\dots,x_r=y)$. Recursively, construct a D-orbit $(x_0=u_0,\dots,u_r=:v)$ so that, for all $0\le j\le r$, $s_U(x_j,u_j)\le 2d\sum_{i=1}^{j}(1/k^i)$. In particular, $s_U(y,v)\le 2d/(k-1)$. □

4.6. Balanced measure of Barnsley-Demko. $\mathrm{Sym}^a(U)$ denotes the quotient of U^a by the action of the symmetric group S_a. In other words, $\mathrm{Sym}^a(U)$ is the space of effective divisors on U of degree a. Define a complete distance δ on $\mathrm{Sym}^a(U)$ as follows. Given $A=\sum_{j=1}^a(x_j)$ and $B=\sum_{j=1}^a(u_j)$, $\delta(A,B):=(1/a)\min_{\sigma\in S_a}(\sum_{j=1}^a s_U(x_j,u_{\sigma(j)}))$. Clearly, the map $(U,s_U)\xrightarrow{I}(\mathrm{Sym}^a(U),\delta)$, where $I(x)=a(x)$, is an isometric embedding. Let $\mathrm{Sym}(U):=\oplus_a\mathrm{Sym}^a(U)$.

Let $V\overset{i}{\hookrightarrow}U$ be a Stolz subdomain of a hyperbolic planar domain. Recall that $k=1/\mathrm{Lip}(i)>1$. Let $C\subset U\times V$ be a contractive curve, with projections $C\xrightarrow{p}U$ and $C\xrightarrow{q}V$. Let d be the topological degree of p. Define $\mathrm{Sym}^a(U)\xrightarrow{\mathscr{S}}\mathrm{Sym}^{da}(U)$, $\mathscr{S}(D):=i_*q_*p^*D$, and $\mathrm{Sym}^a(U)\xrightarrow{\gamma}[0,\infty)$, $\gamma(D):=\delta(dD,\mathscr{S}D)$.

Lemma 4.8. *With the notations above,* $\mathrm{Lip}(\mathscr{S})\le 1/k$. *Consequently, $\gamma(\mathscr{S})\le(1/k)\gamma$.*

Proof. Fix $A,B\in\mathrm{Sym}^a(U)$. Choose a simply connected domain $W\subset U$ with $\partial W\supset\mathrm{supp}(A)\cup\mathrm{supp}(B)\supset B(p)\cap\overline{W}$, and such that all the sections $W\xrightarrow{f_l}V$ of C, $1\le l\le d$, extend continuously to $\overline{W}$. We have $\delta(A,B)=(1/a)\sum_{j=1}^a s_U(x_j,u_j)$, with

$$A=\sum_{j=1}^a(x_j),\qquad B=\sum_{j=1}^a(u_j),\qquad \mathscr{S}A=\sum_{j=1}^a\sum_{l=1}^d(f_lx_j),\qquad \mathscr{S}B=\sum_{j=1}^a\sum_{l=1}^d(f_lu_j). \tag{4.6}$$

Then

$$\delta(\mathscr{S}A,\mathscr{S}B)\le\frac{1}{ad}\sum_{j=1}^a\sum_{l=1}^d s_U(f_lx_j,f_lu_j)\le\frac{1}{kad}\sum_{j=1}^a\sum_{l=1}^d s_U(x_j,u_j)=\frac{1}{k}\delta(A,B). \tag{4.7}$$

Consequently,

$$\gamma(\mathcal{S}A) = \delta(d\mathcal{S}A, \mathcal{S}\mathcal{S}A) = \delta(\mathcal{S}(dA), \mathcal{S}\mathcal{S}A) \le \frac{1}{k}\delta(dA, \mathcal{S}A) = \frac{1}{k}\gamma(A). \tag{4.8}$$

□

Lemma 4.9. *Given a continuous function* $\phi \in \mathcal{C}(U)$*, let* $\mathrm{Sym}^a(U) \xrightarrow{\Phi} \mathbb{C}$*,* $\Phi(D) := (1/a)\sum_{x\in D}\phi(x)$*. Then* $\mathrm{Lip}(\Phi) = \mathrm{Lip}(\phi)$ *and* $|\Phi(\mathcal{S}) - \Phi| \le \mathrm{Lip}(\phi)\gamma$*.*

