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Research Article

Existence and Orbital Stability of Cnoidal Waves for a 1D Boussinesq Equation

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We will study the existence and stability of periodic travelling-wave solutions of the nonlinear one-dimensional Boussinesq-type equation $\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + \Phi_t\Phi_{xx} + 2\Phi_x\Phi_{xt} = 0$. Periodic travelling-wave solutions with an arbitrary fundamental period T_0 will be built by using Jacobian elliptic functions. Stability (orbital) of these solutions by periodic disturbances with period T_0 will be a consequence of the general stability criteria given by M. Grillakis, J. Shatah, and W. Strauss. A complete study of the periodic eigenvalue problem associated to the Lame equation is set up.

1. Introduction

In this paper, we consider the existence of periodic travelling-waves solutions and the study of nonlinear orbital stability of these solutions for the one-dimensional Boussinesq-type equation

$$\Phi_{tt} - \Phi_{xx} + a\Phi_{xxxx} - b\Phi_{xxtt} + \Phi_t\Phi_{xx} + 2\Phi_x\Phi_{xt} = 0, \tag{1.1}$$

where a and b are positive numbers.

One can see that this Boussinesq-type equation is a rescaled version of the one-dimensional Benney-Luke equation

$$\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) + \epsilon(\Phi_t\Phi_{xx} + 2\Phi_x\Phi_{xt}) = 0, \tag{1.2}$$

which is derived from evolution of two-dimensional long water waves with surface tension. In this model, $\Phi(x,t)$ represents the nondimensional velocity potential at the bottom fluid boundary, μ represents the long-wave parameter (dispersion coefficient), ϵ represents the amplitude parameter (nonlinear parameter), and $a-b=\sigma-1/3$, with σ being named the Bond number which is associated with surface tension.

An important feature is that the Benney-Luke equation (1.2) reduces to the Korteweg-de-Vries equation (KdV) when we look for waves evolving slowly in time. More precisely, when we seek for a solution of the form

$$\Phi(x,t)=f(X,\tau), \tag{1.3}$$

where $X=x-t$ and $\tau=\epsilon t/2$. In this case, after neglecting $O(\epsilon)$ terms, $\eta=f_X$ satisfies the KdV equation

$$\eta_\tau-\left(\sigma-\frac{1}{3}\right)\eta_{XXX}+3\eta\eta_X=0. \tag{1.4}$$

It was established by Angulo [1] (see also [2]) and Angulo et al. [3] that cnoidal waves solutions of mean zero for the KdV equation exist and they are orbitally stable in $H^1_{\text{per}}[0,T_0]$. The proof of orbital stability obtained by Angulo et al. was based on the general result for stability due to Grillakis et al. [4] together with the classical arguments by Benjamin in [5], Bona [6], and Weinstein [7] (see also Maddocks and Sachs [8]). This approach is used for obtaining stability initially in the space of functions of mean zero,

$$\mathcal{W}^1=\left\{q\in H^1_{\text{per}}([0,T_0]):\int_0^{T_0}q(y)dy=0\right\}. \tag{1.5}$$

The reason to use the space $\mathcal{W}^1$ to study stability is rather simple. Cnoidal wave solutions are not critical points of the action functional on the space $H^1_{\text{per}}([0,T_0])$, however on the space $\mathcal{W}^1$ cnoidal waves solutions are characterized as critical points of the action functional, as required in [4, 7]. The meaning of this is that the mean-zero property makes the first variation effectively zero from the point of view of the constrained variational problem, and so the theories in [4–7] can be applied.

Due to the strong relationship between the Benney-Luke equation (1.1) and the KdV equation (1.4), we are interested in establishing analogous results in terms of existence and stability of periodic travelling-waves solutions as the corresponding results obtained by Angulo et al. in the case of the KdV equation. More precisely, we want to prove existence of periodic travelling-wave solutions for the Benney-Luke equation (1.1) and to study the orbital stability of them.

In this paper, we will study travelling-waves for (1.1) of the form $\Phi(x,t)=\phi_c(x-ct)$ such that $\psi_c\equiv\phi_c'$ is a periodic function with mean zero on an *a priori* fundamental period and for values of c such that $0<c^2<\min\{1,a/b\}$. So, ϕ_c will be a periodic function. The profile ϕ_c has to satisfy the equation

$$(c^2-1)\phi_c'+(a-bc^2)\phi_c'''-\frac{3c}{2}(\phi_c')^2=A_0, \tag{1.6}$$

where A_0 is an integration constant. So, by following the paper of Angulo et al., we obtain that ψ_c is of type *cnoidal* and it is given by the formula

$$\psi_c(x) = -\frac{1}{3c}\left[\beta_2 + (\beta_3 - \beta_2)\operatorname{cn}^2\left(\frac{1}{\sqrt{a-bc^2}}\sqrt{\frac{\beta_3-\beta_1}{12}}x;k\right)\right] \tag{1.7}$$

with $\beta_1 < \beta_2 < 0 < \beta_3$, $\beta_1 + \beta_2 + \beta_3 = 3(1-c^2)$. Moreover, for T_0 appropriate, this solution has minimal period T_0 and mean zero on $[0, T_0]$. So, we obtain by using the Jacobian Elliptic function *dnoidal*, $\operatorname{dn}(\cdot;k)$, that (1.6) has a periodic solution of the form

$$\phi_c(x) = -\frac{\beta_1}{3c}x - \frac{\beta_3-\beta_2}{3cL_0k^2}\int_0^{L_0x}\operatorname{dn}^2(u;k)\,du + M, \tag{1.8}$$

for appropriate constants M and L_0.

