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ANALOGUES OF SOME TAUBERIAN THEOREMS FOR STRETCHINGS

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Abstract. We investigate the effect of four-dimensional matrix transformation on new classes of double sequences. Stretchings of a double sequence is defined, and this definition is used to present a four-dimensional analogue of D. Dawson's copy theorem for stretching of a double sequence. In addition, the multidimensional analogue of D. Dawson's copy theorem is used to characterize convergent double sequences using stretchings.

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1. Introduction. In this paper, RH-regular matrices and the stretching of double sequences are used to characterize P-convergent sequences. To achieve this goal we begin by defining an ϵ-Pringsheim-copy and a stretching of double sequences. In addition, the copy theorem of Dawson in [1] will be extended as follows: if each of A and T is an RH-regular matrix, and x is any bounded double complex sequence with ϵ being any bounded positive term double sequence with $P\text{-}\lim_{i,j}\epsilon_{i,j} = 0$, then there exists a stretching y of x such that $T(Ay)$ exists and contains an ϵ-Pringsheim-copy of x. By using this extended copy theorem some natural implications and variations of this extended copy theorem will be presented.

2. Definitions, notations, and preliminary results

Definition 2.1 (see [3]). A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k,l > N$. We will describe such an x more briefly as "P-convergent."

Definition 2.2 (see [3]). A double sequence x is called definite divergent, if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1$, $k \geq n_2$.

Definition 2.3. The double sequence $[y]$ is a double subsequence of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{matrix} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{matrix} \tag{2.1}$$

The double sequence x is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. A two-dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. The Silverman-Toeplitz theorem [5, 6] characterizes the regularity of two-dimensional matrix transformations. In [4], Robison presented a four-dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. The definition of regularity for four-dimensional matrices will be stated below along with the Robison-Hamilton characterization of the regularity of four-dimensional matrices.

Definition 2.4. The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 2.5 (see [2, 4]). *The four-dimensional matrix A is RH-regular if and only if*

(RH_1) $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
(RH_2) $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
(RH_3) $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*
(RH_4) $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*
(RH_5) $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*
(RH_6) *there exist finite positive integers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

Example 2.6. The sequences $[y_{n,k}] = 1$ and $[y_{n,k}] = -1$ for each n and k are both subsequences of the double sequence whose n,kth term is $x_{n,k} = (-1)^n$. In addition to the two subsequences given, every double sequence of 1's and -1's is a subsequence of this x.

Example 2.7. As another example of a subsequence of a double sequence, we define x as follows:

$$x_{n,k} := \begin{cases} 1, & \text{if } n = k, \\ \dfrac{1}{n}, & \text{if } n < k, \\ n, & \text{if } n > k. \end{cases} \tag{2.2}$$

Then the double sequence

$$y_{n,k} := \begin{Bmatrix} \frac{1}{2} & 4 & \frac{1}{10} & 20 & \cdot & \cdot \\ 8 & 6 & \frac{1}{12} & 22 & \cdot & \cdot \\ \frac{1}{18} & \frac{1}{16} & \frac{1}{14} & 24 & \cdot & \cdot \\ 32 & 30 & 28 & 26 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \end{Bmatrix} \tag{2.3}$$

is clearly a subsequence of x.

Remark 2.8. Note that if the double sequence x contains at most a finite number of unbounded rows and/or columns, then every subsequence of x is bounded. In addition, the finite number of unbounded rows and/or columns does not affect the P-convergence or P-divergence of x and its subsequences.

Definition 2.9. A number β is called a Pringsheim limit point of the double sequence $x = [x_{n,k}]$ provided that there exists a subsequence $y = [y_{n,k}]$ of $[x_{n,k}]$ that has Pringsheim limit $\beta : P\text{-}\lim y_{n,k} = \beta$.

Example 2.10. Define the double sequence x by

$$x_{n,k} := \begin{cases} (-1)^n, & \text{if } n = k, \\ (-2)^n, & \text{if } n = k+1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

This double sequence has five Pringsheim limit points, namely $-2, -1, 0, 1$, and 2.

Remark 2.11. The definition of a Pringsheim limit point can also be stated as follows: β is a Pringsheim limit point of x provided that there exist two increasing index sequences $\{n_i\}$ and $\{k_i\}$ such that $\lim_i x_{n_i,k_i} = \beta$.

