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ON FUZZY DOT SUBALGEBRAS OF BCH-ALGEBRAS

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ABSTRACT. We introduce the notion of fuzzy dot subalgebras in BCH-algebras, and study its various properties.

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1. Introduction. In [4], Hu and Li introduced the notion of BCH-algebras which are a generalization of BCK/BCI-algebras. In 1965, Zadeh [6] introduced the concept of fuzzy subsets. Since then several researchers have applied this notion to various mathematical disciplines. Jun [5] applied it to BCH-algebras, and he considered the fuzzification of ideals and filters in BCH-algebras. In this paper, we introduce the notion of a fuzzy dot subalgebra of a BCH-algebra as a generalization of a fuzzy subalgebra of a BCH-algebra, and then we investigate several basic properties related to fuzzy dot subalgebras.

2. Preliminaries. A BCH-algebra is an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following conditions:

(i) $x*x=0$,
(ii) $x*y=0=y*x$ implies $x=y$,
(iii) $(x*y)*z=(x*z)*y$ for all $x,y,z\in X$.

In any BCH-algebra X, the following hold (see [2]):

(P1) $x*0=x$,
(P2) $x*0=0$ implies $x=0$,
(P3) $0*(x*y)=(0*x)*(0*y)$.

A BCH-algebra X is said to be *medial* if $x*(x*y)=y$ for all $x,y\in X$. A nonempty subset S of a BCH-algebra X is called a *subalgebra* of X if $x*y\in S$ whenever $x,y\in S$. A map f from a BCH-algebra X to a BCH-algebra Y is called a *homomorphism* if $f(x*y)=f(x)*f(y)$ for all $x,y\in X$.

We now review some fuzzy logic concepts. A fuzzy subset of a set X is a function $\mu:X\to[0,1]$. For any fuzzy subsets μ and ν of a set X, we define

$$\mu\subseteq\nu\Longleftrightarrow\mu(x)\le\nu(x)\quad\forall x\in X,$$
$$(\mu\cap\nu)(x)=\min\{\mu(x),\nu(x)\}\quad\forall x\in X. \tag{2.1}$$

Let $f:X\to Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X.

The fuzzy subset ν of Y defined by

$$\nu(y) := \begin{cases} \sup_{x\in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing,\ \forall y \in Y, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

is called the *image* of μ under f, denoted by $f[\mu]$. If ν is a fuzzy subset of Y, the fuzzy subset μ of X given by $\mu(x) = \nu(f(x))$ for all $x \in X$ is called the *preimage* of ν under f and is denoted by $f^{-1}[\nu]$.

A fuzzy relation μ on a set X is a fuzzy subset of $X \times X$, that is, a map $\mu : X \times X \to [0,1]$. A fuzzy subset μ of a BCH-algebra X is called a *fuzzy subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3. Fuzzy product subalgebras. In what follows let X denote a BCH-algebra unless otherwise specified.

Definition 3.1. A fuzzy subset μ of X is called a *fuzzy dot subalgebra* of X if $\mu(x * y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in X$.

Example 3.2. Consider a BCH-algebra $X = \{0, a, b, c\}$ having the following Cayley table (see [1]):

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	c	0	c
c	c	0	0	0

Define a fuzzy set μ in X by $\mu(0) = 0.5$, $\mu(a) = 0.6$, $\mu(b) = 0.4$, $\mu(c) = 0.3$. It is easy to verify that μ is a fuzzy dot subalgebra of X.

Note that every fuzzy subalgebra is a fuzzy dot subalgebra, but the converse is not true. In fact, the fuzzy dot subalgebra μ in Example 3.2 is not a fuzzy subalgebra since

$$\mu(a * a) = \mu(0) = 0.5 < 0.6 = \mu(a) = \min\{\mu(a), \mu(a)\}. \tag{3.1}$$

Proposition 3.3. *If μ is a fuzzy dot subalgebra of X, then*

$$\mu(0) \geq (\mu(x))^2, \qquad \mu(0^n * x) \geq (\mu(x))^{2n+1}, \tag{3.2}$$

for all $x \in X$ and $n \in \mathbb{N}$ where $0^n * x = 0 * (0 * (\cdots (0 * x) \cdots))$ in which 0 occurs n times.

