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THE PALEY-WIENER-LEVINSON THEOREM REVISITED

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ABSTRACT. In this paper a new proof of the Paley-Wiener-Levinson theorem is presented. This proof is based upon the isometry between the Paley-Wiener space and that of the square-integrable functions in $[-\pi,\pi]$, on one hand, and a Titchmarsh's theorem which allows recovering bandlimited, entire functions from their zeros, on the other hand.

KEY WORDS AND PHRASES. Nonuniform sampling, Lagrange type interpolation series, Riesz basis, entire functions of exponential type.

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1 Introduction

The aim of this paper is twofold: first, it provides a new — somehow simpler — proof of the Paley-Wiener-Levinson (PWL) theorem, and second, it makes clear the relationship between recovering finite-energy, bandlimited functions from an infinite set of samples or from its real zeros (zero crossings, in technical jargon), two well-known tools in signal processing [1, 2, 3].

If B_π denotes the space of $[-\pi,\pi]$-bandlimited L^2-functions, the classic Whittaker-Shannon-Kotel'nikov (WSK) theorem states that any $f \in B_\pi$ can be written as

$$f(z)=\sum_{n=-\infty}^{\infty} f(n)\frac{\sin\pi(z-n)}{\pi(z-n)}, \quad z\in\mathbb{C}, \tag{1.1}$$

which can also be written

$$f(z)=\sum_{n=-\infty}^{\infty} f(n)\frac{G(z)}{G'(n)(z-n)}, \tag{1.2}$$

if $G(z)=\sin\pi z/\pi$. The latter expression exhibits the Lagrange type interpolatory character of the WSK result. Equation (1.1) expresses the possibilty of recovering a certain kind of signal from a sequence of *regularly spaced* samples.

From a practical point of view it is interesting to have a similar result, but for a sequence of samples taken with a *nonuniform* distribution along the real line (a straightforward application of this result would be the recovering of signals from samples affected by time-jitter error, i.e., taken at points $t_n = n+\delta_n$, with δ_n some measurement uncertainty). An appropriate question to get such a result would be how close should the sample points be to the regular sample points so that a similar equation to (1.2) still holds. A first answer to this question was given by Paley and Wiener [4], who proved that if the sequence of sample points, $\{t_n\}_{n\in\mathbf{Z}}$, satisfies

$$D\equiv\sup_{n\in\mathbf{Z}}|t_n-n|<\tau, \tag{1.3}$$

where $\tau = 1/\pi^2$, and the sequence is symmetric, i.e., $t_{-n} = t_n$ ($n \geq 1$), then any $f \in B_\pi$ can be expressed as

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)}, \tag{1.4}$$

where now

$$G(z) = (z - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{t_n^2}\right). \tag{1.5}$$

Later on, Levinson [5] extended condition (1.3) to $\tau = 1/4$ and nonsymmetric sequences. This result is related with the "maximum" perturbation of the Hilbert basis $\{e^{-inx}\}_{n\in\mathbf{Z}}$ of the square-integrable function space $L^2[-\pi, \pi]$, in such a way that the perturbed sequence $\{e^{-it_nx}\}_{n\in\mathbf{Z}}$ is a Riesz basis of the same space. Kadec proved that Levinson's result, $\tau = 1/4$, is the best possible, in the sense that if $D = 1/4$ counterexamples can be found. See [6] for details.

The problem of signal recovering has also been considered from a different point of view. It is well-known from the classic Paley-Wiener theorem that $[-\pi, \pi]$-bandlimited L^2-function space coincides with that of the entire functions of exponential type at most π whose restriction to $\mathbb{R}$ belongs to $L^2(\mathbb{R})$. Although entire functions are not completely determined by the location of their zeros, as can be seen from the Hadamard factorization theorem [6], bandlimited functions are, as can be deduced from a Titchmarsh's theorem [7, 8] to which I will refer later on. A $[a, b]$-bandlimited function is uniquely determined by its zeros up to an exponential factor depending on the spectral interval. If the spectral interval is of the form $[-a, a]$, this exponential factor reduces to a constant.

A good survey of all these results can be found in Ref. [9].

As explained in the beginning, the aim of this paper is to combine the ideas of perturbing the Hilbert basis $\{e^{-inx}\}_{n\in\mathbf{Z}}$ to get a Riesz basis with those of recovering a bandlimited signal from its zero crossings, into a new proof of the PWL interpolation theorem.

