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DEGREE OF APPROXIMATION OF CONJUGATE OF A FUNCTION BELONGING TO Lip$(\xi(t),p)$ CLASS BY MATRIX SUMMABILITY MEANS OF CONJUGATE FOURIER SERIES

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Abstract. We determine the degree of approximation of conjugate of a function belonging to Lip$(\xi(t),p)$ class by matrix summability means of a conjugate series of a Fourier series.

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1. Introduction. Bernstein [2], Alexits [1], Sahney and Goel [14], and Chandra [4] have determined the degree of approximation of a function belonging to Lip α by $(C,1)$, (C,δ), (N,p_n), and $(\overline{N},p_n)$ means of its Fourier series. Working in the same direction Sahney and Rao [15] and Khan [6] have studied the degree of approximation of functions belonging to Lip(α,p) by (N,p_n) and (N,p,q) means, respectively. The (N,p,q) summability reduces to (N,p_n) summability for $q_n=1$ for all n, and to $(\overline{N},q_n)$ means when $p_n=1$ for all n. After quite a good amount of work on degree of approximation of function by different summability means of its Fourier series, for the first time in 1981, Qureshi [12, 13] discussed the degree of approximation of conjugate of a function belonging to Lip α and Lip(α,p) by (N,p_n) means of conjugate Fourier series. But nothing seems to have been done so far to obtain the degree of approximation of conjugate of a function belonging to Lip$(\xi(t),p)$ class by matrix means of conjugate Fourier series. The Lip$(\xi(t),p)$ class is a generalization of Lip α and Lip(α,p). Matrix means includes as special cases the method of $(C,1)$, (C,δ), (N,p_n), $(\overline{N},p_n)$, and (N,p,q) means. In an attempt to make an advance study in this direction, we, in this paper, establish a theorem on degree of approximation of conjugate of a function of Lip$(\xi(t),p)$ class by matrix summability means of conjugate series of a Fourier series then both the results of Qureshi [12, 13] come out as particular cases of our theorem.

2. Definitions and notations. Let f be periodic with period 2π and integrable in the Lebesgue sense. Let its Fourier series be given by

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx). \tag{2.1}$$

The conjugate series of (2.1) is given by

$$\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)=\sum_{n=1}^{\infty}B_n(x). \tag{2.2}$$

Let $\{p_n\}$ be a nonnegative nonincreasing generating sequence for (N, p_n) method such that

$$P_n = P(n) = p_0 + p_1 + p_2 + \cdots + p_n \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty. \tag{2.3}$$

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman Toeplitz [16], that is,

$$\begin{gathered} \sum_{k=0}^{n} a_{n,k} \longrightarrow 1, \quad \text{as } n \longrightarrow \infty, \qquad a_{n,k} = 0, \quad \text{for } k > n, \\ \sum_{k=0}^{n} |a_{n,k}| \le M, \quad \text{a finite constant.} \end{gathered} \tag{2.4}$$

Let $\sum_{m=0}^{\infty} u_m$ be an infinite series such that

$$s_k = u_0 + u_1 + u_2 + \cdots + u_k = \sum_{m=0}^{k} u_m, \tag{2.5}$$

that is, s_k denotes the kth partial sum of the series $\sum_{m=0}^{\infty} u_m$.

The sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k} s_{n-k} \tag{2.6}$$

defines the sequence $\{t_n\}$ of matrix means of the sequence $\{s_n\}$ generated by the sequence of the coefficients $(a_{n,k})$. The series $\sum u_n$ is said to be summable to the sum "S" by matrix method if $\lim_{n\to\infty} t_n$ exists and equal to S (see Zygmund [17]) and we write

$$t_n \longrightarrow S(T), \quad \text{as } n \longrightarrow \infty. \tag{2.7}$$

