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ON Q-ALGEBRAS

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ABSTRACT. We introduce a new notion, called a Q-algebra, which is a generalization of the idea of $BCH/BCI/BCK$-algebras and we generalize some theorems discussed in BCI-algebras. Moreover, we introduce the notion of "quadratic" Q-algebra, and show that every quadratic Q-algebra $(X;*,e)$, $e \in X$, has a product of the form $x * y = x - y + e$, where $x, y \in X$ when X is a field with $|X| \geq 3$.

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1. Introduction. Imai and Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (see [4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers and Kim (see [8]) introduced the notion of d-algebras, that is, (I) $x * x = e$; (IX) $e * x = e$; (VI) $x * y = e$ and $y * x = e$ imply $x = y$, which is another useful generalization of BCK-algebras, after which they investigated several relations between d-algebras and BCK-algebras, as well as other relations between d-algebras and oriented digraphs. At the same time, Jun, Roh, and Kim [6] introduced a new notion, called a BH-algebra, that is, (I) $x * x = e$; (II) $x * e = x$; (VI) $x * y = e$ and $y * x = e$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and they showed that there is a maximal ideal in bounded BH-algebras. We introduce a new notion, called a Q-algebra, which is a generalization of $BCH/BCI/BCK$-algebras and generalize some theorems from the theory of BCI-algebras. Moreover, we introduce the notion of "quadratic" Q-algebra, and obtain the result that every quadratic Q-algebra $(X;*,e)$, $e \in X$, is of the form $x * y = x - y + e$, where $x, y \in X$ and X is a field with $|X| \geq 3$, that is, the product is linear in a special way.

2. Q-algebras. A *Q-algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying axioms:

(I) $x * x = 0$,

(II) $x * 0 = x$,

(III) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

For brevity we also call X a *Q-algebra*. In X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$. Recently, Ahn and Kim [1] introduced the notion of QS-algebras. A Q-algebra X is said to be a *QS-algebra* if it satisfies the additional relation:

(IV) $(x * y) * (x * z) = z * y$, for any $x, y, z \in X$.

Example 2.1. Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz \mid z \in \mathbb{Z}\}$ where $n \in \mathbb{Z}$. Then $(\mathbb{Z};-,0)$ and $(n\mathbb{Z};-,0)$ are Q-algebras, where "$-$" is the usual subtraction of integers.

Example 2.2. Let $X := \{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X;*,0)$ is a Q-algebra, which is not a $BCH/BCI/BCK$-algebra.

Neggers and Kim [7] introduced the related notion of B-algebra, that is, algebras $(X;*,0)$ which satisfy (I) $x*x=0$; (II) $x*0=x$; (V) $(x*y)*z = x*(z*(0*y))$, for any $x,y,z \in X$. It is easy to see that B-algebras and Q-algebras are different notions. For example, Example 2.2 is a Q-algebra, but not a B-algebra, since $(3*2)*1 = 0 \neq 3 = 3*(1*(0*2))$. Consider the following example. Let $X := \{0,1,2,3,4,5\}$ be a set with the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X;*,0)$ is a B-algebra (see [7]), but not a Q-algebra, since $(5*3)*4 = 3 \neq 4 = (5*4)*3$.

Proposition 2.3. *If $(X;*,0)$ is a Q-algebra, then*
(VII) $(x*(x*y))*y = 0$, *for any $x,y \in X$.*

Proof. By (I) and (III), $(x*(x*y))*y = (x*y)*(x*y) = 0$. □

We now investigate some relations between Q-algebras and BCH-algebras (also BCK/BCI-algebras). The following theorems are easily proven, and we omit their proofs.

Theorem 2.4. *Every BCH-algebra X is a Q-algebra. Every Q-algebra X satisfying condition (VI) is a BCH-algebra.*

Theorem 2.5. *Every Q-algebra satisfying condition (IV) and (VI) is a BCI-algebra.*

Theorem 2.6. *Every Q-algebra X satisfying conditions (V), (VI), and* (VIII) $(x*y)*x = 0$ *for any $x,y \in X$, is a BCK-algebra.*

Theorem 2.7. *Every Q-algebra X satisfying $x*(x*y) = x*y$ for all $x,y,z \in X$, is a trivial algebra.*

Proof. Putting $x = y$ in the equation $x*(x*y) = x*y$, we obtain $x*0 = 0$. By (II) $x = 0$. Hence X is a trivial algebra. □

The following example shows that a Q-algebra may not satisfy the associative law.