We will show that the periodic travelling-wave solutions ϕ_c are orbitally stable with regard to the periodic flow generated by (1.1) provided that $0 < |c| < 1 < \sqrt{a/b}$, which corresponds to the Bond number $\sigma > 1/3$ and when for θ small, $0 < |c| < c_* + \theta < \sqrt{a/b} < 1$, which corresponds to the Bond number $\sigma < 1/3$. Here c_* is a specific positive constant (see Theorem 4.3). These conditions of stability are needed to assure the convexity of the function d defined by

$$d(c) = \frac{1}{2}\int_0^{T_0}(1-c^2)\psi_c^2 + (a-bc^2)\left(\psi_c'\right)^2 + c\psi_c^3\,dx, \tag{1.9}$$

where $\psi_c = \phi_c'$ and ϕ_c is a travelling-wave solution of (1.6) of *cnoidal* type, with speed c and period T_0.

Unfortunately from our approach, it is not clear if our waves are stable for the full interval $0 < |c| < \sqrt{a/b} < 1$.

We recall that in a recent paper, Quintero [9] established orbital stability/instability of solitons (solitary wave solutions) for the Benney-Luke equation (1.1) for $0 < c^2 < \min\{1, a/b\}$ by using the variational characterization of d. Orbital stability of the soliton was obtained when $0 < c < 1 < \sqrt{a/b}$ and orbital instability of the soliton was obtained when $0 < c_0 < c < \sqrt{a/b} < 1$ for some positive constant c_0.

Our result of stability of periodic travelling-wave solutions for (1.1) follows from studying the same problem to the Boussinesq system associated with (1.1),

$$\begin{aligned} q_t &= r_x, \\ r_t &= B^{-1}(q_x - aq_{xxx}) - B^{-1}(rq_x + 2qr_x), \end{aligned} \tag{1.10}$$

where $q = \Phi_x$, $r = \Phi_t$, and $B = 1 - b\partial_x^2$. More exactly, we will obtain an existence and uniqueness result for the Cauchy problem associated with system (1.10) in $H^1_{\text{per}}([0, T_0]) \times H^1_{\text{per}}([0, T_0])$ and also that the periodic travelling-wave solutions $(\psi_c, -c\psi_c)$ are orbitally stable by the flow of (1.10) with periodic initial disturbances restrict to the space $\mathcal{W}^1 \times H^1_{\text{per}}([0, T_0])$. In this point, we take advantage of the Grillakis et al.'s stability theory. More concretely, the stability result relies on the convexity of d defined in (1.9) and on a complete spectral analysis of the periodic eigenvalue problem of the linear operator

$$\mathscr{L}_{\mathrm{cn}} = -(a-bc^2)\frac{d^2}{dx^2} + (1-c^2) + 3c\psi_c, \tag{1.11}$$

which is related with the second variation of the action functional associated with system (1.10). We will show that $\mathscr{L}_{\mathrm{cn}}$ has exactly its three first eigenvalues simple, the eigenvalue zero being the second one with eigenfunction ψ_c' and the rest of the spectrum consists of a discrete set of double eigenvalues. This spectral description follows from a careful analysis of the classical *Lame periodic eigenvalue problem*

$$\begin{gathered}\frac{d^2}{dx^2}\Lambda + [\gamma - 12k^2\mathrm{sn}^2(x;k)]\Lambda = 0,\\ \Lambda(0) = \Lambda(2K(k)), \qquad \Lambda'(0) = \Lambda'(2K(k)),\end{gathered} \tag{1.12}$$

where $K = K(k)$ represents the complete elliptic integral of first kind defined by

$$K(k) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1.13}$$

We will show here that (1.12) has the three first eigenvalues simple and the remainder of eigenvalues are double. The exact value of these eigenvalues as well as its corresponding eigenfunctions are given.

We note that our stability results cannot be extended to more general periodic perturbations, for instance, by disturbances of period $2T_0$. In fact, it is well known that problem (1.12) has exactly four intervals of instability, and so when we consider the periodic problem in (1.12) but now with boundary conditions $\Lambda(0) = \Lambda(4K(k))$, $\Lambda'(0) = \Lambda'(4K(k))$, we obtain that the seven first eigenvalues are simple. So, it follows that the linear operator $\mathscr{L}_{\mathrm{cn}}$ with domain $H^1_{\mathrm{per}}([0,2T_0])$ will have exactly three negative eigenvalues which are simple. Hence, since the function d defined above is still convex with the integral in (1.9) defined in $[0,2T_0]$, we obtain that the general stability approach in [4, 10] cannot be applied in this case.