2.1. Particular cases. Seven important cases of matrix means are

(1) $(C,1)$ means when $a_{n,k} = 1/(n+1)$.
(2) Harmonic means when $a_{n,k} = 1/(n-k+1)\log n$.
(3) (C,δ) means when $a_{n,k} = \binom{n-k+\delta-1}{\delta-1} / \binom{n+\delta}{\delta}$.
(4) (H,p) means when $a_{n,k} = 1/\log^{p-1}(n+1)\prod_{q=0}^{p-1}\log^{q}(k+1)$.
(5) Nörlund means [11] when $a_{n,k} = p_{n-k}/P_n$ where $P_n = \sum_{k=0}^{n} p_k$, $q_n = 1$ for all n.
(6) Riesz means $(\overline{N}, p_n)$ [5] when $a_{n,k} = p_k/P_n$, $q_n = 1$ for all n.
(7) Generalized Nörlund mean (N, p, q) [3] when $a_{n,k} = p_{n-k} q_k / R_n$ where

$$R_n = \sum_{k=0}^{n} p_k q_{n-k}. \tag{2.8}$$

In particular cases (5), (6), and (7), $\{p_n\}$ and $\{q_n\}$ are two nonnegative monotonic nonincreasing sequences of real constants.

We define the norm

$$\|f\|_p = \left\{ \int_0^{2\pi} |f(x)|^p \, dx \right\}^{1/p}, \quad p \ge 1 \tag{2.9}$$

and let the degree of approximation be given by (see Zygmund [17])

$$E_n(f) = \operatorname*{Min}_{T_n} \|f - T_n\|_p, \tag{2.10}$$

where $T_n(x)$ is some nth degree trigonometric polynomial.

A function $f \in \operatorname{Lip}\alpha$ if

$$f(x+t) - f(x) = O(t^\alpha), \quad \text{for } 0 < \alpha \le 1 \tag{2.11}$$

and $f \in \operatorname{Lip}(\alpha, p)$ if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{1/p} = O(t^\alpha), \quad 0 < \alpha \le 1, \; p \ge 1 \tag{2.12}$$

(see [10, Definition 5.38]).

Given a positive increasing function $\xi(t)$ and an integer $p > 1$, then $f(x) \in \operatorname{Lip}(\xi(t), p)$ if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right\}^{1/p} = O(\xi(t)), \quad \text{(see [8]).} \tag{2.13}$$

In case $\xi(t) = t^\alpha$, we notice that $\operatorname{Lip}(\xi(t), p)$ class coincides with known $\operatorname{Lip}(\alpha, p)$ class [10].

We use the following notations:

$$\begin{gathered}
\psi(t) = f(x+t) - f(x-t), \\
A_{n,\tau} = \sum_{k=0}^{\tau} a_{n,n-k}, \\
\tau = \text{Integral part of } \frac{1}{t} = \left[\frac{1}{t}\right], \\
\overline{K}_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin t/2}.
\end{gathered} \tag{2.14}$$

3. Known theorems. Qureshi [12] has proved the following theorem.

Theorem 3.1. *If the sequence $\{p_n\}$ satisfies the following conditions:*

$$n|p_n| < C|P_n|, \qquad \sum_{k=1}^{n} k|p_k - p_{k-1}| < C|P_n|, \tag{3.1}$$

then the degree of approximation of a function $\tilde{f}(x)$, conjugate to a periodic function f with period 2π and belonging to the class $\operatorname{Lip}\alpha$, $0 < \alpha < 1$ by Nörlund means of its conjugate series, is given by

$$|\tilde{f}(x) - \tilde{t}_n(x)| = O\left(\frac{1}{P_n} \sum_{k=1}^{n} \frac{P_k}{k^{\alpha+1}}\right), \tag{3.2}$$

where $\tilde{t}_n(x)$ are the (N, p_n) means of series (2.2).

Qureshi [13] has proved another theorem in the following form.