Proof. Since $x * x = 0$ for all $x \in X$, it follows that

$$\mu(0) = \mu(x * x) \geq \mu(x) \cdot \mu(x) = (\mu(x))^2 \tag{3.3}$$

for all $x \in X$. The proof of the second part is by induction on n. For $n = 1$, we have $\mu(0 * x) \geq \mu(0) \cdot \mu(x) \geq (\mu(x))^3$ for all $x \in X$. Assume that $\mu(0^k * x) \geq (\mu(x))^{2k+1}$ for all $x \in X$. Then

$$\begin{aligned}\mu(0^{k+1} * x) &= \mu(0 * (0^k * x)) \geq \mu(0) \cdot \mu(0^k * x) \\ &\geq (\mu(x))^2 \cdot (\mu(x))^{2k+1} = (\mu(x))^{2(k+1)+1}.\end{aligned} \tag{3.4}$$

Hence $\mu(0^n * x) \geq (\mu(x))^{2n+1}$ for all $x \in X$ and $n \in \mathbb{N}$. □

Proposition 3.4. *Let μ be a fuzzy dot subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}(\mu(x_n))^2 = 1$, then $\mu(0) = 1$.*

Proof. According to Proposition 3.3, $\mu(0) \geq (\mu(x_n))^2$ for each $n \in \mathbb{N}$. Since $1 \geq \mu(0) \geq \lim_{n\to\infty}(\mu(x_n))^2 = 1$, it follows that $\mu(0) = 1$. □

Theorem 3.5. *If μ and ν are fuzzy dot subalgebras of X, then so is $\mu \cap \nu$.*

Proof. Let $x, y \in X$, then

$$\begin{aligned}(\mu \cap \nu)(x * y) &= \min\{\mu(x * y), \nu(x * y)\} \\ &\geq \min\{\mu(x) \cdot \mu(y), \nu(x) \cdot \nu(y)\} \\ &\geq (\min\{\mu(x), \nu(x)\}) \cdot (\min\{\mu(y), \nu(y)\}) \\ &= ((\mu \cap \nu)(x)) \cdot ((\mu \cap \nu)(y)).\end{aligned} \tag{3.5}$$

Hence $\mu \cap \nu$ is a fuzzy dot subalgebra of X. □

Note that a fuzzy subset μ of X is a fuzzy subalgebra of X if and only if a nonempty level subset

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\} \tag{3.6}$$

is a subalgebra of X for every $t \in [0,1]$. But, we know that if μ is a fuzzy dot subalgebra of X, then there exists $t \in [0,1]$ such that

$$U(\mu; t) := \{x \in X \mid \mu(x) \geq t\} \tag{3.7}$$

is not a subalgebra of X. In fact, if μ is the fuzzy dot subalgebra of X in Example 3.2, then

$$U(\mu; 0.4) = \{x \in X \mid \mu(x) \geq 0.4\} = \{0, a, b\} \tag{3.8}$$

is not a subalgebra of X since $b * a = c \notin U(\mu; 0.4)$.

Theorem 3.6. *If μ is a fuzzy dot subalgebra of X, then*

$$U(\mu; 1) := \{x \in X \mid \mu(x) = 1\} \tag{3.9}$$

is either empty or is a subalgebra of X.

Proof. If x and y belong to $U(\mu; 1)$, then $\mu(x * y) \geq \mu(x) \cdot \mu(y) = 1$. Hence $\mu(x * y) = 1$ which implies $x * y \in U(\mu; 1)$. Consequently, $U(\mu; 1)$ is a subalgebra of X. □

Theorem 3.7. *Let X be a medial BCH-algebra and let μ be a fuzzy subset of X such that*

$$\mu(0*x) \geq \mu(x), \qquad \mu(x*(0*y)) \geq \mu(x)\cdot\mu(y), \tag{3.10}$$

for all $x, y \in X$. Then μ is a fuzzy dot subalgebra of X.