2 Recovering bandlimited L^2-functions

Let us consider the space of $[-\pi, \pi]$-bandlimited L^2-functions

$$\begin{aligned} B_\pi &= \left\{ f \in L^2(\mathbb{R}) \,/\, \|f\|_2 \equiv \left(\int_{\mathbb{R}} |f(x)|^2 dx\right)^{1/2} < \infty \text{ and } \operatorname{supp} \hat{f} \subseteq [-\pi, \pi] \right\} \\ &= \left\{ f \text{ entire of exponential type at most } \pi, \text{ with } f|_{\mathbb{R}} \in L^2(\mathbb{R}) \right\}, \end{aligned}$$

where the last equality is the statement of the classic Paley-Wiener theorem. Provided with the inner product $\langle f, g\rangle_{B_\pi} = \int_{\mathbb{R}} f\bar{g}$, the space B_π is a separable Hilbert space, isometrically isomorphic to $L^2[-\pi, \pi]$. The isomorphism is precisely the Fourier transform

$$\begin{array}{ccc} B_\pi & \xrightarrow{\mathcal{F}} & L^2[-\pi, \pi] \\ f & \longrightarrow & \hat{f} \end{array}, \qquad f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(t)\, e^{izt} dt. \tag{2.1}$$

The following properties can be established:

(a) The energy of $f \in B_\pi$ is contained in its samples $\{f(n)\}_{n\in\mathbf{Z}}$:

$$\|f\|_{B_\pi}^2 = \int_{\mathbb{R}} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(t)|^2 dt = \|\hat{f}\|_{L^2[-\pi,\pi]}^2 = \sum_{n=-\infty}^{\infty} |f(n)|^2,$$

since $\{f(n)\}_{n\in\mathbf{Z}}$ are the Fourier coefficients of the 2π-periodic extension of $\hat{f}$ in the exponential trigonometric basis.

(b) Since $\{e^{-inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-\pi,\pi]$, so is

$$\mathcal{F}^{-1}\left(\{e^{-inx}\}_{n\in\mathbb{Z}}\right)=\left\{\frac{\sin\pi(z-n)}{\pi(z-n)}\right\}_{n\in\mathbb{Z}}=\{T_n\mathrm{sinc}\,z\}_{n\in\mathbb{Z}},$$

where $T_af(x)\equiv f(x-a)$ is the translation operator. Therefore, any $f\in B_\pi$ can be expanded as the cardinal series

$$f(z)=\sum_{n=-\infty}^{\infty}c_n\frac{\sin\pi(z-n)}{\pi(z-n)},\qquad c_n=\langle f,T_n\mathrm{sinc}\,\rangle_{B_\pi}.$$

(c) Convergence in the norm of B_π implies uniform convergence in horizontal strips in $\mathbb{C}$, because

$$|f(z)|\le e^{\pi|y|}\|f\|_{B_\pi},\quad z=x+iy.$$

This follows, in a straightforward way, from the isometry and Cauchy-Schwarz inequality:

$$|f(x+iy)|\le\frac{1}{2\pi}\int_{-\pi}^{\pi}|\widehat{f}(t)|\,e^{-ty}dt\le\frac{e^{\pi|y|}}{2\pi}\int_{-\pi}^{\pi}|\widehat{f}(t)|\,dt\le e^{\pi|y|}\|f\|_{B_\pi}.$$

(d) The sinc function is the reproducing kernel of B_π: for $f\in B_\pi$ and $x\in\mathbb{R}$,

$$\begin{aligned}f(x)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{f}(\xi)\,e^{ix\xi}d\xi=\langle\widehat{f},e^{-ix\xi}\rangle_{L^2[-\pi,\pi]}=\langle f,T_x\mathrm{sinc}\rangle_{B_\pi}\\&=\int_{\mathbb{R}}f(t)\mathrm{sinc}\,(t-x)\,dt=(f*\mathrm{sinc}\,)(x).\end{aligned}$$

By taking $z=n\in\mathbb{Z}$ in (b) and using (c), it follows that $c_n=f(n)$. This is a proof of the classic WSK theorem:

THEOREM 2.1 (WKS theorem) *Every $f\in L^2(\mathbb{R})$ bandlimited to $[-\pi,\pi]$ can be reconstructed from its samples at the integers $\{f(n)\}_{n\in\mathbb{Z}}$ via the formula*

$$f(z)=\sum_{n=-\infty}^{\infty}f(n)\frac{\sin\pi(z-n)}{\pi(z-n)},$$

where the convergence is uniform in horizontal strips of $\mathbb{C}$ (in particular in $\mathbb{R}$).

By means of this theorem we have a tool for recovering bandlimited signals from a sequence of samples; but, as commented in the Introduction, these signals can also be recovered from their zeros (zero crossings in the real case). The following Titchmarsh's theorem [7] provides the mathematical foundation for this:

THEOREM 2.2 (Titchmarsh theorem) *Let $F\in L^1[a,b]$ and define the entire function f to be*

$$f(z)=\int_a^b F(w)\,e^{zw}dw.$$

Then f has infinitely many zeros, $\{z_n\}_{n\in\mathbb{N}}$, with nondecreasing absolute values, such that

$$f(z)=f(0)\,e^{\frac{a+b}{2}z}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right),$$

where the infinite product is conditionally convergent.

In the above theorem, it is assumed that a and b are the effective lower and upper limits of the integral, in the sense that there are no numbers $\alpha > a$ and $\beta < b$ such that $F(\omega) = 0$ (a.e.) in $[a, \alpha]$ or $[\beta, b]$.