Proof. Fix two divisors in $\mathrm{Sym}^a(U)$, $A = \sum_{j=1}^a (x_j)$ and $B = \sum_{j=1}^a (u_j)$, with $\delta(A,B) = (1/a)\sum_{j=1}^a s_U(x_j,u_j)$. We have

$$|\Phi A - \Phi B| = \frac{1}{a}\sum_{i=1}^{a} \left|\phi(x_j) - \phi(u_j)\right| \le \frac{\mathrm{Lip}(\phi)}{a}\sum_{i=1}^{a} s_U(x_j,u_j) = \mathrm{Lip}(\phi)\delta(A,B). \tag{4.9}$$

Since $\Phi I = \phi$, $\mathrm{Lip}(\phi) = \mathrm{Lip}(\Phi)$. Also,

$$\left|\Phi(\mathcal{S}A) - \Phi A\right| = \left|\Phi(\mathcal{S}A) - \Phi(dA)\right| \le \mathrm{Lip}(\Phi)\delta(\mathcal{S}A, dA) = \mathrm{Lip}(\phi)\gamma(A). \tag{4.10}$$

□

For a compact K, $\|\cdot\|_K$ denotes the L^∞-norm on $\mathcal{C}(K)$.

Corollary 4.10. *Given* $\phi \in \mathcal{C}(U)$ *and* $D \in \mathrm{Sym}(U)$*,* $\mu_\phi := \lim_{r\to\infty}\Phi(\mathcal{S}^r D)$ *exists in* $\mathbb{C}$ *and does not depend on* D*. The linear functional* $\mathcal{C}(U) \ni \phi \mapsto \mu_\phi \in \mathbb{C}$ *is real and nonnegative, with* $\mu_1 = 1$*. Also,* $|\mu_\phi| \le \|\phi\|_{A_C}$ *for all* $\phi \in \mathcal{C}(U)$*.*

Proof. Assume that $\mathrm{Lip}(\phi) < \infty$. Then

$$\left|\Phi\mathcal{S}^{r+1}D - \Phi\mathcal{S}^r D\right| \le \mathrm{Lip}(\phi)\gamma(\mathcal{S}^r D) \le \frac{\mathrm{Lip}(\phi)\gamma(D)}{k^r}, \tag{4.11}$$

so the limit exists in $\mathbb{C}$. For $A \in \mathrm{Sym}^a(U)$ and $B \in \mathrm{Sym}^b(U)$,

$$\begin{aligned} \left|\Phi\mathcal{S}^r A - \Phi\mathcal{S}^r B\right| &= \left|\Phi\mathcal{S}^r(bA) - \Phi\mathcal{S}^r(aB)\right| \le \mathrm{Lip}\left(\Phi\mathcal{S}^r\right)\delta(bA,aB) \\ &= \frac{\mathrm{Lip}(\phi)\delta(bA,aB)}{k^r} \longrightarrow 0, \end{aligned} \tag{4.12}$$

so the limit is independent of the divisor.

Let $K = \overline{\bigcup_r \tilde{\mathcal{K}}_C^r(\mathrm{supp}(D))}$. Then K is compact in U, and $\tilde{\mathcal{K}}_C(K) \subset K$. Given ϕ_1 and ϕ_2 in $\mathcal{C}(K)$, $\Phi_1(\mathcal{S}^r D)$ and $\Phi_2(\mathcal{S}^r D)$ are well defined. Moreover, for all $r \ge 0$, $|\Phi_1(\mathcal{S}^r D) - \Phi_2(\mathcal{S}^r D)| \le \|\phi_1 - \phi_2\|_K$. The subspace of $\mathcal{C}(K)$ formed by the functions that admit a Lipschitz extension to U is dense in $\mathcal{C}(K)$. It follows that, for arbitrary $\phi \in \mathcal{C}(U)$, $\lim_r \Phi(\mathcal{S}^r D)$ exists in $\mathbb{C}$. To see that this limit is independent of the divisor, take $A, B \in \mathrm{Sym}(U)$, put $K = \overline{\bigcup_r \tilde{\mathcal{K}}_C^r(\mathrm{supp}(A+B))}$, and approximate ϕ in $\mathcal{C}(K)$ with functions that admit Lipschitz extensions to U.