Definition 2.12. A double sequence x is divergent in the Pringsheim sense (P-divergent) provided that x does not converge in the Pringsheim sense (P-convergent).

Remark 2.13. Definition 2.12 can also be stated as follows: a double sequence x is P-divergent provided that either x contains at least two subsequences with distinct finite Pringsheim limit points or x contains an unbounded subsequence. Also note that, if x contains an unbounded subsequence then x also contains a definite divergent subsequence.

Example 2.14. This is an example of a convergent double sequence whose terms form an unbounded set

$$x_{n,k} := \begin{cases} k, & \text{if } n = 1, \\ n, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Example 2.15. This is an example of an unbounded divergent double sequence with three finite Pringsheim limit points, namely $-1, 0$, and 1:

$$x_{n,k} := \begin{cases} k+1, & \text{if } n = 1, \\ (-1)^{n+1}, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Example 2.16. This is an example of a double sequence which contains an unbounded subsequence

$$x_{n,k} := \begin{cases} n, & \text{if } n = k, \\ -n, & \text{if } n = k+1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Example 2.17. For an example of a definite divergent sequence take $x_{n,k} = n$ for each n and k; then it is also clear that x contains an unbounded subsequence.

The following propositions are easily verified.

Proposition 2.18. *If $x = [x_{n,k}]$ is P-convergent to L then x cannot converge to a limit M, where $M \neq L$.*

Proposition 2.19. *If $x = [x_{n,k}]$ is P-convergent to L, then any subsequence of x is also P-convergent to L.*

Remark 2.20. For an ordinary single-dimensional sequence, any sequence is a subsequence of itself. This, however, is not the case in the two-dimensional plane, as illustrated by the following example.

Example 2.21. The sequence

$$x_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 1, & \text{if } n = 0,\ k = 1, \\ 1, & \text{if } n = 1,\ k = 0, \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

contains only two subsequences, namely, $[y_{n,k}] = 0$ for each n and k, and

$$z_{n,k} := \begin{cases} 1, & \text{if } n = k = 0, \\ 0, & \text{otherwise;} \end{cases} \tag{2.9}$$

neither subsequences is x.

The following propositions are easily verified.

Proposition 2.22. *If every subsequence of $x = [x_{k,l}]$ is P-convergent, then x is P-convergent.*

Proposition 2.23. *The double sequence x is P-convergent to L if and only if every subsequence of x is P-convergent to L.*

Definition 2.24. The double sequence y contains an *ϵ-Pringsheim-copy* of x provided that y contains a subsequence y_{n_i,k_j} such that $|y_{n_i,k_j} - x_{i,j}| < \epsilon_{i,j}$, for $i, j = 1, 2, \ldots$.

Example 2.25. Let

$$x_{n,k} := \begin{cases} (-1)^n, & \text{if } k = n, \\ 0, & \text{otherwise,} \end{cases} \tag{2.10}$$

and let $P\text{-}\lim_{n,k} \epsilon_{n,k} = 0$ with

$$y_{n,k} := \begin{cases} (-1)^n, & \text{if } k = n, \\ \epsilon_{n,k}, & \text{otherwise.} \end{cases} \tag{2.11}$$

Observe that, not only does y contain an ϵ-Pringsheim-copy of x, but y itself is an ϵ-Pringsheim-copy of x.

Definition 2.26. The double sequence y is a *stretching* of x provided that there exist two increasing index sequences $\{R_i\}_{i=0}^{\infty}$ and $\{S_j\}_{j=0}^{\infty}$ of integers such that

$$y_{n,k} := \begin{cases} R_0 = S_0 = 1, & \\ x_{n,i}, & \text{if } R_{i-1} \le k < R_i, \\ x_{j,k}, & \text{if } S_{j-1} \le n < S_j, \\ i,j = 1,2\ldots. & \end{cases} \tag{2.12}$$

Remark 2.27. This definition demonstrates the procedure which is used to construct a stretching of a double sequence x. This procedure uses a sequence of stages to construct the stretching of x. These stages are constructed using a sequence of abutting rows and columns of x. These rows and columns are constructed as follows.

Stage 1. Begin by repeating the first row of x R_1 times and denote the resulting double sequence by $y^{1,0}$ then repeat the first column of $y^{1,0}$ S_1 times resulting in $y^{1,1}$.