If f is bandlimited to $[-a, a]$, then

$$f(z) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

provided $f(0) \neq 0$, or

$$f(z) = Az^m \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right),$$

if $z = 0$ is a zero of f of order m.

Notice that the zeros in Titchmarsh theorem may be complex. This poses a difficulty from a technical viewpoint, as complex zeros are harder to detect than real zeros; but whenever they are real, this theorem provides a useful tool for signal recovering, usually referred to as *real-zero interpolation* [2, 10].

3 The PWL interpolation theorem

In what follows $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ will denote a sequence of real numbers such that

$$D = \sup_{n\in\mathbb{Z}} |t_n - n| < \frac{1}{4}.$$

Let us define

$$G(z) = (z - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{t_n}\right)\left(1 - \frac{z}{t_{-n}}\right),$$

an entire, well-defined function (whose set of zeros is $\{t_n\}_{n\in\mathbb{Z}}$) as it will be made clear along the proof of the following theorem.

THEOREM 3.1 (PWL theorem) *Any $f \in B_\pi$ can be recovered from its sample values $\{f(t_n)\}_{n\in\mathbb{Z}}$ by means of the Lagrange type interpolation series*

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)},$$

which is uniformly convergent in horizontal strips of $\mathbb{C}$ (in particular in $\mathbb{R}$).

PROOF: By Kadec's $\frac{1}{4}$-theorem (p. 42 of Ref. [6]), $\{e^{-it_n\xi}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2[-\pi, \pi]$. Consequently it will admit a unique biorthogonal basis $\{h_n(\xi)\}_{n\in\mathbb{Z}}$ (p. 28 of Ref. [6]), i.e., for every $m, n \in \mathbb{Z}$,

$$\langle h_n, e^{-it_m\xi}\rangle_{L^2[-\pi,\pi]} = \delta_{nm} \quad \text{(Kronecker's symbol)}.$$

Thus, every $\hat{f} \in L^2[-\pi, \pi]$ can be expressed as

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \langle \hat{f}, h_n\rangle_{L^2[-\pi,\pi]} e^{-it_n\xi} = \sum_{n=-\infty}^{\infty} \langle \hat{f}, e^{-it_n\xi}\rangle_{L^2[-\pi,\pi]} h_n(\xi).$$

By using the isometry $\mathcal{F}^{-1}$, we have in B_π

$$f(z) = \sum_{n=-\infty}^{\infty} \langle \hat{f}, h_n\rangle_{L^2[-\pi,\pi]} \mathcal{F}^{-1}\left(e^{-it_n\xi}\right)(z) = \sum_{n=-\infty}^{\infty} \langle \hat{f}, e^{-it_n\xi}\rangle_{L^2[-\pi,\pi]} \mathcal{F}^{-1}(h_n)(z).$$

By setting $g_n \equiv \mathcal{F}^{-1}(h_n)$ and taking into acount that $\langle \widehat{f}, h_n \rangle_{L^2[-\pi,\pi]} = \langle f, g_n \rangle_{B_\pi}$ and that, by property (d) of section 2, $\langle \widehat{f}, e^{-it_n\xi} \rangle_{L^2[-\pi,\pi]} = \langle f, T_{t_n}\mathrm{sinc}\,\rangle_{B_\pi} = f(t_n)$, we can rewrite

$$f(z) = \sum_{n=-\infty}^{\infty} \langle f, g_n \rangle_{B_\pi} (T_{t_n}\mathrm{sinc}\,)(z) = \sum_{n=-\infty}^{\infty} f(t_n) g_n(z) .$$

Now,

$$g_n(z) = \mathcal{F}^{-1}(h_n)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h_n(\xi)\, e^{iz\xi} d\xi$$

is an entire function, bandlimited to $[-\pi, \pi]$ whose zeros are $\{t_m\}_{m \neq n}$, and therefore, by Titchmarsh theorem,

$$g_n(z) = A_n \frac{G(z)}{z - t_n} .$$

(Notice that by setting $n = 0$, for instance, the above formula shows that $G(z)$ is an entire function, as stated at the begining of this section.) Since $g_n(t_n) = 1$, then $A_n = 1/G'(t_n)$; thus

$$f(z) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{G(z)}{G'(t_n)(z - t_n)} ,$$

which is convergent in the norm of B_π, and, by property (c) of section 2, uniformly in horizontal strips of $\mathbb{C}$. ■

Although not important for the proof, we have obtained, as a byproduct, the interesting result that $\{(T_{t_n}\mathrm{sinc}\,)(z)\}_{n \in \mathbb{Z}}$ and $\{g_n(z)\}_{n \in \mathbb{Z}}$ are biorthogonal Riesz bases in B_π.

The irregular sampling problem has also been considered within more general contexts, as bandlimited L^p-functions [11], for instance, where there is a similar theorem which has been proved with complex variables techniques. One of the most striking differences is that the sampling is somewhat more sensitive to noise, in the sense that $\{t_n\}_{n \in \mathbb{Z}}$ must satisfy the stronger restriction $D < 1/2p$ for $2 \leq p < \infty$.

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