By definition, $\mu_\phi \in \mathbb{R}$ when ϕ is real, $\mu_\phi \ge 0$ when $\phi \ge 0$, and $\mu_1 = 1$. Choosing D with $\mathrm{supp}(D) \subset A_C$, we get $|\mu_\phi| \le \|\phi\|_{A_C}$. □

We can reformulate this corollary as follows.

Proposition 4.11. *There exists a (unique) probability measure μ_C on A_C with $\lim_{r\to\infty}\Phi(\mathcal{S}^r D) = \int_{A_C}\phi\, d\mu_C$ for all $\phi\in\mathcal{C}(U)$ and all $D\in\mathrm{Sym}(U)$.*

By definition, the measure μ_C describes the frequency with which Borel sets are visited by the total orbit of any point.

Remark 4.12. The operator $\mathcal{C}(A_C)\xrightarrow{T}\mathcal{C}(A_C)$, $(T\phi)(x) = \Phi(\mathcal{S}(x))$ is linear and continuous, with $\|T\| = 1$. With $\psi := T\phi$, we get $\Psi(\mathcal{S}^r(x)) = \Phi(\mathcal{S}^{r+1}(x))$, hence $\mu_\psi = \mu_\phi$. In other words, μ_C satisfies the *functional equation* $T^*\mu_C = \mu_C$, where T^* denotes the dual of T.

Remark 4.13. Corollary 4.4 and Remark 4.12 imply that $\mathrm{supp}(\mu_C) = A_C$.

5. Examples

5.1. Mixed iteration. Let V be Stolz in a hyperbolic planar domain U. Given finitely many holomorphic functions $U\xrightarrow{f_j}V$, $1\le j\le n$, the union $C(f_1,\dots,f_n)$ of their graphs is a contractive curve in $U\times V$. Let $A(f_1,\dots,f_n)$ denote the associated attractor. For a word $w = (w_1,\dots,w_l)\in\{1,\dots,n\}^r$ of length $r(w) = r$, let $U\xrightarrow{f_w}V$, $f_w = f_{w_r}\circ\cdots\circ f_{w_1}$. Since $\mathrm{Lip}(f_w)\le(1/k)^{l(w)}$, with $k = k(U,V) > 1$, f_w has a unique fixed point $x_w\in U$. We get $A(f_1,\dots,f_n) = \overline{\bigcup_w\{x_w\}}$.

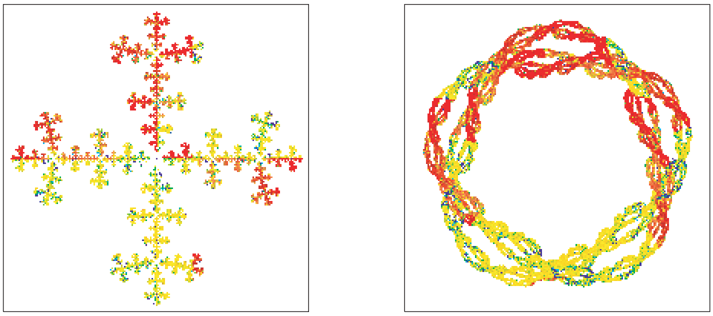
Remark 5.1. It is important that U be connected: let $f(x) = x^2(x-2)^2 = g(x)/2$, and $A = \{0,1,2\}$. We have $f(A)\cup g(A) = A$, $f'_{|A} = 0 = g'_{|A}$, and $\overline{f(U)\cup g(U)}\subset U$, where U is the union of the three disjoint disks of radius 0.1 and centers 0, 1, 2. Thus, A can be viewed as the attractor associated to the mixed iteration of f and g on U. But the point 2 is not periodic. Of course, no *domain* containing the points 0 and 1 can be f-invariant.

Remark 5.2. Let $C\subset U\times V$ be a contractive curve, and assume that $\beta(V) = V$, where $\beta(x+\mathrm{i}y) = x-\mathrm{i}y$. Let $\beta(C) := \{(u,\beta(v)) : (u,v)\in C\}$. If $U_0\xrightarrow{f}V$ is a section of C, then $U_0\xrightarrow{\beta f}V$ is a section of $\beta(C)$, and all sections of $\beta(C)$ are obtained in this way. $(V,ds_V)\xrightarrow{\beta}(V,ds_V)$ is an isometry, hence $\mathrm{Lip}(\beta f) = \mathrm{Lip}(f)$. Therefore, *given two contractive curves C and D in $U\times V$, with $\beta(V) = V$, the previous results still hold for the "mixed curve" $C\cup\beta(D)$.*

Example 5.3. Given finitely many $a_j\in\mathbb{C}$, let $m = \min(|a_j|)$ and $M = \max(|a_j|)$. If $m^2 - m > M$, then $\bigcup_j(y^2 = x - a_j)$ is contractive in $\Delta(0,m)\times\Delta(0,(m+M)^{1/2})$. The "cross" in Figure 5.1a is the associated attractor, with $a_1 = 2.01 = -a_2$. The color intensity indicates the density of the Barnsley-Demko measure.