Example 2.8. (a) Let $X := \{0,1,2\}$ with the table as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then X is a Q-algebra, but associativity does not hold, since $(0*1)*2 = 0 \neq 1 = 0*(1*2)$.

(b) Let $\mathbb{Z}$ and $\mathbb{R}$ be the set of all integers and real numbers, respectively. Then $(\mathbb{Z};-,0)$ and $(\mathbb{R};\div,1)$ are nonassociative Q-algebras where "$-$" is the usual subtraction and "$\div$" is the usual division.

Theorem 2.9. *Every Q-algebra $(X;*,0)$ satisfying the associative law is a group under the operation "$*$".*

Proof. Putting $x = y = z$ in the associative law $(x*y)*z = x*(y*z)$ and using (I) and (II), we obtain $0*x = x*0 = x$. This means that 0 is the zero element of X. By (I), every element x of X has as its inverse the element x itself. Therefore $(X;*)$ is a group. □

3. The G-part of Q-algebras.

In this section, we investigate the properties of the G-part in Q-algebras.

Lemma 3.1. *If $(X;*,0)$ is a Q-algebra and $a*b = a*c$, $a,b,c \in X$, then $0*b = 0*c$.*

Proof. By (I) and (II) $(a*b)*a = (a*a)*b = 0*b$ and $(a*c)*a = (a*a)*c = 0*c$. Since $a*b = a*c$, $0*b = 0*c$. □

Definition 3.2. Let $(X;*,0)$ be a Q-algebra. For any nonempty subset S of X, we define

$$G(S) := \{x \in S \mid 0*x = x\}. \tag{3.1}$$

In particular, if $S = X$ then we say that $G(X)$ is the *G-part* of X.

Corollary 3.3. *A left cancellation law holds in $G(X)$.*

Proof. Let $a,b,c \in G(X)$ with $a*b = a*c$. By Lemma 3.1, $0*b = 0*c$. Since $b,c \in G(X)$, we obtain $b = c$. □

Proposition 3.4. *Let $(X;*,0)$ be a Q-algebra. Then $x \in G(X)$ if and only if $0*x \in G(X)$.*

Proof. If $x \in G(X)$, then $0*x = x$ and $0*(0*x) = 0*x$. Hence $0*x \in G(X)$.

Conversely, if $0*x \in G(x)$, then $0*(0*x) = 0*x$. By applying Corollary 3.3, we obtain $0*x = x$. Therefore $x \in G(X)$. □

For any Q-algebra $(X;*,0)$, the set

$$B(X) := \{x \in X \mid 0*x = 0\} \tag{3.2}$$

is called the *p-radical* of X. If $B(X) = \{0\}$, then we say that X is a *p-semisimple* Q-algebra. The following property is obvious.

(IX) $G(X) \cap B(X) = \{0\}$.

Proposition 3.5. *If $(X;*,0)$ is a Q-algebra and $x,y \in X$, then*

$$y \in B(X) \Longleftrightarrow (x*y)*x = 0. \tag{3.3}$$

Proof. By (I) and (III) $(x*y)*x = (x*x)*y = 0*y = 0$ if and only if $y \in B(X)$. □

Definition 3.6. Let $(X;*,0)$ be a Q-algebra and $I(\neq \emptyset) \subseteq X$. The set I is called an *ideal* of X if for any $x,y,z \in X$,

(1) $0 \in I$,

(2) $x*y \in I$ and $y \in I$ imply $x \in I$.

Obviously, $\{0\}$ and X are ideals of X. We call $\{0\}$ and X the *zero ideal* and the *trivial ideal* of X, respectively. An ideal I is said to be *proper* if $I \neq X$.

In Example 2.2 the set $I := \{0,1,2\}$ is an ideal of X.

Proposition 3.7. *Let $(X;*,0)$ be a Q-algebra. Then $B(X)$ is an ideal of X.*

Proof. Since $(0*0)*0 = 0$, by Proposition 3.5, $0 \in B(X)$. Let $x*y \in B(X)$ and $y \in B(X)$. Then by Proposition 3.5, $((x*y)*x)*(x*y) = 0$. By (III), $((x*y)*(x*y))*x = 0*x = 0$. Hence $x \in B(X)$. Therefore $B(X)$ is an ideal of X. □

Proposition 3.8. *If S is a subalgebra of a Q-algebra $(X;*,0)$, then $G(X) \cap S = G(S)$.*

Proof. It is obvious that $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0*x = x$ and $x \in S \subseteq X$. Then $x \in G(X)$ and so $x \in G(X) \cap S$, which proves the proposition. □

Theorem 3.9. *Let $(X;*,0)$ be a Q-algebra. If $G(X) = X$, then X is p-semisimple.*