This paper is organized as follows. In Section 2, we establish the Hamiltonian structure for (1.10). In Section 3, we build periodic travelling-waves of fundamental period T_0 using Jacobian elliptic functions, named cnoidal waves, with the property of having mean zero in $[0,T_0]$. We also prove the existence of a smooth curve of cnoidal wave solutions for (1.10) with a fixed period T_0 and the mean-zero property in $[0,T_0]$. In Section 4, we study the periodic eigenvalue problem associated with the linear operator in (1.11). We also prove the convexity of the function d in a different fashion as it was done by Angulo et al. in [3, KdV equation (1.3)]. In Section 5, we discuss the main issue regarding orbital stability for the Boussinesq system (1.10). This requires proving the existence and uniqueness results of global mild solutions for this system, and applying Grillakis, Shatah, and Strauss stability methods, as done in [3]. Finally, in Section 6, we state the orbital stability of periodic wave solutions of the Benney-Luke equation, by showing the equivalence between the Cauchy problem for the Benney-Luke equation (1.1) and the Boussinesq system (1.10).

2. Hamiltonian structure

The Boussinesq system (1.10) can be written as a Hamiltonian system in the new variables

$$(q,p) \equiv \left(q, Br + \frac{1}{2}q^2\right) \tag{2.1}$$

as

$$\begin{aligned} q_t &= \partial_x B^{-1}\left(p - \frac{1}{2}q^2\right), \\ p_t &= \partial_x(Aq - rq), \end{aligned} \tag{2.2}$$

with $A = 1 - a\partial_x^2$ and $B = 1 - b\partial_x^2$. This system arises as the Euler-Lagrange equation for the action functional

$$\mathcal{S} = \int_{t_0}^{t_1} \mathcal{Q}\begin{pmatrix} q \\ p \end{pmatrix} dt, \tag{2.3}$$

where the Lagrangian $\mathcal{Q}$ and the Hamiltonian are given, respectively, by

$$\begin{aligned} \mathcal{Q}\begin{pmatrix} q \\ p \end{pmatrix} &= \frac{1}{2}\int_0^{T_0} \left\{B^{-1}\left(p - \frac{1}{2}q^2\right)\left(p - \frac{1}{2}q^2\right) - qAq + B^{-1}\left(p - \frac{1}{2}q^2\right)q^2\right\} dx, \\ \mathcal{H}\begin{pmatrix} q \\ p \end{pmatrix} &= \frac{1}{2}\int_0^{T_0} \left\{\left(p - \frac{1}{2}q^2\right)B^{-1}\left(p - \frac{1}{2}q^2\right) + qAq\right\} dx. \end{aligned} \tag{2.4}$$

In this way, we obtain the canonical Hamiltonian form

$$\partial_x \mathcal{H}_p = q_t, \qquad \partial_x \mathcal{H}_q = p_t, \tag{2.5}$$

and the Hamiltonian system in the variable $V = \binom{q}{p}$ as

$$V_t = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \mathcal{H}'(V). \tag{2.6}$$

We observe that the Hamiltonian in (2.4) is formally conserved in time for solutions of system (2.2), since

$$\begin{aligned} \frac{d}{dt}\mathcal{H}(V) &= \int_0^{T_0} \{\mathcal{H}_q q_t + \mathcal{H}_p p_t\} dx \\ &= \int_0^{T_0} \{\mathcal{H}_q \partial_x \mathcal{H}_p + \mathcal{H}_p \partial_x \mathcal{H}_q\} dx = \int_0^{T_0} \partial_x\{\mathcal{H}_q \mathcal{H}_p\} dx. \end{aligned} \tag{2.7}$$

So, the Hamiltonian

$$\mathcal{H}\begin{pmatrix} q \\ r \end{pmatrix} = \frac{1}{2}\int_0^{T_0} \{rBr + qAq\} dx \tag{2.8}$$

associated to (1.10) is formally conserved in time. Moreover, since the Hamiltonian is translation-invariant, then by Noether's theorem there is an associated momentum functional $\mathcal{N}$ which is also conserved in time. This functional has the form

$$\mathcal{N}\begin{pmatrix} q \\ r \end{pmatrix} = \int_0^{T_0} \left(Br + \frac{1}{2}q^2\right) q\,dx. \tag{2.9}$$