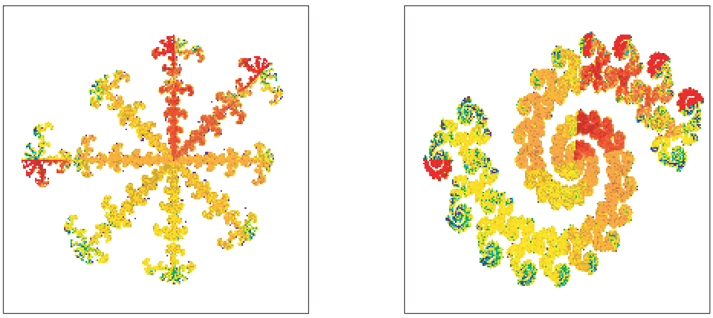
Example 5.4. Given k, a in Δ, the Möbius map $kM_a(z) = k((z-a)/(1-\overline{a}z))$ contracts the Poincaré metric. Fix k_1, k_2, k, a in Δ, and consider the mixed iteration in Δ of k_1M_0, k_2M_0, and kM_a. The point $(0,0)$ is a node of the corresponding curve, and $0\in A(k_1M_0,k_2M_0,kM_a)$. Figure 5.2 shows the associated attractor when $a = 0.9 = k$ and $k_1 = 0.9\exp(\mathrm{i}(\pi/r)) = -k_2$, with $r = 4$ (Figure 5.2a), and $r = 9$ (Figure 5.2b).

5.2. Contractive Blaschke maps. Given an effective divisor $\alpha = \sum d_j(a_j)$ in Δ, let $M_\alpha(z) := \prod_j((z-a_j)/(1-\overline{a_j}z))^{d_j}$.



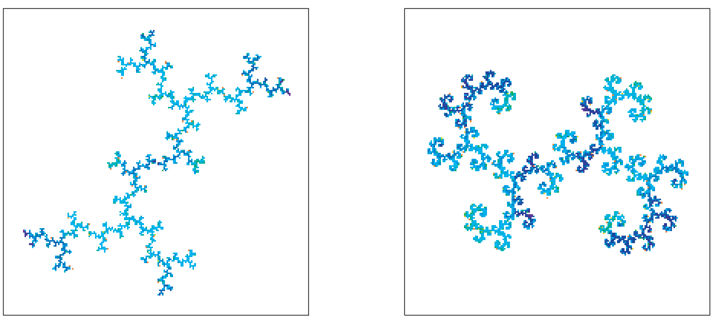
(a) (b)

FIGURE 5.1. Mixed Julia set (a), and Ochs' example (b).



(a) (b)

FIGURE 5.2. Mixed iteration of contractive Möbius maps.



(a) (b)

FIGURE 5.3. Mixed iteration of contractive Blaschke maps.

Example 5.5. If $|k| < 1$ and $e \le \min(d_j)$, the curve $(y^e = kM_\alpha(x))$ is contractive in $\Delta \times \Delta(0, |k|^{1/e})$. Figure 5.3 shows the associated attractor, with parameters $e = 2$, $\alpha = 3(0.7\mathrm{i})$, $k = 0.73$ (Figure 5.3a), and $e = 2$, $\alpha = 2(0.9) + 3(0.3 + \mathrm{i}0.5)$, $k = 0.75$ (Figure 5.3b).

5.3. An example by Ochs. The following example is taken from [5]. Consider the curve $C = (y^e = k\prod_{i=1}^{d}(x - a_i))$, with $e > d$; $\max(|a_i|) = 1$; and $|k| > e^e/d^d(e-d)^{e-d}$. Then C is contractive in a suitable product of annuli centered at 0. Figure 5.1b shows the corresponding attractor, for $C = (y^5 = 29\mathrm{i}(x-1)^2)$.

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