Internat. J. Math. & Math. Sci.
VOL. 20 NO. 3 (1997) 511-516

NONINCLUSION THEOREMS FOR SUMMABILITY MATRICES

J. A. FRIDY
Department of Mathematics and Computer Science
Kent State University
Kent, Ohio 44242

(Received November 30, 1995)

ABSTRACT. For both ordinary convergence and ℓ^1-summability explicit sufficient conditions on a matrix have long been known that ensure that the summability method is strictly stronger than the identity map The main results herein show that a matrix that satisfies those conditions can be included by another matrix *only* if the other matrix satisfies those same conditions.

KEY WORDS AND PHRASES: regular matrix, $\ell - \ell$ matrix, (summability) inclusion, Silverman-Töeplitz conditions, Knopp-Lorentz conditions.
1991 AMS SUBJECT CLASSIFICATION CODES: 40D25, 40C05

1. INTRODUCTION AND TERMINOLOGY

Let x denote a complex number sequence $\{x_k\}_{k=1}^{\infty}$ and A denote an infinite matrix $[a_{nk}]$ with complex entries; then Ax is the transformed sequence whose n-th term is given by $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$. Let c denote the set of convergent sequences, and $c_A = \{x : Ax \in c\}$ Similarly, $\ell^1 = \left\{x : \sum_{k=1}^{\infty} |x_k| < \infty\right\}$ and $\ell_A = \{x : Ax \in \ell^1\}$ The matrix A is called *regular* if $c \subseteq c_A$, and A is *stronger than (ordinary) convergence* if $c \subsetneqq c_A$ Similarly, A is called an $\ell - \ell$ matrix if $\ell^1 \subseteq \ell_A$, and A is *stronger than the identity* (map) if $\ell^1 \subsetneqq \ell_A$. If A and B are matrices such that $\lim Ax = L$ implies $\lim Bx = L$, then we say "B includes A," and this clearly implies that $c_A \subseteq c_B$. In the $\ell - \ell$ case we simply write $\ell_A \subseteq \ell_B$ with no verbal phase describing it.

There is previous work giving explicit conditions on A to imply that $c_A = c = c_I$ or $\ell_A = \ell^1 = \ell_I$ (See, e.g., the Mercerian-type theorems in [1], [3], and [4]. In [2] and [5] conditions on A were given that ensure that $c \subsetneqq c_A$ and $\ell^1 \subsetneqq \ell_A$, respectively Explicit conditions are not known for making general comparisons of c_A and c_B or of ℓ_A and ℓ_B (except when $B = I$) In this paper we address the general inclusion question. The principal results show that if A satisfies the conditions of [2] or [5] that ensure that A is stronger than I, then A can be included by B *only* if B also satisfies those same conditions

For the reader's convenience we state the theorems due to Silverman-Toeplitz [6, page 43] and Knopp-Lorentz [7] that characterize regular matrices and $\ell - \ell$ matrices, respectively

SILVERMAN-TÖEPLITZ THEOREM. The matrix A is regular if and only if the following conditions are satisfied:

(i) for each k, $\lim_n a_{nk} = 0$,

(ii) for each n, $\sum_{k=1}^{\infty} a_{nk}$ converges and $\lim_n \sum_{k=1}^{\infty} a_{nk} = 1$,

(iii) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$

KNOPP-LORENTZ THEOREM. The matrix A is an $\ell - \ell$ matrix if and only if the following condition is satisfied:

(iv) $\sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty$.

2. COMPARISON OF REGULAR MATRICES

In [2] Agnew proved the following simple criterion for establishing $c \subsetneq c_A$

THEOREM 2.1. If A is regular and satisfies the condition

$$\lim_{n,k} a_{nk} = 0, \tag{2.1}$$

then $c \subsetneq c_A$.

The double limit in (2.1) is taken in the Pringsheim sense: if $\epsilon > 0$ then there exists an N such that $|a_{nk}| < \epsilon$ whenever *both* $n > N$ and $k > N$. This sets the stage for the first of our "noninclusion theorems."

THEOREM 2.2. If A and B are regular matrices such that A satisfies (2.1) and B does not, i e.,

$$\lim_{n,k} b_{nk} \neq 0, \tag{2.2}$$

then $c_A \not\subseteq c_B$.