Theorem 3.2. *If $f(x)$ is periodic and belongs to the class* $\mathrm{Lip}(\alpha,p)$ *for* $0<\alpha\le 1$, *and if the sequence* $\{p_n\}$ *is as defined in (2.3) with other requirements therein and if*

$$\int_1^n\left(\frac{(p(y)^q)}{y^{q\alpha+2-\delta q-q}}\right)=O\left(\frac{p(n)}{n^{\alpha-1/q-\delta-1}}\right), \tag{3.3}$$

then

$$||\tilde{t}_n-\tilde{f}||_p=O\left(\frac{1}{n^{\alpha-1/p}}\right), \tag{3.4}$$

where $\tilde{t}_n$ *are the* (N,p_n) *means of the series (2.2) and* $1/p+1/q=1$ *such that* $1\le p\le\infty$.

4. Main theorem.

Our object of this paper is to prove the following theorem.

Theorem 4.1. *If* $T=(a_{n,k})$ *is an infinite regular triangular matrix such that the elements* $a_{n,k}$ *is nonnegative and nondecreasing with* k, *then the degree of approximation of a function* $\tilde{f}(x)$, *conjugate to a* 2π*-periodic function* f *belonging to* $\mathrm{Lip}(\xi(t),p)$ *class by matrix summability means of its conjugate series is given by*

$$||\tilde{t}_n(x)-\tilde{f}(x)||=O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right) \tag{4.1}$$

provided $\xi(t)$ *satisfies the following conditions:*

$$\left\{\int_0^{1/n}\left(\frac{t\,|\psi(t)|}{\xi(t)}\right)^p dt\right\}^{1/p}=O\left(\frac{1}{n}\right), \tag{4.2}$$

$$\left\{\int_{1/n}^{\pi}\left(\frac{t^{-\delta}\psi(t)}{\xi(t)}\right)^p dt\right\}^{1/p}=O\left(n^{\delta}\right), \tag{4.3}$$

where δ *is an arbitrary number such that* $q(1-\delta)-1>0$, *conditions (4.2) and (4.3) hold uniformly in* x,

$$\tilde{t}_n(x)=\sum_{k=0}^{n}a_{n,n-k}\bar{s}_{n-k}(x), \tag{4.4}$$

that is, matrix means of conjugate Fourier series (2.2), $1/p+1/q=1$, *such that* $1\le p\le\infty$ *and*

$$\tilde{f}(x)=-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t\,dt. \tag{4.5}$$

5. Lemmas.

For the proof of our theorem the following lemmas are required.

Lemma 5.1 [9]. *If* $a_{n,k}$ *is nonnegative and nonincreasing with* k, *then for* $0\le a\le b\le\infty$, $a\le t\le\pi$ *and any* n,

$$\left|\sum_{k=a}^{b}a_{n,n-k}e^{i(n-k)t}\right|=O(A_{n,\tau}). \tag{5.1}$$

Lemma 5.2. *Under the conditions of Theorem 4.1 on* $(a_{n,k})$ *for* $0 < 1/n \le t \le \pi$,

$$\overline{K}_n(t) = O\left(\frac{A_{n,\tau}}{t}\right). \tag{5.2}$$

Proof. Since for $0 < 1/n \le t \le \pi$, $\sin(t/2) < t$, therefore for $t > 0$ and $\tau \le n$, we have,

$$\begin{aligned} |\overline{K}(t)| &= \left| \frac{1}{2\pi} \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)} \right| \\ &\le \left| \frac{1}{2\pi} \operatorname{Re} \sum_{k=0}^{n} \frac{a_{n,n-k} e^{i(n-k-1/2)t}}{\sin(t/2)} \right| \\ &= O\left(\frac{1}{t} \left| \sum_{k=0}^{n} a_{n,n-k} e^{i(n-k)t} \right| \left| e^{-it/2} \right| \right) \\ &= O\left(\frac{1}{t} \left| \sum_{k=0}^{n} a_{n,n-k} e^{i(n-k)t} \right| \right) \\ &= O\left(\frac{A_{n,\tau}}{t}\right) \end{aligned} \tag{5.3}$$

by Lemma 5.1. □

6. Proof of the main theorem. Let $\overline{s}_n(x)$ denote the nth partial sum of series (2.2), then, following [7], we have

$$\overline{s}_n(x) - \left(-\frac{1}{2\pi}\int_0^{\pi} \psi(t) \cot\frac{1}{2}t\, dt\right) = \frac{1}{2\pi}\int_0^{\pi} \psi(t) \frac{\cos(n+1/2)t}{\sin t/2}\, dt. \tag{6.1}$$