Stage 2. Begin by repeating the R_1+1 row of $y^{1,1}$, R_2-R_1 times which yields $y^{2,1}$ then repeat the S_1+1 column of $y^{2,1}$, S_2-S_1 times which yields $y^{2,2}$.

$\vdots$

Stage i. Begin by repeating the $1+\sum_{p=1}^{i-1} R_p$ row of $y^{i-1,i-1}$, R_i-R_{i-1} times which yields $y^{i,i-1}$ then repeat the $1+\sum_{q=1}^{i-1} S_q$ column of $y^{i,i-1}$, S_i-S_{i-1} times which yields $y^{i,i}$. Note that in each stage we repeat the number of rows and then repeat the number of columns. However the resulting stretching y of x is the same, if we first repeat the number of columns and then repeat the numbers of rows. Also note that every sequence itself is a stretching of itself and the sequences that induce this kind of stretching are $R_i = i$ and $S_j = j$.

Example 2.28. The sequence

$$\begin{matrix} x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\ x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\ x_{1,1} & x_{1,1} & x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} & x_{1,3} & x_{1,3} & x_{1,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{2,1} & x_{2,1} & x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} & \cdots \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\ x_{3,1} & x_{3,1} & x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \end{matrix} \tag{2.13}$$

is a stretching of x induced by $R_i = 3i$ and $S_j = 3j$.

3. Main results. The following theorem is given its name because of its similarity to the copy theorem of Dawson in [1].

Theorem 3.1 (extended copy theorem)**.** *If each of A and T is an RH-regular matrix, and x is any bounded double complex sequence with ϵ being any bounded positive term* *double sequence with $P\text{-}\lim_{i,j}\epsilon_{i,j}=0$, then there exists a stretching y of x such that $T(Ay)$ exists and contains an ϵ-Pringsheim-copy of x.*

Proof. We begin by introducing a few notations which are used only in this proof. Let

$$\begin{gathered}
\|A\|:=\sup_{m,n>\bar{B}}\left(\sum_{k,l}|a_{m,n,k,l}|\right)<K_A,\qquad \|T\|:=\sup_{m,n>\bar{B}}\left(\sum_{k,l}|t_{m,n,k,l}|\right)<K_T,\\
M_{i,j}:=1+\sum_{k,l=1}^{i,j}|x_{k,l}|,\qquad \delta_{i,j}:=\min_{i,j}\left\{\frac{\epsilon_{k,l}}{1}\le k\le i\cup 1\le l\le j\right\},\\
K:=K_A+K_T+\max_{i,j}\left\{\frac{\epsilon_{k,l}}{1}\le k\le i\cup 1\le l\le j\right\}+1,\qquad Q_{i,j}:=KM_{i,j}+1,\\
c_{i,j}(r,s):=\left\{\frac{(k,l)}{1}\le k<r_i\cup 1\le l<s_j\right\},\\
\bar{c}_{i,j}(r,s):=\left\{\frac{(k,l)}{r_i}\le k<\infty\cup s_j\le l<\infty\right\},\qquad \bar{b}_{i,j}(r,s):=c_{i,j}(r,s)\backslash c_{i-1,j-1}(r,s).
\end{gathered}\tag{3.1}$$

Then by (RH_2) there exist m_{α_1} and n_{β_1} such that for $m>m_{\alpha_1}>\bar{B}$ and $n>n_{\beta_1}>\bar{B}$, where $\bar{B}$ is defined by the sixth RH-condition,

$$\left|\sum_{k,l=1}^{\infty,\infty}a_{m,n,k,l}-1\right|<\frac{\delta_{\alpha_1,\beta_1}}{16Q_{\alpha_1,\beta_1}}.\tag{3.2}$$

Also by (RH_1) and (RH_2) there exist a_{α_1} and b_{β_1} such that

$$\sum_{(k,l)\in c_{\alpha_1,\beta_1}(m,n)}|t_{a_{\alpha_1},b_{\beta_1},k,l}|<\frac{\delta_{\alpha_1,\beta_1}}{8Q_{\alpha_1,\beta_1}},\qquad \left|\sum_{k,l=1}^{\infty,\infty}t_{a_{\alpha_1},b_{\beta_1},k,l}-1\right|<\frac{\delta_{\alpha_1,\beta_1}}{8Q_{\alpha_1,\beta_1}}.\tag{3.3}$$