Proof. Assume that $G(X) = X$. By (X), $\{0\} = G(X) \cap B(X) = X \cap B(X) = B(X)$. Hence X is p-semisimple. □

Theorem 3.10. *If $(X;*,0)$ is a Q-algebra of order 3, then $|G(X)| \neq 3$, that is, $G(X) \neq X$.*

Proof. For the sake of convenience, let $X = \{0,a,b\}$ be a Q-algebra. Assume that $|G(X)| = 3$, that is, $G(X) = X$. Then $0*0 = 0$, $0*a = a$, and $0*b = b$. From $x*x = 0$ and $x*0 = x$, it follows that $a*a = 0$, $b*b = 0$, $a*0 = a$, and $b*0 = b$. Now let $a*b = 0$. Then 0, a, and b are candidates of the computation. If $b*a = 0$, then

$a*b=0=b*a$ and so $(a*b)*a=(b*a)*a$. By (III), $(a*a)*b=(b*a)*a$. Hence $0*b=0*a$. By the cancellation law in $G(X)$, $b=a$, a contradiction. If $b*a=a$, then $a=b*a=(0*b)*a=(0*a)*b=a*b=0$, a contradiction. For the case $b*a=b$, we have $b=b*a=(0*b)*a=(0*a)*b=a*b=0$, which is also a contradiction. Next, if $a*b=a$, then $(a*(a*b))*b=(a*a)*b=0*b=b\neq 0$. This leads to the conclusion that Proposition 2.3 does not hold, a contradiction. Finally, let $a*b=b$. If $b*a=0$, then $b=a*b=(0*a)*b=(0*b)*a=b*a=0$, a contradiction. If $b*a=a$, $b=a*b=(0*a)*b=(0*b)*a=b*a=0$, a contradiction. For the case $b*a=b$, we have $a=0*a=(b*b)*a=(b*a)*b=b*b=0$, which is again a contradiction. This completes the proof. □

Proposition 3.11. *If $(X;*,0)$ is a Q-algebra of order 2, then in every case the G-part $G(X)$ of X is an ideal of X.*

Proof. Let $|X|=2$. Then either $G(X)=\{0\}$ or $G(X)=X$. In either case, $G(X)$ is an ideal of X. □

Theorem 3.12. *Let $(X;*,0)$ be a Q-algebra of order 3. Then $G(X)$ is an ideal of X if and only if $|G(X)|=1$.*

Proof. Let $X:=\{0,a,b\}$ be a Q-algebra. If $|G(X)|=1$, then $G(X)=\{0\}$ is the trivial ideal of X.

Conversely, assume that $G(X)$ is an ideal of X. By Theorem 3.10, we know that either $|G(X)|=1$ or $|G(X)|=2$. Suppose that $|G(X)|=2$. Then either $G(X)=\{0,a\}$ or $G(X)=\{0,b\}$. If $G(X)=\{0,a\}$, then $b*a\notin G(X)$ because $G(X)$ is an ideal of X. Hence $b*a=b$. Then $a=0*a=(b*b)*a=(b*a)*b=b*b=0$, which is a contradiction. Similarly, $G(X)=\{0,b\}$ leads to a contradiction. Therefore $|G(X)|\neq 2$ and so $|G(X)|=1$. □

Definition 3.13. An ideal I of a Q-algebra $(X;*,0)$ is said to be *implicative* if $(x*y)*z\in I$ and $y*z\in I$, then $x*z\in I$, for any $x,y,z\in X$.

Theorem 3.14. *Let $(X;*,0)$ be a Q-algebra and let I be an implicative ideal of X. Then I contains the G-part $G(X)$ of X.*

Proof. If $x\in G(X)$, then $(0*x)*x=x*x=0\in I$ and $x*x=0\in I$. Since I is implicative, it follows that $x=0*x\in I$. Hence $G(X)\subseteq I$. □

Definition 3.15. Let X and Y be Q-algebras. A mapping $f:X\to Y$ is called a *homomorphism* if

$$f(x*y)=f(x)*f(y),\quad \forall x,y\in X. \tag{3.4}$$

A homomorphism f is called a *monomorphism* (resp., *epimorphism*) if it is injective (resp., surjective). A bijective homomorphism is called an *isomorphism*. Two Q-algebras X and Y are said to be *isomorphic*, written by $X\cong Y$, if there exists an isomorphism $f:X\to Y$. For any homomorphism $f:X\to Y$, the set $\{x\in X\mid f(x)=0\}$ is called the *kernel* of f, denoted by $\mathrm{Ker}(f)$ and the set $\{f(x)\mid x\in X\}$ is called the *image* of f, denoted by $\mathrm{Im}(f)$. We denote by $\mathrm{Hom}(X,Y)$ the set of all homomorphisms of Q-algebras from X to Y.