Next we are interested in finding periodic travelling-waves solutions for system (1.10), in other words, solutions of the form $(q,r) = (\psi(x-ct), g(x-ct))$. By substituting, we have that the couple (ψ, g) satisfies the nonlinear system

$$g = -c\psi + A_0, \tag{2.10}$$

$$c^2(1 - b\partial_x^2)\psi = (1 - a\partial_x^2)\psi + \frac{3c}{2}\psi^2 - A_0\psi + \mathcal{A}, \tag{2.11}$$

with A_0 and $\mathcal{A}$ integration constants. Now, since our approach of stability is based on the context of the stability theory of Grillakis et al. (see proof of our Theorem 5.1), we need to show that (ψ, g) satisfies the equation

$$\delta\mathcal{F}\begin{pmatrix} \psi_c \\ g \end{pmatrix} = \begin{pmatrix} \mathcal{A} \\ 0 \end{pmatrix} \tag{2.12}$$

with

$$\mathcal{F} = \mathcal{H} + c\mathcal{N}, \tag{2.13}$$

therefore it follows from (2.10) that we must have $A_0 = 0$. In other words, we have to solve the system

$$g = -c\psi, \tag{2.14}$$

$$(1 - c^2)\psi + (bc^2 - a)\psi'' + \frac{3c}{2}\psi^2 = \mathcal{A}. \tag{2.15}$$

On the other hand, if we look for periodic travelling-wave solutions $\Phi(x,t) = \phi(x-ct)$ for (1.1), then $\eta \equiv \phi'$ has mean zero and satisfies equation

$$(1 - c^2)\eta + (bc^2 - a)\eta'' + \frac{3c}{2}\eta^2 = \mathcal{A}_1, \tag{2.16}$$

where $\mathcal{A}_1$ is an integration constant. Note that if η is a periodic solution with mean zero on $[0, L]$, then $\mathcal{A}_1 \neq 0$ and ϕ is periodic of period L. As a consequence of this, we have to look for periodic solutions ψ with mean zero for (2.15), and so $\mathcal{A} \neq 0$. This simple observation shows that $V_c = \binom{\psi_c}{-c\psi_c}$ cannot be a critical point of the action functional $\mathcal{F}$. This shows the need to adapt Grillakis et al.'s stability result to the present case (see Theorem 5.1). More precisely, we need in our stability theory to have $\mathcal{F}'(V_c)\vec{v} = \langle(\mathcal{A}, 0), \vec{v}\rangle = 0$, for $\vec{v} = (f, g)$. So, we need to have $f \in \mathcal{W}^1$.

3. Existence of a smooth curve of cnoidal waves with mean zero

In this section, we are interested in building explicit travelling-wave solutions for (1.1) and (1.10). Our analysis will show that the initial profile of ϕ_c can be taken as periodic or not, with a periodic derivative ψ_c of *cnoidal* form. Our main interest here will be the construction of a smooth curve $c \to \psi_c$ of periodic travelling-wave with a fixed fundamental period L and mean zero on $[0,L]$, so we will have that ϕ_c is periodic. More precisely, our main theorem is the following.

Theorem 3.1. *For every $T_0 > 0$, there are smooth curves*

$$c \in I = \left(-\sqrt{\min\left\{1,\frac{a}{b}\right\}},\sqrt{\min\left\{1,\frac{a}{b}\right\}}\right) \setminus \{0\} \longrightarrow \psi_c \in H^1_{\text{per}}([0,T_0]) \tag{3.1}$$

of solutions of the equation

$$(a-bc^2)\psi_c'' - (1-c^2)\psi_c - \frac{3c}{2}\psi_c^2 = A_{\psi_c}, \tag{3.2}$$

where each ψ_c has fundamental period T_0 and mean zero on $[0,T_0]$. Moreover, there are smooth curves $c \in I \to \beta_i(c)$, $i = 1,2,3$, such that

$$A_{\psi_c} = \frac{-3c}{2T_0}\int_0^{T_0}\psi_c^2(\xi)d\xi = \frac{1}{3c}\frac{1}{6}\sum_{i<j}\beta_i(c)\beta_j(c), \tag{3.3}$$

and ψ_c has the cnoidal form

$$\psi_c(x) = -\frac{1}{3c}\left[\beta_2 + (\beta_3-\beta_2)\,\text{cn}^2\left(\frac{1}{\sqrt{a-bc^2}}\sqrt{\frac{\beta_3-\beta_1}{12}}x;k\right)\right] \tag{3.4}$$

with $\beta_1 < \beta_2 < 0 < \beta_3$, $\beta_1+\beta_2+\beta_3 = 3(1-c^2)$ and $k^2 = (\beta_3-\beta_2)/(\beta_3-\beta_1)$.

The proof of Theorem 3.1 is based on the techniques developed by Angulo et al. in [3], so we use the implicit function theorem together with the theory of complete elliptic integrals and Jacobi elliptic functions. We divide the proof of Theorem 3.1 in several steps. The following two subsections will show the construction of cnoidal waves solutions with mean zero. Sections 3.3 and 3.4 will give the proof of the theorem. Section 3.5 gives a more careful study of the modulus function k.