Proof. Since X is medial, we have $0*(0*y) = y$ for all $y \in X$. Hence

$$\mu(x*y) = \mu(x*(0*(0*y))) \geq \mu(x)\cdot\mu(0*y) \geq \mu(x)\cdot\mu(y) \tag{3.11}$$

for all $x, y \in X$. Therefore μ is a fuzzy dot subalgebra of X. □

Theorem 3.8. *Let $g : X \to Y$ be a homomorphism of BCH-algebras. If ν is a fuzzy dot subalgebra of Y, then the preimage $g^{-1}[\nu]$ of ν under g is a fuzzy dot subalgebra of X.*

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{aligned} g^{-1}[\nu](x_1*x_2) &= \nu(g(x_1*x_2)) = \nu(g(x_1)*g(x_2)) \\ &\geq \nu(g(x_1))\cdot\nu(g(x_2)) = g^{-1}[\nu](x_1)\cdot g^{-1}[\nu](x_2). \end{aligned} \tag{3.12}$$

Thus $g^{-1}[\nu]$ is a fuzzy dot subalgebra of X. □

Theorem 3.9. *Let $f : X \to Y$ be an onto homomorphism of BCH-algebras. If μ is a fuzzy dot subalgebra of X, then the image $f[\mu]$ of μ under f is a fuzzy dot subalgebra of Y.*

Proof. For any $y_1, y_2 \in Y$, let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, and $A_{12} = f^{-1}(y_1 * y_2)$. Consider the set

$$A_1 * A_2 := \{x \in X \mid x = a_1 * a_2 \text{ for some } a_1 \in A_1,\ a_2 \in A_2\}. \tag{3.13}$$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that

$$f(x) = f(x_1*x_2) = f(x_1)*f(x_2) = y_1*y_2, \tag{3.14}$$

that is, $x \in f^{-1}(y_1*y_2) = A_{12}$. Hence $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f[\mu](y_1*y_2) &= \sup_{x\in f^{-1}(y_1*y_2)} \mu(x) = \sup_{x\in A_{12}} \mu(x) \\ &\geq \sup_{x\in A_1*A_2} \mu(x) \geq \sup_{x_1\in A_1,\ x_2\in A_2} \mu(x_1*x_2) \\ &\geq \sup_{x_1\in A_1,\ x_2\in A_2} \mu(x_1)\cdot\mu(x_2). \end{aligned} \tag{3.15}$$

Since $\cdot : [0,1]\times[0,1] \to [0,1]$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\tilde{x}_1 \geq \sup_{x_1\in A_1}\mu(x_1) - \delta$ and $\tilde{x}_2 \geq \sup_{x_2\in A_2}\mu(x_2) - \delta$, then $\tilde{x}_1 \cdot \tilde{x}_2 \geq \sup_{x_1\in A_1}\mu(x_1)\cdot\sup_{x_2\in A_2}\mu(x_2) - \varepsilon$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu(a_1) \geq$

$\sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\mu(a_2) \geq \sup_{x_2 \in A_2} \mu(x_2) - \delta$. Then

$$\mu(a_1) \cdot \mu(a_2) \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) - \varepsilon. \tag{3.16}$$

Consequently,

$$\begin{aligned} f[\mu](y_1 * y_2) &\geq \sup_{x_1 \in A_1,\, x_2 \in A_2} \mu(x_1) \cdot \mu(x_2) \\ &\geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) \\ &= f[\mu](y_1) \cdot f[\mu](y_2), \end{aligned} \tag{3.17}$$

and hence $f[\mu]$ is a fuzzy dot subalgebra of Y. □

Definition 3.10. Let σ be a fuzzy subset of X. The *strongest fuzzy σ-relation* on X is the fuzzy subset μ_σ of $X \times X$ given by $\mu_\sigma(x, y) = \sigma(x) \cdot \sigma(y)$ for all $x, y \in X$.

Theorem 3.11. *Let μ_σ be the strongest fuzzy σ-relation on X, where σ is a fuzzy subset of X. If σ is a fuzzy dot subalgebra of X, then μ_σ is a fuzzy dot subalgebra of $X \times X$.*