Now

$$\begin{aligned} &\sum_{k=0}^{n} a_{n,n-k} \left\{ \overline{s}_{n-k}(x) - \left(-\frac{1}{2\pi}\int_0^{\pi} \psi(t) \cot\frac{1}{2}t\, dt\right)\right\} \\ &\qquad = \frac{1}{2\pi}\int_0^{\pi} \psi(t) \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)}\, dt \end{aligned} \tag{6.2}$$

or

$$\begin{aligned} \tilde{t}_n(x) - \tilde{f}(x) &= \frac{1}{2\pi}\int_0^{\pi} \psi(t) \sum_{k=0}^{n} a_{n,n-k} \frac{\cos(n-k-1/2)t}{\sin(t/2)}\, dt \\ &= \int_0^{\pi} \psi(t) \overline{K}_n(t)\, dt \\ &= \int_0^{1/n} \psi(t) \overline{K}_n(t)\, dt + \int_{1/n}^{\pi} \psi(t) \overline{K}_n(t)\, dt \\ &= I_1 + I_2. \end{aligned} \tag{6.3}$$

Applying Hölder's inequality and the fact that $\psi(t) = W(\mathrm{Lip}\,\xi(t), p)$, we get

$$
\begin{aligned}
I_1 &= \int_0^{1/n} \psi(t)\overline{K}_n(t)\,dt \\
&\le O\left[\int_0^{1/n}\left\{\frac{t\,|\psi(t)|}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_0^{1/n}\left\{\frac{\overline{K}_n(t)\xi(t)}{t}\right\}^q dt\right]^{1/q} \\
&= o\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left\{\frac{\xi(t)}{t}\frac{1}{2\pi}\left|\sum_{k=0}^{n} a_{n,n-k}\frac{\cos(n-k-1/2)t}{\sin(t/2)}\right|\right\}^q dt\right]^{1/q} \quad \text{by (4.2)} \\
&= O\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left\{\frac{\xi(t)}{t}\sum_{k=0}^{n}\frac{a_{n,n-k}}{t}\right\}^q\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left(\frac{\xi(t)}{t^2}\right)^q dt\right]^{1/q} \\
&= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left[\int_1^{1/n}\frac{dt}{t^{2q}}\right]^{1/q} \quad \text{by mean value theorem} \\
&= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left[\left\{\frac{t^{-2q+1}}{-2q+1}\right\}_1^{1/n}\right]^{1/q} \\
&= O\left(\frac{\xi(1/n)}{n}\right)O\left(n^{2-1/q}\right) \\
&= O\left(\xi\left(\frac{1}{n}\right)n^{1-1/q}\right), \\
I_1 &= O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right) \quad \left(\text{since } \frac{1}{p}+\frac{1}{q}=1\right).
\end{aligned}
\tag{6.4}
$$

Consider I_2

$$
\begin{aligned}
I_2 &= \left[\int_{1/n}^{\pi}\left\{\frac{|t^{-\delta}\psi(t)|}{\xi(t)}\right\}^p dt\right]^{1/p}\left[\int_{1/n}^{\pi}\left\{\frac{\overline{K}_n(t)\xi(t)}{t^{-\delta}}\right\}^q dt\right]^{1/q} \\
&= O\left[\int_{1/n}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^p dt\right]^{1/p} O\left[\int_{1/n}^{\pi}\left\{\frac{\xi(t)A_{n,\tau}}{t^{-\delta+1}}\right\}^q dt\right]^{1/q} \quad \text{by Lemma 5.2} \\
&= O(n^{\delta})\cdot O\left[\int_{1/n}^{\pi}\left\{\frac{\xi(t)A_{n,\tau}}{t^{-\delta+1}}\right\}^q dt\right]^{1/q} \quad \text{by condition (4.3)} \\
&= O(n^{\delta})\cdot O\left[\int_{1/n}^{n}\left\{\frac{\xi(1/y)A_{n,[y]}}{y^{\delta-1}}\right\}^q \frac{dy}{y^2}\right]^{1/q} \\
&= O(n^{\delta})\cdot O\left(\xi\left(\frac{1}{n}\right)A_{n,n}\right)\left[\int_1^{n}\left\{\frac{dy}{y^{q(\delta-1)+2}}\right\}\right]^{1/q} \quad \text{by mean value theorem}
\end{aligned}
$$