3.1. Building periodic solution. One can see directly that travelling-waves solutions for (1.1), that is, solutions of the form $\Phi(x,t) = \phi(x-ct)$, have to satisfy the equation

$$(c^2-1)\phi'' + (a-bc^2)\phi^{(4)} - 3c\phi''\phi' = 0. \tag{3.5}$$

Integrating over $[0,x]$, we find that ϕ satisfies equation

$$(c^2-1)\phi' + (a-bc^2)\phi''' - \frac{3c}{2}(\phi')^2 = A_0, \tag{3.6}$$

and so $\psi \equiv \phi'$ satisfies equation

$$(c^2-1)\psi + (a-bc^2)\psi'' - \frac{3c}{2}\psi^2 = A_0, \tag{3.7}$$

where A_0 is an integration constant. Note that for periodic travelling-wave solution ϕ with a specific period L, we have that ψ has mean zero on $[0,L]$, therefore A_0 needs to be nonzero. Moreover, if ψ is a periodic solution with mean zero on $[0,L]$, then $A_0 \neq 0$ and ϕ is periodic of period L.

Next we scale function ψ. Defining

$$\varphi(x) = -\beta\psi(\theta x), \quad \text{with } \beta = 3c, \qquad \theta^2 = a - bc^2, \tag{3.8}$$

we have that φ satisfies the ordinary differential equation

$$\varphi'' + \frac{1}{2}\varphi^2 - (1-c^2)\varphi = A_\varphi \tag{3.9}$$

with $A_\varphi = -3cA_0$. For $0 < c^2 < 1$, a class of periodic solutions to (3.9) called *cnoidal waves* was found already in the 19th century work of Boussinesq [11, 12] and Korteweg and de Vries [13]. It may be written in terms of the Jacobi elliptic function as

$$\varphi_c(x) \equiv \varphi(x) = \beta_2 + (\beta_3 - \beta_2)\operatorname{cn}^2\left(\sqrt{\frac{\beta_3-\beta_1}{12}}x; k\right), \tag{3.10}$$

where

$$\beta_1 < \beta_2 < \beta_3, \qquad \beta_1 + \beta_2 + \beta_3 = 3(1-c^2), \qquad k^2 = \frac{\beta_3-\beta_2}{\beta_3-\beta_1}. \tag{3.11}$$

Here is a classical argument leading exactly to these formulas. Fix $c \in (-1,1)$ and multiply (3.9) by the integrating factor φ', a second exact integration is possible, yielding the first-order equation

$$3(\varphi')^2 = -\varphi^3 + 3(1-c^2)\varphi^2 + 6A_\varphi\varphi + 6B_\varphi, \tag{3.12}$$

where B_φ is another constant of integration. Suppose φ to be a nonconstant, smooth, periodic solution of (3.12). The formula (3.12) may be written as

$$[\varphi'(z)]^2 = \frac{1}{3}F_\varphi(\varphi(z)) \tag{3.13}$$

with $F_\varphi(t) = -t^3 + 3(1-c^2)t^2 + 6A_\varphi t + 6B_\varphi$ a cubic polynomial. If F_φ has only one real root β, say, then $\varphi'(z)$ can vanish only when $\varphi(z) = \beta$. This means that the maximum value of φ which takes on its period domain $[0,\widetilde{T}]$ is the same, with $\widetilde{T} = T/\theta$, as its minimum value there, and so φ is constant, contrary to presumption. Therefore F_φ must have three real roots, say $\beta_1 < \beta_2 < \beta_3$ (the degenerate cases will be considered presently). Note that for the existence of these different zeros, it is necessary to have that $(1-c^2)^2+2A_\varphi>0$. So, we have

$$F_\varphi(t)=(t-\beta_1)(t-\beta_2)(\beta_3-t), \tag{3.14}$$

where we have incorporated the minus sign into the third factor. Of course, we must have

$$\begin{gathered}\beta_1+\beta_2+\beta_3=3(1-c^2),\\ -\frac{1}{6}(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)=A_\varphi,\\ \frac{1}{6}\beta_1\beta_2\beta_3=B_\varphi.\end{gathered} \tag{3.15}$$

It follows immediately from (3.13)-(3.14) that φ must take values in the range $\beta_2\le\varphi\le\beta_3$. Normalize φ by letting $\rho=\varphi/\beta_3$, so that (3.13)-(3.14) become

$$(\rho')^2=\frac{\beta_3}{3}(\rho-\eta_1)(\rho-\eta_2)(1-\rho), \tag{3.16}$$

where $\eta_i=\beta_i/\beta_3,\ i=1,2$. The variable ρ lies in the interval $[\eta_2,1]$. By translation of the spatial coordinates, we may locate a maximum value of ρ at $x=0$. As the only critical points of ρ for values of ρ in $[\eta_2,1]$ are when $\rho=\eta_2<1$ and when $\rho=1$, it must be the case that $\rho(0)=1$. One checks that $\rho''>0$ when $\rho=\eta_2$ and $\rho''<0$ when $\rho=1$. Thus it is clear that our putative periodic solution must oscillate monotonically between the values $\rho=\eta_2$ and $\rho=1$. A simple analysis would now allow us to conclude that such periodic solutions exist, but we are pursuing the formula (3.10), not just existence.