Proof. Assume that σ is a fuzzy dot subalgebra of X. For any x_1, x_2, y_1, $y_2 \in X$, we have

$$\begin{aligned} \mu_\sigma((x_1, y_1) * (x_2, y_2)) &= \mu_\sigma(x_1 * x_2, y_1 * y_2) \\ &= \sigma(x_1 * x_2) \cdot \sigma(y_1 * y_2) \\ &\geq (\sigma(x_1) \cdot \sigma(x_2)) \cdot (\sigma(y_1) \cdot \sigma(y_2)) \\ &= (\sigma(x_1) \cdot \sigma(y_1)) \cdot (\sigma(x_2) \cdot \sigma(y_2)) \\ &= \mu_\sigma(x_1, y_1) \cdot \mu_\sigma(x_2, y_2), \end{aligned} \tag{3.18}$$

and so μ_σ is a fuzzy dot subalgebra of $X \times X$. □

Definition 3.12. Let σ be a fuzzy subset of X. A fuzzy relation μ on X is called a *fuzzy σ-product relation* if $\mu(x, y) \geq \sigma(x) \cdot \sigma(y)$ for all $x, y \in X$.

Definition 3.13. Let σ be a fuzzy subset of X. A fuzzy relation μ on X is called a *left fuzzy relation* on σ if $\mu(x, y) = \sigma(x)$ for all $x, y \in X$.

Similarly, we can define a right fuzzy relation on σ. Note that a left (resp., right) fuzzy relation on σ is a fuzzy σ-product relation.

Theorem 3.14. *Let μ be a left fuzzy relation on a fuzzy subset σ of X. If μ is a fuzzy dot subalgebra of $X \times X$, then σ is a fuzzy dot subalgebra of X.*

Proof. Assume that a left fuzzy relation μ on σ is a fuzzy dot subalgebra of $X \times X$. Then

$$\begin{aligned} \sigma(x_1 * x_2) = \mu(x_1 * x_2, y_1 * y_2) &= \mu((x_1, y_1) * (x_2, y_2)) \\ \geq \mu(x_1, y_1) \cdot \mu(x_2, y_2) &= \sigma(x_1) \cdot \sigma(x_2) \end{aligned} \tag{3.19}$$

for all $x_1, x_2, y_1, y_2 \in X$. Hence σ is a fuzzy dot subalgebra of X. □

Theorem 3.15. *Let μ be a fuzzy relation on X satisfying the inequality $\mu(x,y) \leq \mu(x,0)$ for all $x,y \in X$. Given $z \in X$, let σ_z be a fuzzy subset of X defined by $\sigma_z(x) = \mu(x,z)$ for all $x \in X$. If μ is a fuzzy dot subalgebra of $X \times X$, then σ_z is a fuzzy dot subalgebra of X for all $z \in X$.*

Proof. Let $z,x,y \in X$, then

$$\begin{aligned}\sigma_z(x*y) &= \mu(x*y,z) = \mu(x*y,z*0) \\ &= \mu\big((x,z)*(y,0)\big) \geq \mu(x,z)\cdot\mu(y,0) \\ &\geq \mu(x,z)\cdot\mu(y,z) = \sigma_z(x)\cdot\sigma_z(y),\end{aligned} \tag{3.20}$$

completing the proof. □

Theorem 3.16. *Let μ be a fuzzy relation on X and let σ_μ be a fuzzy subset of X given by $\sigma_\mu(x) = \inf_{y\in X}\mu(x,y)\cdot\mu(y,x)$ for all $x \in X$. If μ is a fuzzy dot subalgebra of $X \times X$ satisfying the equality $\mu(x,0) = 1 = \mu(0,x)$ for all $x \in X$, then σ_μ is a fuzzy dot subalgebra of X.*

Proof. For any $x,y,z \in X$, we have

$$\begin{aligned}\mu(x*y,z) &= \mu(x*y,z*0) = \mu\big((x,z)*(y,0)\big) \\ &\geq \mu(x,z)\cdot\mu(y,0) = \mu(x,z), \\ \mu(z,x*y) &= \mu(z*0,x*y) = \mu\big((z,x)*(0,y)\big) \\ &\geq \mu(z,x)\cdot\mu(0,y) = \mu(z,x).\end{aligned} \tag{3.21}$$