Proposition 3.16. *Suppose that $f : X \to X'$ is a homomorphism of Q-algebras. Then*
(1) $f(0) = 0'$,
(2) f *is isotone, that is, if $x * y = 0$, $x, y \in X$, then $f(x) * f(y) = 0'$.*

Proof. Since $f(0) = f(0 * 0) = f(0) * f(0) = 0'$, (1) holds. If $x, y \in X$ and $x \le y$, that is, $x * y = 0$, then by (1), $f(x) * f(y) = f(x * y) = f(0) = 0'$. Hence $f(x) \le f(y)$, proving (2). □

Theorem 3.17. *Let $(X; *, 0)$ and $(X; *', 0')$ be Q-algebras and let B be an ideal of Y. Then for any $f \in \mathrm{Hom}(X, Y)$, $f^{-1}(B)$ is an ideal of X.*

Proof. By Proposition 3.16(1), $0 \in f^{-1}(B)$. Assume that $x * y \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $f(x) * f(y) = f(x * y) \in B$. It follows from the fact that B is an ideal of Y that $f(x) \in B$, that is, $x \in f^{-1}(B)$. This means that $f^{-1}(B)$ is an ideal of X. The proof is complete. □

Since $\{0'\}$ is an ideal of X', $\mathrm{Ker}(f) = f^{-1}(\{0'\})$ for any $f \in \mathrm{Hom}(X, Y)$. Hence we obtain the following corollary.

Corollary 3.18. *The kernel* $\mathrm{Ker}(f)$ *is an ideal of X.*

4. The quadratic Q-algebras.

Let X be a field with $|X| \ge 3$. An algebra $(X; *)$ is said to be *quadratic* if $x * y$ is defined by $x * y := a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$, where $a_1, \dots, a_6 \in X$, for any $x, y \in X$. A quadratic algebra $(X; *)$ is said to be *quadratic Q-algebra* (resp., *QS-algebra*) if it satisfies conditions (I), (II), and (III) (resp., (IV)).

Theorem 4.1. *Let X be a field with $|X| \ge 3$. Then every quadratic Q-algebra $(X; *, e)$, $e \in X$, has the form $x * y = x - y + e$ where $x, y \in X$.*

Proof. Define

$$x * y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F. \tag{4.1}$$

Consider (I).

$$e = x * x = (A + B + C)x^2 + (D + E)x + F. \tag{4.2}$$

Let $x := 0$ in (4.2). Then we obtain $F = e$. Hence (4.1) turns out to be

$$x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + e. \tag{4.3}$$

If $y := x$ in (4.3), then

$$e = x * x = (A + B + C)x^2 + (D + E)x + e, \tag{4.4}$$

for any $x \in X$, and hence we obtain $A + B + C = 0 = D + E$, that is, $E = -D$ and $B = -A - C$. Hence (4.3) turns out to be

$$x * y = (x - y)(Ax - Cy + D) + e. \tag{4.5}$$

Let $y := e$ in (4.5). Then by (II) we have

$$x = x * e = (x - e)(Ax - Ce + D) + e, \tag{4.6}$$

that is, $(Ax-Ce+D-1)(x-e)=0$. Since X is a field, either $x-e=0$ or $Ax-Ce+D-1=0$. Since $|X|\geq 3$, we have $Ax-Ce+D-1=0$, for any $x\neq e$ in X. This means that $A=0, 1-D+Ce=0$. Thus (4.5) turns out to be

$$x*y=(x-y)+C(x-y)(e-y)+e. \tag{4.7}$$

To satisfy condition (III) we consider $(x*y)*z$ and $(x*z)*y$.

$$\begin{aligned}(x*y)*z &= (x*y-z)+C(x*y-z)(e-z)+e\\ &= (x-y-z)+C(x-y)(e-z)+2e\\ &\quad +C[(x-y)+C(x-y)(e-y)+(e-z)](e-z)\\ &= (x-y-z)+C(x-y)(2e-y-z)+2e\\ &\quad +C^2(x-y)(e-y)(e-z)+C(e-z)^2.\end{aligned} \tag{4.8}$$

Interchange y with z in (4.8). Then

$$\begin{aligned}(x*z)*y &= (x-z-y)+C(x-z)(2e-z-y)+2e\\ &\quad +C^2(x-z)(e-z)(e-y)+C(e-y)^2.\end{aligned} \tag{4.9}$$

By (4.8) and (4.9) we obtain

$$0=(x*y)*z-(x*z)*y=C^2(e-y)(e-z)(z-y). \tag{4.10}$$

Since X is a field with $|X|\geq 3$, we obtain $C=0$. This means that every quadratic Q-algebra $(X;*,e)$, has the form $x*y=x-y+e$ where $x,y\in X$, completing the proof. □

Example 4.2. Let $\mathbb{R}$ be the set of all real numbers. Define $x*y:=x-y+\sqrt{2}$. Then $(\mathbb{R};*,\sqrt{2})$ is a quadratic Q-algebra.