PROOF. First note that since the rows of A are null sequences, (2.1) implies that

$$\lim_k \left\{ \max_n |a_{nk}| \right\} = 0. \tag{2.3}$$

Also, (2.2) allows us to choose increasing sequences ν' and κ' of row and column indices satisfying

$$|b_{\nu'(m),\kappa'(m)}| \geq \delta > 0 \quad \text{for all } m. \tag{2 4}$$

Then use (2.3) to choose a subsequence of those pairs $< \nu''(m), \kappa''(m) >$ such that

$$\max_n |a_{n,\kappa''(m)}| < 2^{-m} \quad \text{for all } m. \tag{2 5}$$

Next, using conditions ST(i) we choose a further subsequence $< \nu(m), \kappa(m) >$ so that for each m, $k < \kappa(m)$ and $n > \nu(m)$ imply

$$|a_{nk}| < 2^{-m} \quad \text{and} \quad |b_{nk}| < 2 - m. \tag{2 6}$$

We also use the assumption that the rows of B tend to zero (from ST(ii)) tọ choose $< \nu(m), \kappa(m) >$ so that

$$|b_{nk}| < 2^{-m}$$

whenever $k > \kappa(m)$ and $n < \nu(m)$. Define the sequence x by

$$x_k := \begin{cases} m, & \text{if } k = \kappa(m) \text{ for } m = 1, 2, ..., \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

For $n > \nu(m)$, (2.5), (2.6), and (2.7) yield

$$\begin{aligned} |(Ax)_n| &= \left| \sum_{j=0}^{\infty} a_{n,\kappa(j)}(j) \right| \\ &\leq \sum_{j \leq m} 2^{-m}(j) + \sum_{j > m} 2^{-j}(j) \\ &\leq 2^{-m} \sum_{j \leq m} j + R_m, \end{aligned}$$

say As $m \to \infty$ these expressions both tend to zero ($R_m \to 0$ because the series $\sum 2^{-j}j$ is convergent); hence, $\lim Ax = 0$ and $x \in c_A$. For Bx we have

$$\begin{aligned}\left|(Bx)_{\nu(m)}\right| &\geq |b_{\nu(m),\kappa(m)}x_{\kappa(m)}| - \sum_{j\neq m} |b_{\nu(m),\kappa(j)}|(j) \\ &= |b_{\nu(m),\kappa(m)}m| - \sum_{j<m} 2^{-m}(j) - \sum_{j>m} 2^{-j}(j) \\ &= |b_{\nu(m),\kappa(m)}|m - 2^{-m-1}m(m+1) - R_m.\end{aligned}$$

The latter two terms tend to zero as above, and by (2.4) the first term is unbounded, hence, $x \notin c_B$, and the proof is complete.

REMARK. In the proof of Theorem 2.2 we did not use the full strength of the regularity hypothesis. It would have sufficed to assume only that the rows and columns of A and B tend to zero.

To illustrate Theorem 2.2 we can take A to be any Cesàro matrix C_j for $j > 0$, or any Euler-Knopp matrix E_r for $0 < r < 1$. (They all satisfy (2.1).) Then B could be any Nörlund matrix N_p with p finitely nonzero (see [6, page 64]), or any weighted mean $\overline{N}_p$ with $p \in \ell^1$ (see [6, page 57]), they satisfy (2.2). Therefore none of the latter matrices includes any of the former.

One might note the similarity of form between Theorem 2.2 and Theorem 2.0.3 of [8] where Wilansky proved that if A is conull and B is not, then $c_A \not\subseteq c_B$. The conservative matrix A is *conull* provided that

$$\lim_n \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} \left(\lim_n a_{nk}\right) = 0.$$

3. COMPARISON OF $\ell - \ell$ MATRICES

In [5] the following theorem was proved, giving a sufficient condition for an $\ell - \ell$ matrix to be stronger than the identity matrix

THEOREM 3.1. If A is an $\ell - \ell$ matrix for which there exists an integer m such that

$$\liminf_k \sum_{n=m}^{\infty} |a_{nk}| = 0, \tag{3.1}$$

then $\ell^1 \subsetneq \ell_A$.

We next give an $\ell - \ell$ analogue of Theorem 2.2.

THEOREM 3.2. If A and B are $\ell - \ell$ matrices such that A satisfies (3.1) and B does not, then $\ell_A \not\subseteq \ell_B$

Actually, we shall prove somewhat more.

THEOREM 3.3. Let A be an $\ell - \ell$ matrix for which there is an integer μ and a sequence $\{k(j)\}_{j=1}^{\infty}$ of column indices such that

$$\lim_j \sum_{n=\mu}^{\infty} |a_{n,k(j)}| = 0; \tag{3.2}$$

if B is a matrix satisfying

$$\lim_j \sum_{n=\mu}^{\infty} |b_{n,k(j)}| \neq 0, \tag{3.3}$$

then $\ell_A \not\subseteq \ell_B$.