In addition, there exist $\bar{m}_{\alpha_1},\bar{n}_{\beta_1},\alpha_2$, and β_2 such that if $1\le\psi\le a_{\alpha_1}$ and $1\le\omega\le b_{\beta_1}$, then

$$\sum_{(k,l)\in\bar{c}_{\alpha_1,\beta_1}(\bar{m},\bar{n})}|t_{\psi,\omega,k,l}|<\frac{\delta_{\alpha_1,\beta_1}}{16Q_{\alpha_2,\beta_2}}.\tag{3.4}$$

Also, there exist $r_{\alpha_1}>1$ and $s_{\beta_1}>1$ such that if $1\le m\le\bar{m}_{\alpha_1}$ and $1\le n\le\bar{n}_{\beta_1}$ then

$$\sum_{(k,l)\in\bar{c}_{\alpha_1,\beta_1}(r,s)}|a_{m,n,k,l}|\le\frac{\delta_{\alpha_1,\beta_1}}{16Q_{\alpha_2,\beta_2}}.\tag{3.5}$$

Now, without loss of generality, we set $\alpha_p=p$ and $\beta_q=q$. Having chosen

$$\left\{\begin{matrix}m_p,\bar{m}_p,a_p,r_p\\ n_q,\bar{n}_q,b_q,s_q\end{matrix}\right\}_{p=0,q=0}^{i-1,j-1}\tag{3.6}$$

with $m_0=n_0=\bar{m}_0=\bar{n}_0=a_0=b_0=r_0=s_0=1$, now choose $m_i>\bar{m}_{i-1}$ and $n_j>\bar{n}_{j-1}$ such that if $m>m_i$ and $n>n_j$ then

$$\left|\sum_{(k,l)\in\bar{c}_{i-1,j-1}(r,s)}a_{m,n,k,l}-1\right|<\frac{\delta_{i,j}}{16Q_{i,j}2^{i+j}},\tag{3.7}$$

$$\sum_{(k,l)\in c_{i-1,j-1}(r,s)} |a_{m,n,k,l}| < \frac{\delta_{i,j}}{8Q_{i-1,j-1}2^{i+j}}. \tag{3.8}$$

Also choose $a_i > a_{i-1}$ and $b_j > b_{j-1}$ such that

$$\sum_{(k,l)\in c_{i,j}(m,n)} |t_{a_i,b_j,k,l}| < \frac{\delta_{i,j}}{8Q_{i,j}}, \qquad \left| \sum_{(k,l)\in \bar{c}_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}}. \tag{3.9}$$

Next choose $\bar{m}_i > m_i$ and $\bar{n}_j > n_j$ such that if $1 \le \psi \le a_i$ and $1 \le \omega \le b_j$ then

$$\sum_{(k,l)\in \bar{c}_{i,j}(\bar{m},\bar{n})} |t_{\psi,\omega,k,l}| < \frac{\delta_{i,j}}{2^{2+i+j}Q_{i+1,j+1}}. \tag{3.10}$$

Then choose $r_i > r_{i-1}$ and $s_j > s_{j-1}$ such that if $1 \le m \le \bar{m}_i$ and $1 \le n \le \bar{n}_j$ then

$$\sum_{(k,l)\in \bar{c}_{i,j}(r,s)} |a_{m,n,k,l}| < \frac{\delta_{i,j}}{2^{4+i+j}Q_{i+1,j+1}}, \tag{3.11}$$

where $m_i, n_j, \bar{m}_i, \bar{n}_j, r_i$, and s_j are chosen using (RH_1), (RH_2), (RH_3), and (RH_4) such that if $1 \le p \le j-1$ and $1 \le q \le i-1$ the following is obtained:

$$\left| \sum_{(k,l)\in \bar{b}_{p,j}(r,s)} a_{m,n,k,l} \right| \le \frac{\delta_{p,j}}{8Q_{p,j}2^{p+j}}, \qquad \left| \sum_{(k,l)\in \bar{b}_{i,q}(r,s)} a_{m,n,k,l} \right| \le \frac{\delta_{i,q}}{8Q_{i,q}2^{i+q}}. \tag{3.12}$$