$$= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)\right)\left[\left\{\frac{y^{-q(\delta-1)-1}}{-q(\delta-1)-1}\right\}_1^n\right]^{1/q}$$

$$= O\left(n^{\delta}\xi\left(\frac{1}{n}\right)\right)O\left(n^{-\delta+1-1/q}\right)$$

$$= O\left(\xi\left(\frac{1}{n}\right)n^{1-1/q}\right)$$

$$I_2 = O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right) \quad \left(\text{since } \frac{1}{p}+\frac{1}{q}=1\right). \tag{6.5}$$

By combining (6.3), (6.4), and (6.5) we have

$$|\tilde{t}_n(x)-\tilde{f}(x)| = O\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right), \tag{6.6}$$

therefore

$$\begin{aligned}\|\tilde{t}_n(x)-\tilde{f}(x)\|_p &= O\left[\left\{\int_0^{2\pi}\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right)^p dx\right\}^{1/p}\right]\\ &= O\left[\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right)\left(\int_0^{2\pi} dx\right)^{1/p}\right]\\ &= O\left[\xi\left(\frac{1}{n}\right)n^{1/p}\right].\end{aligned} \tag{6.7}$$

This completes the proof of the theorem. □

7. Applications. The following corollaries can be derived from the main theorem.

Corollary 7.1. *If $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$, then the degree of approximation of a function $\tilde{f}(x)$, conjugate to 2π-periodic function f belonging to the class* $\mathrm{Lip}(\alpha,p)$ *is given by*

$$|\tilde{t}_n(x)-\tilde{f}(x)| = O\left(\frac{1}{n^{\alpha-1/p}}\right). \tag{7.1}$$

Proof. We have

$$\|\tilde{t}_n(x)-\tilde{f}(x)\|_p = O\left\{\int_0^{2\pi}|\tilde{t}_n(x)-\tilde{f}(x)|^p dx\right\}^{1/p} \tag{7.2}$$

or

$$\left(\xi\left(\frac{1}{n}\right)n^{1/p}\right)^p = O\left\{\int_0^{2\pi}|\tilde{t}_n(x)-\tilde{f}(x)|^p dx\right\}^{1/p} \tag{7.3}$$

or

$$O(1) = O\left\{\int_0^{2\pi}|\tilde{t}_n(x)-\tilde{f}(x)|^p dx\right\}^{1/p} O\left(\frac{1}{\xi(1/n)n^{1/p}}\right). \tag{7.4}$$

Hence

$$|\tilde{t}_n(x)-\tilde{f}(x)| = O\left[\xi\left(\frac{1}{n}\right)n^{1/p}\right] \tag{7.5}$$

for if not the right-hand side will be $O(1)$, therefore

$$|\tilde{t}_n(x)-\tilde{f}(x)| = O\left[\left(\frac{1}{n}\right)^{\alpha} n^{1/p}\right] = O\left(\frac{1}{n^{\alpha-1/p}}\right). \tag{7.6}$$

This completes the proof. □

Corollary 7.2. *If $p \to \infty$ in Corollary 7.1, then for $0 < \alpha < 1$,*

$$|\tilde{t}_n(x)-\tilde{f}(x)|_p = O\left(\frac{1}{n^{\alpha}}\right). \tag{7.7}$$

Remark 7.3. An independent proof of Corollary 7.1 can be derived along the same lines as the theorem.