It follows that

$$\begin{aligned}\mu(x*y,z)\cdot\mu(z,x*y) &\geq \mu(x,z)\cdot\mu(z,x) \\ &\geq \big(\mu(x,z)\cdot\mu(z,x)\big)\cdot\big(\mu(y,z)\cdot\mu(z,y)\big)\end{aligned} \tag{3.22}$$

so that

$$\begin{aligned}\sigma_\mu(x*y) &= \inf_{z\in X}\mu(x*y,z)\cdot\mu(z,x*y) \\ &\geq \Big(\inf_{z\in X}\mu(x,z)\cdot\mu(z,x)\Big)\cdot\Big(\inf_{z\in X}\mu(y,z)\cdot\mu(z,y)\Big) \\ &= \sigma_\mu(x)\cdot\sigma_\mu(y).\end{aligned} \tag{3.23}$$

This completes the proof. □

Definition 3.17 (see Choudhury et al. [3])**.** A *fuzzy map* f from a set X to a set Y is an ordinary map from X to the set of all fuzzy subsets of Y satisfying the following conditions:

(C1) for all $x \in X$, there exists $y_x \in X$ such that $(f(x))(y_x) = 1$,

(C2) for all $x \in X$, $f(x)(y_1) = f(x)(y_2)$ implies $y_1 = y_2$.

One observes that a fuzzy map f from X to Y gives rise to a unique ordinary map $\mu_f : X \times X \to I$, given by $\mu_f(x,y) = f(x)(y)$. One also notes that a fuzzy map from X to Y gives a unique ordinary map $f_1 : X \to Y$ defined as $f_1(x) = y_x$.

Definition 3.18. A fuzzy map f from a BCH-algebra X to a BCH-algebra Y is called a *fuzzy homomorphism* if

$$\mu_f(x_1 * x_2, y) = \sup_{y = y_1 * y_2} \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2) \tag{3.24}$$

for all $x_1, x_2 \in X$ and $y \in Y$.

One notes that if f is an ordinary map, then the above definition reduces to an ordinary homomorphism. One also observes that if a fuzzy map f is a fuzzy homomorphism, then the induced ordinary map f_1 is an ordinary homomorphism.

Proposition 3.19. *Let $f : X \to Y$ be a fuzzy homomorphism of BCH-algebras. Then*

(i) $\mu_f(x_1 * x_2, y_1 * y_2) \geq \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2)$ *for all* $x_1, x_2 \in X$ *and* $y_1, y_2 \in Y$.

(ii) $\mu_f(0, 0) = 1$.

(iii) $\mu_f(0 * x, 0 * y) \geq \mu_f(x, y)$ *for all* $x \in X$ *and* $y \in Y$.

(iv) *if Y is medial and $\mu_f(x, y) = t \neq 0$, then $\mu_f(0, y_x * y) = t$ for all $x \in X$ and $y \in Y$, where $y_x \in Y$ with $\mu_f(x, y_x) = 1$.*

Proof. (i) For every $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have

$$\begin{aligned} \mu_f(x_1 * x_2,\ y_1 * y_2) &= \sup_{y_1 * y_2 = \tilde{y}_1 * \tilde{y}_2} \mu_f(x_1, \tilde{y}_1) \cdot \mu_f(x_2, \tilde{y}_2) \\ &\geq \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2). \end{aligned} \tag{3.25}$$

(ii) Let $x \in X$ and $y_x \in Y$ be such that $\mu_f(x, y_x) = 1$. Using (I) and (i), we get

$$\mu_f(0, 0) = \mu_f(x * x, y_x * y_x) \geq \mu_f(x, y_x) \cdot \mu_f(x, y_x) = 1 \tag{3.26}$$

and so $\mu_f(0, 0) = 1$.

(iii) The proof follows from (i) and (ii).

(iv) Assume that Y is medial and $\mu_f(x, y) = t \neq 0$ for all $x \in X$ and $y \in Y$, and let $y_x \in Y$ be such that $\mu_f(x, y_x) = 1$. Then

$$\begin{aligned} \mu_f(0, y_x * y) &= \mu_f(x * x, y_x * y) \geq \mu_f(x, y_x) \cdot \mu_f(x, y) \\ &= t = \mu_f(x, y) = \mu_f(x * 0, y_x * (y_x * y)) \\ &\geq \mu_f(x, y_x) \cdot \mu_f(0, y_x * y) = \mu_f(0, y_x * y), \end{aligned} \tag{3.27}$$

and hence $\mu_f(0, y_x * y) = t$. This completes the proof. □

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