Therefore by (3.9) and (3.10) we have

$$\left| \sum_{(k,l)\in c_{i,j}(\bar{m},\bar{n})\setminus c_{i,j}(m,n)} t_{a_i,b_j,k,l} - 1 \right| \le \frac{\delta_{i,j}}{4Q_{i,j}}, \tag{3.13}$$

and by (3.7), (3.8), and (3.11) we also obtain

$$\left| \sum_{(k,l)\in \bar{b}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| < \frac{\delta_{i,j}}{8Q_{i,j}2^{i+j}}, \tag{3.14}$$

where $m_i \le m \le \bar{m}_i$ and $n_j \le n \le \bar{n}_j$. Let $\{y_{k,l}\}$ be the stretching of x induced by $\{r_i\}$ and $\{s_j\}$. Since

$$\begin{aligned}(Ay)_{m,n} - x_{i,j} = &\sum_{k,l=1}^{r_{i-1}-1,s_{j-1}-1} a_{m,n,k,l}y_{k,l} + \sum_{(k,l)\in \bar{b}_{i,j}(r,s)} a_{m,n,k,l}y_{k,l} - x_{i,j} \\ &+ \sum_{p,q=i+1,j+1}^{\infty,\infty} \sum_{(k,l)\in \bar{b}_{p,q}(r,s)} a_{m,n,k,l}y_{k,l},\end{aligned} \tag{3.15}$$

if $i,j > 1$, with $m_i \le m \le \bar{m}_i$ and $n_j \le n \le \bar{n}_j$ the following is obtained:

$$\left| \sum_{k,l=1}^{r_{i-1}-1,s_{j-1}-1} a_{m,n,k,l}y_{k,l} \right| \le \max\left\{ \frac{|x_{k,l}|}{1} \le k \le i-1 \cup 1 \le l \le j-1 \right\} \sum_{k,l=1}^{r_{i-1}-1,s_{j-1}-1} |a_{m,n,k,l}y_{k,l}|. \tag{3.16}$$

By (3.8),

$$\left| \sum_{k,l=1}^{r_{i-1}-1,s_{j-1}-1} a_{m,n,k,l} y_{k,l} \right| \le \max\left\{ \frac{|x_{k,l}|}{1} \le k \le i-1 \cup 1 \le l \le j-1 \right\} \frac{\delta_{i,j}}{8Q_{i-1,j-1}}. \tag{3.17}$$

Since

$$Q_{i-1,j-1} = K\left(1 + \sum_{k,l=1}^{i-1,j-1} |x_{k,l}|\right) + 1 \ge K \max\left\{ \frac{|x_{k,l}|}{1} \le k \le i-1 \cup 1 \le l \le j-1 \right\}, \tag{3.18}$$

the following holds:

$$\left| \sum_{k,l=1}^{r_{i-1}-1,s_{j-1}-1} a_{m,n,k,l} y_{k,l} \right| \le \frac{\delta_{i,j}}{8K}, \tag{3.19}$$

the following also is obtained:

$$\begin{aligned} \left| \sum_{p,q=i+1,j+1}^{\infty,\infty} \sum_{(k,l)\in \bar{b}_{p,q}(r,s)} a_{m,n,k,l} y_{k,l} \right| &\le \sum_{p,q=i+1,j+1}^{\infty,\infty} |x_{k,l}| \sum_{(k,l)\in \bar{b}_{p,q}(r,s)} |a_{m,n,k,l}| \\ &\le \frac{\delta_{i,j}}{2^4 K} \sum_{p,q=i+1,j+1}^{\infty,\infty} \frac{1}{2^{p+q}} \le \frac{\delta_{i,j}}{8K}, \end{aligned} \tag{3.20}$$

because

$$\sum_{k,l=r_p,s_q}^{\infty,\infty} |a_{m,n,k,l}| \le \frac{\delta_{p-1,q-1}}{2^{4+p+q} Q_{p,q}}, \qquad \frac{|x_{p,q}|}{Q_{p,q}} < \frac{1}{K}. \tag{3.21}$$