Example 4.3. Let $\mathcal{K}:=\mathrm{GF}(p^n)$ be a Galois field. Define $x*y:=x-y+e$, $e\in\mathcal{K}$. Then $(\mathcal{K};*,e)$ is a quadratic Q-algebra.

Theorem 4.4. *Let X be a field with $|X|\geq 3$. Then every quadratic Q-algebra on X is a (quadratic) QS-algebra.*

Proof. Let $(X;*,e)$ be a quadratic Q-algebra. Then $x*y=x-y+e$ for any $x,y\in X$, and hence

$$\begin{aligned}(x*y)*(x*z) &= (x-y+e)*(x-z+e)\\ &= (x-y+e)-(x-z+e)+e\\ &= z-y+e=z*y,\end{aligned} \tag{4.11}$$

completing the proof. □

Remark 4.5. Usually a nonquadratic Q-algebra need not be a QS-algebra. See the following example.

Example 4.6. Consider the Q-algebra $(X;*,0)$ in Example 2.2. This algebra is not a QS-algebra, since $(3*1)*(3*2) = 3 \neq 0 = 2*1$.

Corollary 4.7. *Let X be a field with $|X| \geq 3$. Then every quadratic Q-algebra on X is a BCI-algebra.*

Proof. It is an immediate consequences of Theorems 2.5 and 4.4. □

Theorem 4.8. *Let X be a field with $|X| \geq 3$. Then every quadratic Q-algebra $(X;*,e)$ is p-semisimple. Furthermore, if $\operatorname{char}(X) \neq 2$, then $G(X) = B(X)$.*

Proof. Notice that $B(X) = \{x \in X \mid e*x = e\} = \{x \in X \mid e-x+e = e\} = \{x \in X \mid e-x = 0\} = \{e\}$, that is, $(X;*,e)$ is p-semisimple. Also, if $\operatorname{char}(X) \neq 2$, then 2 is invertible in X and $G(X) = \{x \in X \mid e*x = x\} = \{x \in X \mid e-x+e = x\} = \{x \in X \mid 2e = 2x\} = \{x \in X \mid e = x\} = \{e\}$. Of course, if $\operatorname{char}(X) = 2$, then $2e = 2x = 0$ for all $x \in X$, whence $G(X) = X$. □

This shows that there is a large class of examples of p-semisimple QS-algebras obtained as quadratic Q-algebras.

Theorem 4.9. *Let X be a field with $|X| \geq 3$. Then every quadratic Q-algebra on X is isomorphic to every other such algebra defined on X.*

Proof. Let $x*y := x-y+e_1$ and $x*'y := x-y+e_2$, where $e_1, e_2 \in X$. Let $\pi(x) := x+(e_2-e_1)$, for all $x \in X$. Then $\pi(x*y) = [(x-y)+e_1]+(e_2-e_1) = (x-y)+e_2 = (x+(e_2-e_1))+(y+(e_2-e_1))+e_2 = \pi(x)*'\pi(y)$, whence the fact that $\pi^{-1}(x) = x+(e_1-e_2)$ yields the conclusion that π is an isomorphism of Q-algebras. □

Theorem 4.10. *Let X be a field with $|X| \geq 3$. Then every quadratic Q-algebra $(X;*,e)$ determines the abelian group $(X,+)$ via the definition $x+y = x*(e-y)$.*

Proof. Note that $x*(e-y) = x-(e-y)+e = x+y$ returns the additive operation of the field X, which is an abelian group. □

Not every quadratic Q-algebra $(X;*,e)$, $e \in X$, on a field X with $|X| \geq 3$ need be a BCK-algebra, since $((x*y)*(x*z))*(z*y) = e+(y-z) \neq e$ in general.

Problem 4.11. Construct a cubic Q-algebra which is not quadratic. Verify that among such cubic Q-algebras there are examples which are not QS-algebras. Furthermore, the question whether there are non-p-semisimple cubic Q-algebras is also of interest.

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