PROOF. First note that we may assume that the rows of B satisfy

$$\lim_j b_{n,k(j)} = 0 \quad \text{for each } n. \tag{3.4}$$

For, if not, then there is an n^* and a subsequence $\{k'(j)\}$ such that

$$|b_{n^*,k'(j)}| \geq \epsilon > 0 \quad \text{for every } j. \tag{3 5}$$

Property (3.2) allows us to choose a further subsequence $\{k''(j)\}$ such that

$$\sum_{n=\mu}^{\infty} |a_{n,k''(j)}| < \frac{1}{j^2} \quad \text{for each } j.$$

Define

$$x_k := \begin{cases} 1, & \text{if } k = k''(j) \text{ for } j = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

This yields

$$\begin{aligned} \sum_{n \geq \mu} |(Ax)_n| &= \sum_{n=\mu}^{\infty} \left| \sum_{j=1}^{\infty} a_{n,k''(j)} \right| \\ &\leq \sum_{j=1}^{\infty} \sum_{n=\mu}^{\infty} |a_{n,k''(j)}| \\ &\leq \sum_{j=1}^{\infty} \frac{1}{j^2}, \end{aligned}$$

while (3.5) implies that the series

$$(Bx)_{n^*} = \sum_{j=1}^{\infty} b_{n^*,k''(j)}$$

is nonconvergent. Thus, as in the proof of Theorem 1 of [5], we can choose x so that $x \in \ell_A$ but Bx is not defined.

Assume that (3.2), (3.3), and (3.4) hold We shall find an x in ℓ_A that is not in ℓ_B. Using (3 3) and replacing $\{k(j)\}$ with one of its (appropriately chosen) subsequences $\{k(i)\}$, we can assume without loss of generality that

$$\sum_{n=1}^{\infty} |b_{n,k(i)}| \geq 2\delta > 0 \quad \text{for each } i. \tag{3.6}$$

Replacing $\{k(i)\}$ with yet another of its subsequences $\{k(p)\}$ we can get ,

$$\sum_{n=\mu}^{\infty} |a_{n,k(p)}| < t_p \quad \text{for each } p, \tag{3 7}$$

where $t \in \ell^1$.

Next we construct an increasing sequence $\{\nu(m)\}$ of row indices and a further subsequence $\{\kappa(m)\}$ of $\{k(p)\}$ to define the sequence x that we seek First, take $\nu(-1) = 0$; then use (3 6) to choose $\kappa(1)$ satisfying

$$\sum_{n=1}^{\infty} |b_{n,\kappa(1)}| \geq 2\delta,$$

and choose $\nu(1)$ so that

$$\sum_{n \leq \nu(1)} |b_{n,\kappa(1)}| \geq \delta \quad \text{and} \quad \sum_{n > \nu(1)} |b_{n,\kappa(1)}| < t_1.$$

After $\kappa(1) < \ldots < \kappa(m-1)$ and $\nu(1) < \ldots < \nu(m-1)$ have been selected use (3 4) to choose $\kappa(m) > \kappa(m-1)$ such that

$$\sum_{n=1}^{\nu(m-1)} |b_{n,\kappa(m)}| < t_m, \tag{3.8}$$

and by (3.6),

$$\sum_{n=1}^{\infty} |b_{n,\kappa(m)}| \geq 2\delta.$$

Then select $\nu(m) > \nu(m-1)$ so that

$$\sum_{n=1+\nu(m-1)}^{\nu(m)} |b_{n,\kappa(m)}| \geq \delta \tag{3.9}$$

and

$$\sum_{n=\nu(m)}^{\infty} |b_{n,\kappa(m)}| < t_m. \tag{3.10}$$

Now define x by

$$x_k := \begin{cases} \dfrac{e^{i\theta}}{m}, & \text{if } k = \kappa(m), \text{ for } m = 1, 2, \ldots, \text{ and } \theta \in \mathbb{R} \\ 0, & \text{otherwise.} \end{cases}$$

This yields $x \in \ell_A$ because by (3.7),

$$\sum_{n=\mu}^{\infty} |(Ax)_n| \leq \sum_{n=\mu}^{\infty} \left| \sum_{m=1}^{\infty} a_{n,\kappa(m)} \right| < \sum_{n=\mu}^{\infty} t_m.$$

For Bx, inequalities (3.8), (3.9), and (3.10) give

$$\begin{aligned} \sum_{n=1}^{\nu(n)} |(Bx)_n| &= \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} \left| \sum_{j=1}^{\infty} b_{n,\kappa(j)} \left(\frac{1}{j}\right) \right| \\ &\geq \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} \left\{ |b_{n,\kappa(m)}| \frac{1}{m} - \sum_{j \neq m} |b_{n,\kappa(j)}| \right\} \\ &= \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} |b_{n,\kappa(m)}| \frac{1}{m} - \sum_{m=1}^{N} \sum_{n=1+\nu(m-1)}^{\nu(m)} \sum_{j \neq m} |b_{n,\kappa(j)}| \\ &\geq \delta \sum_{m=1}^{N} \frac{1}{m} - 2 \sum_{j=1}^{\infty} t_{\kappa(j)}. \end{aligned}$$

Hence, $Bx \notin \ell^1$, which establishes the assertion that x is in ℓ_A but not in ℓ_B.