Change variables again by letting

$$\rho=1+(\eta_2-1)\sin^2\varrho \tag{3.17}$$

with $\varrho(0)=0$ and ϱ continuous.

Substituting into (3.16) yields the equation

$$(\varrho')^2=\frac{\beta_3}{12}(1-\eta_1)\left[1-\frac{1-\eta_2}{1-\eta_1}\sin^2\varrho\right]. \tag{3.18}$$

To put this in standard form, define

$$k^2=\frac{1-\eta_2}{1-\eta_1},\qquad \ell=\frac{\beta_3}{12}(1-\eta_2). \tag{3.19}$$

Of course $0\le k^2\le1$ and $\ell>0$. We may solve for ϱ implicitly to obtain

$$F(\varrho;k)=\int_0^{\varrho(x)}\frac{dt}{\sqrt{1-k^2\sin^2 t}}=\sqrt{\ell}\,x. \tag{3.20}$$

The left-hand side of (3.20) is just the standard elliptic integral of the first kind (see [14]). Moreover, the elliptic function $\operatorname{sn}(z;k)$ is, for fixed k, defined in terms of the inverse of the mapping $\varrho \mapsto F(\varrho;k)$. Hence, (3.20) implies that

$$\sin\varrho = \operatorname{sn}(\sqrt{\ell}x;k), \tag{3.21}$$

and therefore

$$\rho = 1 + (\eta_2 - 1)\operatorname{sn}^2(\sqrt{\ell}x;k). \tag{3.22}$$

As $\operatorname{sn}^2 + \operatorname{cn}^2 = 1$, it transpires that $\rho = \eta_2 + (1-\eta_2)\operatorname{cn}^2(\sqrt{\ell}x;k)$, which, when properly unwrapped, is exactly the *cnoidal wave* solution (3.10), or ψ_c has the form

$$\psi_c(x) = -\frac{1}{3c}\left[\beta_2 + (\beta_3 - \beta_2)\operatorname{cn}^2\left(\frac{1}{\sqrt{a-bc^2}}\sqrt{\frac{\beta_3-\beta_1}{12}}x;k\right)\right]. \tag{3.23}$$

Next we consider the degenerate cases. First, fix the value of c and consider whether or not periodic solutions can persist if $\beta_1 = \beta_2$ or $\beta_2 = \beta_3$. As φ can only take values in the interval $[\beta_2, \beta_3]$, we conclude that the second case leads only to the constant solution $\varphi(x) \equiv \beta_2 = \beta_3$. Indeed, the limit of (3.10) as $\beta_2 \to \beta_3$ is uniform in x and is exactly this constant solution. If, on the other hand, c and β_1 are fixed, say, $\beta_2 \downarrow \beta_1$ and $\beta_3 = 3(1 - c^2) - \beta_2 - \beta_1$, then $k \to 1$, the elliptic function cn converges, uniformly on compact sets, to the hyperbolic function sech and (3.10) becomes, in this limit,

$$\varphi(x) = \varphi_\infty + \gamma\operatorname{sech}^2\left(\sqrt{\frac{\gamma}{12}}x\right) \tag{3.24}$$

with $\varphi_\infty = \beta_1$ and $\gamma = \beta_3 - \beta_1$. If $\beta_1 = 0$, we obtain

$$\varphi(x) = 3(1-c^2)\operatorname{sech}^2\left(\frac{\sqrt{1-c^2}}{2}x\right). \tag{3.25}$$

So, by returning to the original function ψ, we obtain the standard solitary-wave solution

$$\psi(x) = -\frac{1-c^2}{c}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{1-c^2}{a-bc^2}}x\right), \tag{3.26}$$

of speed $0 < c^2 < \min\{1, a/b\}$ of the Benney-Luke equation (see [9]).

Next, by returning to original variable ϕ_c, we obtain after integration and using the formula (see [14])

$$\int \operatorname{cn}^2(u;k)du = \frac{1}{k^2}\left[\int_0^u \operatorname{dn}^2(x;k)dx - (1-k^2)u\right] \tag{3.27}$$

that

$$\phi_c(x) = -\frac{\beta_1}{3c}x - \frac{\beta_3-\beta_2}{3cL_0k^2}\int_0^{L_0x}\operatorname{dn}^2(u;k)du + M, \tag{3.28}$$

where M is an integration constant and

$$L_0 = \frac{1}{\sqrt{a-bc^2}}\sqrt{\frac{\beta_3-\beta_1}{12}}. \tag{3.29}$$

3.2. Mean-zero property. Cnoidal waves φ_c having mean zero on their natural minimal period, T_c, for (3.9) are constructed here. The condition of zero mean, namely that

$$\int_0^{T_c} \varphi_c(\xi)d\xi = 0, \tag{3.30}$$

physically amounts to demanding that the wavetrain has the same mean depth as does the undisturbed free surface (this is a very good presumption for waves generated by an oscillating wavemaker in a channel, e.g., as no mass is added in such a configuration). Wavetrains with non-zero mean are readily derived from this special case as will be remarked presently.

