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ON THE EXTREME POINTS OF SOME CLASSES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. Let U be the unit disk, $D \supset U$ an open connected set and $z_0 \in D$. Let also $\mathbf{P}(z_0, c, D)$ be the class of holomorphic functions in D for which $f(z_0) = c$ and $Ref(z) > 0$ in U. We find the extreme points of the class $\mathbf{P}(z_0, c, D)$.

KEY WORDS AN PHRASES. Extreme points, positive real part.
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1. INTRODUCTION.

Let U be the unit disk $\{z : |z| < 1\}, D \supset U$ an open connected set, $z_0 \in D$ and $\mathbf{H}(D)$ be the class of holomorphic functions in D. By $\mathbf{P}(z_0, c, D)$ we denote the class of the functions $f \in \mathbf{H}(D)$ for which $f(z_0) = c$ and $Ref(z) > 0$ in U. Let $\mathbf{EP}(z_0, c, D)$ be the subclass of the extreme points of the above class for $\mathbf{P} = \mathbf{P}(0, 1, U)$ it has proven [1] that

$$\mathbf{EP} = \{(\epsilon + z)(\epsilon - z)^{-1} : \epsilon \in \partial U - D\},$$

In this paper we find the points of the subclass $\mathbf{EP}(z_0, c, D)$.

2. MAIN RESULT.

THEOREM. (i) If $(1 - |z_0|)Rec \leq 0$ then $\mathbf{EP}(z_0, c, D) = \emptyset$. (ii) If $(1 - |z_0|)Rec > 0$ then $f \in \mathbf{EP}(z_0, c, D)$ iff it has the form

$$f(z) = x_1(\frac{\epsilon + z}{\epsilon - z}) + ix_2,$$

where $\epsilon \in \partial U - D, x_1 = Rec[Re(\frac{\epsilon+z_0}{\epsilon-z_0})]^{-1}$ and $x_2 = Imc - x_1.Im(\frac{\epsilon+z_0}{\epsilon-z_0})$

PROOF. Let $f \in \mathbf{P}(z_0, c, D)$ with $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ in U. Let also $r < 1$, S be a complex number and $M > 0$ such that $0 < 2|S| < M$ and $z \in \partial U$. Since

$$[1 \pm \frac{1}{M}(Sz + \overline{S}z^{-1})]Ref(rz) > 0$$

then

$$Re[f(rz) \pm \frac{1}{M}(Szf(rz) + \overline{S}\sum_{n=0}^{\infty}\alpha_n r^n z^{n-1} + S\overline{\alpha}_0 z)] \tag{1}$$

By the maximum principle for harmonic functions it follows that (1) holds for every $z \in U$. Therefore for $r \to 1$ we have $Re(f(z) \pm u_1(z)) > 0$ in U where

$$u_1(z) = \frac{1}{M}[\overline{S}z^{-1}(f(z) - \alpha_0) + S\overline{\alpha}_0 z + Szf(z)] \tag{2}$$

Choosing appropriate $S \neq 0$ we get $Reu_1(z_0) = 0$. Setting $u(z) = u_1(z) - iImu_1(z_0)$ from $u(z_0) = 0$ it follows that $f \pm u \in \mathbf{P}(z_0, c, D)$.

Let now $f \in \mathbf{EP}(z_0, c, D)$. Then it is obvious that $u(z) = 0$ in D. If we set $S = |S|e^{i(\varphi + \frac{\pi}{2})}$ then from equality $u = 0$ we conclude that f has the form

$$f(z) = \frac{\xi_1(1 + z^2 e^{2i\varphi}) + \xi_2 z e^{i\varphi}}{(1 - z^2 e^{2i\varphi})} + i\xi_3 =$$

$$\frac{1}{2}(\xi_1 + \frac{\xi_2}{2})(\frac{1 + e^{i\varphi}z}{1 - e^{i\varphi}z}) + \frac{1}{2}(\xi_1 - \frac{\xi_2}{2})(\frac{1 - e^{i\varphi}z}{1 + e^{i\varphi}z}) + i\xi_3,$$

where $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$.

We now prove that $|\xi_2| = 2\xi_1$. From the Caratheodory' s inequality we have $|f'(0)| \leq 2Ref(0)$ and hence $|\xi_2| \leq 2\xi_1$. If $|\xi_2| < 2\xi_1$ then there are ξ_1^*, ξ_2^* such that $0 < |\xi_1^*| < \xi_1 + \frac{\xi_2}{2}, 0 < |\xi_2^*| < \xi_1 - \frac{\xi_2}{2}$, and $Reu_1^*(z_0) = 0$, where

$$u_1^*(z) = \xi_1^*(\frac{1 + e^{i\varphi}z}{1 - e^{i\varphi}z}) + \xi_2^*(\frac{1 - e^{i\varphi}z}{1 + e^{i\varphi}z})$$

Setting $u^*(z) = u_1^*(z) - iImu_1^*(z_0)$ then $f \pm u^* \in \mathbf{P}(z_0, c, D)$. Since $f \in \mathbf{EP}(z_0, c, D)$ it follows that $u^* = 0$ and hence $\xi_1^* = \xi_2^* = 0$. Therefore if $f \in \mathbf{EP}(z_0, c, D)$ then $|\xi_2| = 2\xi_1$ and hence f has the form

$$f(z) = x_1(\frac{\epsilon + z}{\epsilon - z}) + ix_2, x_1 > 0, x_2 \in \mathbb{R}, \epsilon \in \partial U - D. \quad (4)$$

From (4) we have

$$x_1 = Rec[Re(\frac{\epsilon + z_0}{\epsilon - z_0})]^{-1} > 0 \text{ and hence } (1 - |z_0|)Rec > 0.$$

Let $f \in \mathbf{P}(z_0, c, D)$ and having the form (4). Let also $0 < \lambda < 1$ and $f_1, f_2 \in \mathbf{EP}(z_0, c, D)$ such that $f = \lambda f_1(1 - \lambda)f_2$. Then

$$\frac{\epsilon + z}{\epsilon - z} = \lambda^* g_1(z) + (1 - \lambda^*)g_2(z) \text{ in } U,$$

where

$$\lambda^* = \lambda\frac{Ref_1(0)}{Ref(0)}, \; g_i(z) = \frac{f_i(z) - iImf_i(0)}{Ref_i(0)}, \; i = 1, 2.$$

Since

$$\frac{\epsilon + z}{\epsilon - z} \in \mathbf{EP} \text{ and } g_i \in \mathbf{P}$$

then

$$\frac{\epsilon + z}{\epsilon - z} = g_1(z) = g_2(z) \text{ in } U.$$

From the identity Theorem and the restrictions $f(z_0) = f_1(z_0) = f_2(z_0) = c$, we obtain $f = f_2$ and hence $f \in \mathbf{EP}(z_0, c, D)$.

REFERENCES

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