Therefore by (3.11),

$$\begin{aligned} \left| \sum_{(k,l)\in \bar{b}_{i,j}(r,s)} a_{m,n,k,l} y_{k,l} - x_{i,j} \right| &\le \sum_{q=1}^{i-1} |x_{i,q}| \left| \sum_{(k,l)\in \bar{b}_{i,q}(r,s)} a_{m,n,k,l} \right| \\ &\quad + \sum_{p=1}^{j-1} |x_{p,j}| \left| \sum_{(k,l)\in \bar{b}_{p,j}(r,s)} a_{m,n,k,l} \right| \\ &\quad + |x_{i,j}| \left| \sum_{(k,l)\in \bar{b}_{i,j}(r,s)} a_{m,n,k,l} - 1 \right| \\ &\le \sum_{p,q=1,1}^{i,j} \frac{|x_{i,j}|}{Q_{i,j}} \frac{\delta_{p,q}}{2^{p+q+3}} \le \frac{\delta_{i,j}}{K8} \sum_{p,q=1,1}^{i,j} \frac{1}{2^{p+q}} = \frac{\delta_{i,j}}{K2}. \end{aligned} \tag{3.22}$$

Therefore,

$$|(Ay)_{m,n} - x_{i,j}| \le \frac{\delta_{i,j}}{K8} + \frac{\delta_{i,j}}{K4} + \frac{\delta_{i,j}}{K2} < \frac{\delta_{i,j}}{2K}. \tag{3.23}$$

Note that the inequality (3.23) is true for $m_1 \le m \le \bar{m}_1$ and $n_1 \le n \le \bar{n}_1$, and also this inequality is true for $i, j \ge 1$ with $m_i \le m \le \bar{m}_i$ and $n_j \le n \le \bar{n}_j$. Hence

$$(Ay)_{m,n} = x_{i,j} + u_{i,j}, \tag{3.24}$$

where $|u_{i,j}| \le \delta_{i,j}/2K$. Note that if $\bar{m}_{i-1} \le m \le m_i$ and $\bar{n}_{j-1} \le n \le n_j$, then the following is obtained:

$$\begin{aligned}
|(Ay)_{m,n}| &\le \left| \sum_{k,l=1}^{r_i-1,s_j-1} a_{m,n,k,l} y_{k,l} \right| + \left| \sum_{p,q=i+1,j+1}^{\infty,\infty} \sum_{k,l \in \bar{b}_{p,q}(r,s)} a_{m,n,k,l} y_{k,l} \right| \\
&\le \max\left\{ \frac{|x_{k,l}|}{1} \le k \le i \cup 1 \le l \le j \right\} \sum_{k,l=1}^{r_i-1,s_j-1} |a_{m,n,k,l}| \\
&\quad + \sum_{p,q=i+1,j+1}^{\infty,\infty} |x_{k,l}| \sum_{k,l \in \bar{b}_{p,q}(r,s)} |a_{m,n,k,l}| \\
&\le K m_{i,j} + \sum_{p,q=i+1,j+1}^{\infty,\infty} |x_{k,l}| \frac{\delta_{p,q}}{2^{4+p+q} Q_{p+1,q+1}} \\
&\le K m_{i,j} + \frac{\delta_{i,j}}{K4} \sum_{p,q=i+1,j+1}^{\infty,\infty} \frac{1}{2^{p+q}} \\
&\le K m_{i,j} + 1 = Q_{i,j}.
\end{aligned} \tag{3.25}$$

Also, if $m_{i-1} \le m \le m_i$ and $n_{j-1} \le n \le n_j$ then

$$\begin{aligned}
\left| \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l} y_{k,l} \right| &\le |(Ay)_{m,n} - x_{i,j}| + |x_{i,j}| \\
&\le \frac{\delta_{i,j}}{2K} + K m_{i,j} \le K m_{i,j} + 1 = Q_{i,j}.
\end{aligned} \tag{3.26}$$

By using (3.25) we now show the existence of $T(Ay)$. If $a_{i-1} < m \le a_i$ and $b_{j-1} < n \le b_j$ then

$$\begin{aligned}
\left| \sum_{k,l=\bar{m}_i+1,\bar{n}_j+1}^{\infty,\infty} t_{m,n,k,l} (Ay)_{k,l} \right| &\le \sum_{r,s=i,j}^{\infty,\infty} \sum_{(p,q)\in \bar{b}_{r+1,s+1}(\bar{m},\bar{n})} |t_{m,n,p,q} (Ay)_{p,q}| \\
&\le \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1} \sum_{(p,q)\in \bar{b}_{r+1,s+1}(\bar{m},\bar{n})} |t_{m,n,p,q}| \\
&\le \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1} \frac{\delta_{r,s}}{2^{2+r+s} Q_{r+1,s+1}} \\
&\le \delta_{i,j} \frac{1}{4} \sum_{r,s=1}^{\infty,\infty} \frac{1}{2^{r+s}} < \frac{\delta_{i,j}}{4}.
\end{aligned} \tag{3.27}$$