Let a phase speed c_0 be given with $0 < c_0^2 < \min\{1, a/b\}$, and consider four constants β_1, β_2, β_3 and k as in (3.10). The complete elliptic integral of the first kind (see [1, Chapter 2], or [14]) is the function $K(k)$ defined by the formula

$$K \equiv K(k) \equiv \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{3.31}$$

The fundamental period of the cnoidal wave φ_{c_0} in (3.10) is $T_{c_0} = T_{\varphi_{c_0}}$,

$$T_{c_0} \equiv T_{c_0}(\beta_1,\beta_2,\beta_3) = \frac{4\sqrt{3}}{\sqrt{\beta_3-\beta_1}}K(k), \tag{3.32}$$

with K as in (3.31). The period of cn is $4K(k)$ and cn is antisymmetric about its half period, from which (3.32) follows.

The condition of mean zero of φ_{c_0} over a period $[0, T_{c_0}]$ is easily determined to be

$$0 = \beta_2 + (\beta_3-\beta_2)\frac{1}{2K}\int_0^{2K} \mathrm{cn}^2(\xi;k)d\xi. \tag{3.33}$$

Simple manipulations with elliptic functions put (3.33) into a more useful form, namely

$$\int_0^{2K} \mathrm{cn}^2(\xi;k)d\xi = 2\int_0^{K} \mathrm{cn}^2(u;k)du = \frac{2}{k^2}\left[E(k) - k'^2K(k)\right], \tag{3.34}$$

where $k' = (1-k^2)^{1/2}$ and $E(k)$ is the complete elliptical integral of the second kind defined by the formula

$$E \equiv E(k) \equiv \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}dt. \tag{3.35}$$

Thus the zero-mean value condition is exactly

$$\beta_2 + (\beta_3-\beta_2)\frac{E(k) - k'^2K(k)}{k^2K(k)} = 0. \tag{3.36}$$

Because $(\beta_3-\beta_2)k'^2=(\beta_2-\beta_1)k^2$ and

$$\frac{dK(k)}{dk}=\frac{E(k)-k'^2K(k)}{kk'^2} \tag{3.37}$$

(see [14]), the relation (3.36) has the equivalent form

$$\beta_1K(k)+(\beta_3-\beta_1)E(k)=0, \tag{3.38}$$

$$\frac{dK}{dk}=-\frac{\beta_2}{\beta_3-\beta_2}\frac{k}{k'^2}K. \tag{3.39}$$

We note that by replacing $K(k)$ and $E(k)$, we have that (3.38) is equivalent to have $A(\beta_2,\beta_3)=0$, where

$$A(\beta_2,\beta_3)=\int_0^1\frac{1}{\sqrt{1-t^2}}\frac{\beta_3-(\beta_3-\beta_2)t^2}{\sqrt{2\beta_3+\beta_2-\alpha_0-(\beta_3-\beta_2)t^2}}dt \tag{3.40}$$

with $\beta_1+\beta_2+\beta_3=\alpha_0$, $\alpha_0=3(1-c_0^2)$. Now we are in a good position to prove that under some consideration, φ_{c_0} has mean zero.

Theorem 3.2. *Let $\alpha_0=3(1-c_0^2)$. Then for $\beta_3>\alpha_0$ fixed, there are numbers $\beta_1<\beta_2<0<\beta_3$ satisfying that $\beta_1+\beta_2+\beta_3=\alpha_0$ and the cnoidal wave defined in (3.10), $\varphi_{c_0}=\varphi(\cdot,\beta_1,\beta_2,\beta_3)$ has mean zero in $[0,T_{c_0}]$. Moreover,*

(1) *the map $\beta_2:(\alpha_0,\infty)\to((\alpha_0-\beta_3)/2,0)$, $\beta_3\to\beta_2(\beta_3)$ is continuous,*
(2) *$\lim_{\beta_3\to\alpha_0^+}T_{c_0}=\infty$, and $\lim_{\beta_3\to\infty}T_{c_0}=0$.*

Proof. Let $\beta_3>\alpha_0$ and note that for $t\in[0,1]$ and $(\alpha_0-\beta_3)/2<s<0$,

$$2\beta_3+s-\alpha_0-(\beta_3-s)t^2\geq\beta_3+2s-\alpha_0>0. \tag{3.41}$$

In other words, $A(s,\beta_3)$ is well defined for $s\in I=((\alpha_0-\beta_3)/2,0)$. We observe that $A(0,\beta_3)>0$ and a straightforward computation shows that

$$\lim_{s\to(\alpha_0-\beta_3)/2}A(s,\beta_3)=-\infty. \tag{3.42}$$

In fact, for $s=(\alpha_0-\beta_3)/2$, we have that

$$\frac{\beta_3-(\beta_3-s)t^2}{\sqrt{1-t^2}\sqrt{2\beta_3+s-\alpha_0-(\beta_3-s)t^2}}=\frac{\sqrt{2}\beta_3}{\sqrt{3\beta_3-\alpha_0}}-\frac{\sqrt{2}(\beta_3-\alpha_0)}{2\sqrt{3\beta_3-\alpha_0}}\left(\frac{t^2}{1-t^2}\right). \tag{3.43}$$

Moreover, from (see [1, Theorem 5.6]), we have that $\partial_sA(s,\beta_3)>0$ with $s\in((\alpha_0-\beta_3)/2,0)$. Then we can conclude that there exists a unique $s_0\in((\alpha_0-\beta_3)/2,0)$ such that $A(s_0,\beta_3)=0$.