Note that in defining x we need only have $|x_{\kappa(m)}| \leq 1/m$ in order to have the subsequent inequalities valid It is the convergence of the $\mu - 1$ series

$$\sum_{j=1}^{\infty} a_{n,\kappa(j)} x_{\kappa(j)} = (Ax)_n$$

for $n = 1, 2, \ldots, \mu - 1$ that requires the factor of $e^{i\theta}$ in $x_{\kappa(m)}$ (See Theorem 1 and Lemma 1 of [5])

REMARK. As above with Theorem 1, we have not needed the full strength of the hypotheses, in this case, the assumption that A is an $\ell - \ell$ matrix is stronger than what is needed. Condition (3 2) guarantees that $Ax \in \ell^1$ whenever it exists, so the only concern is that $(Ax)_n$ exists for $n < \mu$ This existence would be guaranteed by assuming only that the row sequences $\{a_{n,k(j)}\}_{j=1}^{\infty}$ for $n < \mu$ are bounded. (See Lemma 1 of [5].)

As an illustration of Theorem 3.3, we give an example of two matrices that are noncomparable in the $\ell - \ell$ sense

EXAMPLE 3.1. Define A by

$$a_{nk} := \begin{cases} 1, & \text{if } n = 1 \text{ and } k = 1, 2, .., \\ \frac{1}{n}, & \text{if } k = n > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Take B to be the Euler-Knopp matrix E_r for some $r \in (0, 1)$.

$$E_r[n, k] := \begin{cases} \binom{n}{k} (1 - r)^{n-k} r^k, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

Then

$$\sum_{n=2}^{\infty} |a_{nk}| = \frac{1}{n}, \quad \text{for } k = 2, 3, ...,$$

so A satisfies condition (3 2). In Theorem 4 of [5], it is noted that for each k,

$$\sum_{n=k}^{\infty} |E_r[n, k]| = \frac{1}{r},$$

so E_r does not satisfy (3.2). Hence, by Theorem 3.2, $\ell_A \not\subseteq \ell_{E_r}$. Although the following does not involve Theorems 3.2 and 3.3, for the sake of completeness we show that $\ell_{E_r} \not\subseteq \ell_A$. This is verified by observing that if $r \in (0, 1)$ and $x_k := (-r)^{-k}$, then $(E_r x)_n = (-r)^n$; therefore $x \in \ell_{E_r}$ But $|(Ax)_n| = |(-r)^{-n}/n| \to \infty$, so $x \notin \ell_A$.

In closing we offer an open question related to Theorem 3.2. Can the absolute sums in conditions (3.2) and (3.3) be weakened to ordinary sums? More precisely, if A satisfies

$$\liminf_k \left| \sum_{n=\mu}^{\infty} a_{nk} \right| = 0$$

and B satisfies

$$\liminf_k \left| \sum_{n=\mu}^{\infty} b_{nk} \right| > 0,$$

does it follow that $\ell_A \not\subseteq \ell_B$?

REFERENCES

[1] AGNEW, R. P., "On equivalence of methods of summation," *Tôhoku Math. J.* **35** (1932), 244-252

[2] ______, "A simple sufficient condition that a method of summability be stronger than convergence," *Bull. Amer. Math. Soc.* **52** (1946), 128-132

[3] ______, "Equivalence of methods for evaluation of sequences," *Proc. Amer. Math. Soc.* **3** (1952), 550-556

[4] FRIDY, J. A., "Mercerian-type theorems for absolute summability," *Portugal. Math.* **33** (1974), 141-145

[5] ______, "Absolute summability matrices that are stronger than the identity mapping," *Proc. Amer. Math. Soc.* **47** (1975), 112-118.

[6] HARDY, G.H , *Divergent Series*, Clarendon Press, Oxford, 1949

[7] KNOPP, K. and LORENTZ, G. G., "Beitràge zur absoluten Limitierung," *Arch. Math.* **2** (1949), 10-16.

[8] WILANSKY, A , "An application of Banach linear functionals to summability," *Trans. Amer. Math. Soc.* **67** (1949), 59-68.