Therefore $T(Ay)$ exists. Also, by (3.25) we now show that $T(Ay)$ contains an ϵ-Pringsheim-copy of x. First note that

$$\left|\sum_{k,l=1}^{\infty,\infty} t_{a_i,b_j,k,l}(Ay)_{k,l} - x_{i,j}\right| \leq \sum_{k,l=1}^{m_i-1,n_j-1} \left|t_{a_i,b_j,k,l}(Ay)_{k,l}\right| + \left|\sum_{(k,l)\in\bar{b}_{i,j}(r,s)} t_{a_i,b_j,k,l}(Ay)_{k,l} - x_{i,j}\right| + \left|\sum_{k,l=\bar{m}_i+1,\bar{n}_j+1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l}\right|, \tag{3.28}$$

with

$$\sum_{k,l=1}^{m_i-1,n_j-1} \left|t_{a_i,b_j,k,l}(Ay)_{k,l}\right| = \sum_{k,l=1}^{m_i-1,n_j-1} \left|t_{a_i,b_j,k,l}\right| Q_{i,j} \leq Q_{i,j}\frac{\delta_{i,j}}{8Q_{i,j}} = \frac{\delta_{i,j}}{8}, \tag{3.29}$$

$$\begin{aligned} \left|\sum_{(k,l)\in\bar{b}_{i,j}(r,s)} t_{a_i,b_j,k,l}(Ay)_{k,l} - x_{i,j}\right| &= \left|\sum_{(k,l)\in\bar{b}_{i,j}(r,s)} t_{a_i,b_j,k,l}(x_{i,j}+u_{i,j}) - x_{i,j}\right| \\ &\leq |x_{i,j}| \sum_{(k,l)\in\bar{b}_{i,j}(r,s)} \left|t_{a_i,b_j,k,l}-1\right| + \sum_{(k,l)\in\bar{b}_{i,j}(r,s)} \left|t_{a_i,b_j,k,l}u_{i,j}\right| \\ &\leq \frac{|x_{i,j}|}{Q_{i,j}}\frac{\delta_{i,j}}{4} + \frac{\delta_{i,j}}{4K}\sum_{(k,l)\in\bar{b}_{i,j}(r,s)} \left|t_{a_i,b_j,k,l}\right| \\ &\leq \frac{\delta_{i,j}}{2}, \end{aligned} \tag{3.30}$$

$$\begin{aligned} \left|\sum_{k,l=\bar{m}_i+1,\bar{n}_j+1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l}\right| &\leq \sum_{r,s=i,j}^{\infty,\infty}\sum_{(p,q)\in\bar{b}_{r+1,s+1}(\bar{m},\bar{n})} \left|t_{a_i,b_j,p,q}(Ay)_{p,q}\right| \\ &\leq \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1}\sum_{(p,q)\in\bar{b}_{r+1,s+1}(\bar{m},\bar{n})} \left|t_{a_i,b_j,p,q}\right| \\ &\leq \sum_{r,s=i,j}^{\infty,\infty} Q_{r+1,s+1}\frac{\delta_{r,s}}{2^{2+r+s}Q_{r+1,s+1}} \leq \frac{\delta_{i,j}}{4}. \end{aligned} \tag{3.31}$$

Hence,

$$\left|\sum_{k,l=1}^{\infty,\infty} t_{m,n,k,l}(Ay)_{k,l} - x_{i,j}\right| \leq \frac{\delta_{i,j}}{4} + \frac{\delta_{i,j}}{2} + \frac{\delta_{i,j}}{8} < \delta_{i,j} \leq \epsilon_{i,j}. \tag{3.32}$$

This completes the proof of the extended copy theorem. □

The next two results are immediate corollaries of the extended copy theorem.

Corollary 3.2. *If T is any RH-regular matrix summability method and A is an RH-regular matrix such that Ay is T-summable for every stretching y of x, then x is P-convergent.*

Corollary 3.3. *If T is any RH-regular matrix summability method and A is an RH-regular matrix such that Ay is absolutely T-summable for every stretching y of x, then x is P-convergent.*

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