8. Particular cases. (1) If $a_{n,k} = p_{n-k}/p_n$, $\xi(t) = t^{\alpha}$, $0 < \alpha < 1$, $p \to \infty$ and using $1/n^{\alpha} \le 1/p_n \sum_{k=1}^{n} p_k/k^{\alpha+1}$ (see [14, Lemma 1]), then the result of Qureshi [12] becomes the particular case of the main theorem.

(2) The result of Qureshi [13] becomes the particular case of our theorem if $(a_{n,k})$ is defined as in case (1) and $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$.

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References

[1] G. Alexits, *Über die Annanherugng einer stetign Function durch die Cesároschen Mittel in hrer Fourier-reihe*, Math. Ann. **100** (1928), 264–277.

[2] S. N. Bernstein, *Sur l'order de la meilleure approximation des fonctions continues par les polynomes de degre donne*, Mem. Cl. Sci. Acad. Roy. Belg. **4** (1912), 1–103.

[3] D. Borwein, *On products of sequences*, J. London Math. Soc. **33** (1958), 352–357. MR 21#244. Zbl 081.28003.

[4] P. Chandra, *On the degree of approximation of functions belonging to the Lipschitz class*, Nanta Math. **8** (1975), no. 1, 88–91. MR 54#13431. Zbl 0319.41015.

[5] G. H. Hardy, *Divergent Series*, Clarendon Press, Oxford, 1949. MR 11#25a. Zbl 032.05801.

[6] H. H. Khan, *On the degree of approximation of functions belonging to class* Lip (α,p), Indian J. Pure Appl. Math. **5** (1974), no. 2, 132–136. MR 52#1130. Zbl 308.41010.

[7] S. Lal, *On K^{λ}-summability of conjugate series of Fourier series*, Bull. Calcutta Math. Soc. **89** (1997), no. 1, 97–104. CMP 1 628 717. Zbl 910.42003.

[8] ———, *On degree of approximation of functions belonging to the weighted $(L^p,\xi(t))$ class by $(C,1)(E,1)$ means*, Tamkang J. Math. **30** (1999), no. 1, 47–52. MR 2000h#41019.

[9] ———, *On the degree of approximation of conjugate of a function belonging to weighted $W(L^p,\xi(t))$ class by matrix summability means of conjugate series of a Fourier series*, Tamkang J. Math. **31** (2000), no. 4, 279–288. MR 2001k:42014.

[10] L. McFadden, *Absolute Nörlund summability*, Duke Math. J. **9** (1942), 168–207. MR 3,295g. Zbl 0061.12106.

[11] N. E. Nölund, *Surene application des fonctions permutables*, Lunds Universitets Arsskrift **16** (1919), 1–10.

[12] K. Qureshi, *On the degree of approximation of functions belonging to the Lipschitz class by means of a conjugate series*, Indian J. Pure Appl. Math. **12** (1981), no. 9, 1120–1123. MR 82k#42005. Zbl 475.42004.

[13] ______, *On the degree of approximation of functions belonging to the class* Lip(α, p), Indian J. Pure Appl. Math. **13** (1982), no. 4, 466–470. MR 83f:41014a. Zbl 494.42004.

[14] B. N. Sahney and D. S. Goel, *On the degree of approximation of continuous functions*, Ranchi Univ. Math. J. **4** (1973), 50–53. MR 50#5332. Zbl 296.41008.

[15] B. N. Sahney and V. Rao, *Error bounds in the approximation of functions*, Bull. Austral. Math. Soc. **6** (1972), 11–18. MR 45#2393. Zbl 0229.42008.

[16] O. Toeplitz, *Überallgemeine lineare Mittel bil dungen*, P. M. F. **22** (1913), 113–119.

[17] A. Zygmund, *Trigonometrical Series*, Monografie Matematyczne, vol. 5, Subwencji Fundusz Kultury Narodowej, Warszawa, 1935. Zbl 0011.01703.

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