The continuity of the map $\beta_2:(\alpha_0,\infty)\to((\alpha_0-\beta_3)/2,0)$, $\beta_3\to\beta_2=\beta_2(\beta_3)$ follows by the implicit function theorem applied to the function $A(s,\beta_3)$.

Now if the fundamental period T_{c_0} of φ_{c_0} is regarded as function of the parameter β_3, then for $\beta_2 = \beta_2(\beta_3)$, we have

$$T_{c_0}(\beta_3) = \frac{4\sqrt{3}}{\sqrt{2\beta_3 + \beta_2 - \alpha_0}} K(k), \quad k^2 = \sqrt{\frac{\beta_3 - \beta_2}{2\beta_3 + \beta_2 - \alpha_0}}. \tag{3.44}$$

Since $K(1) = +\infty$ and $2\beta_3 + \beta_2 - \alpha_0 \to \alpha_0$ as $\beta_3 \to \alpha_0$, we conclude that

$$\lim_{\beta_3 \to \alpha_0^+} T_{c_0}(\beta_3) = +\infty. \tag{3.45}$$

On the other hand, from the fact that E is a decreasing function in k with $E(k) \le E(0) = \pi/2$ and (3.38), we have that

$$K(k) = -\left(\frac{\beta_3 - \beta_1}{\beta_1}\right) E(k) = \left(\frac{2\beta_3 + \beta_2 - \alpha_0}{\beta_3 + \beta_2 - \alpha_0}\right) E(k) \le \frac{\pi}{2}\left(\frac{2\beta_3 + \beta_2 - \alpha_0}{\beta_3 + \beta_2 - \alpha_0}\right). \tag{3.46}$$

Using that $-(\beta_3 - \alpha_0)/2 \le \beta_2 < 0$, we obtain that

$$0 \le T_{c_0}(\beta_3) = \frac{4\sqrt{3}}{\sqrt{2\beta_3 + \beta_2 - \alpha_0}} K(k) \le 2\pi\sqrt{3}\left(\frac{\sqrt{2\beta_3 + \beta_2 - \alpha_0}}{\beta_3 + \beta_2 - \alpha_0}\right) \le 4\pi\sqrt{3}\frac{\sqrt{2\beta_3 - \alpha_0}}{\beta_3 - \alpha_0}. \tag{3.47}$$

So, we conclude that

$$\lim_{\beta_3 \to \infty} T_{c_0}(\beta_3) = 0. \tag{3.48}$$

□

3.3. Fundamental period. The first step to establish the existence of a curve of periodic wave solutions to the Benney-Luke equation with a given period is based on proving the existence of an interval of speed waves for cnoidal waves φ_c in (3.10).

Lemma 3.3. *Let c_0 be a fixed number with $0 < c_0^2 < \min\{1, a/b\}$, consider $\beta_1 < \beta_2 < 0 < \beta_3$ satisfying Theorem 3.2 and $\varphi_{c_0} = \varphi_{c_0}(\cdot, \beta_1, \beta_2, \beta_3)$ with mean zero over $[0, T_{c_0}]$. Define*

$$\lambda(c) = \sqrt{\frac{(a - bc_0^2)(1 - c^2)}{(a - bc^2)(1 - c_0^2)}}, \tag{3.49}$$

with c such that $0 < c^2 < \min\{1, a/b\}$. Then

(1) *there exist an interval $I(c_0)$ around c_0, a ball $B(\vec{\beta})$ around $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$, and a unique smooth function*

$$\begin{aligned} \Pi : I(c_0) &\longrightarrow B(\vec{\beta}), \\ c &\longrightarrow (\alpha_1(c), \alpha_2(c), \alpha_3(c)) \end{aligned} \tag{3.50}$$

such that $\Pi(c_0) = (\beta_1, \beta_2, \beta_3)$ *and* $\alpha_i \equiv \alpha_i(c)$ *with* $\alpha_1 < \alpha_2 < 0 < \alpha_3$ *satisfying*

$$\begin{gathered} \frac{4\sqrt{3}}{\sqrt{\alpha_3 - \alpha_1}} K(k) = \lambda(c) T_{c_0}, \\ \alpha_1 + \alpha_2 + \alpha_3 = \alpha_0, \\ \alpha_1 K(k) + (\alpha_3 - \alpha_1) E(k) = 0, \end{gathered} \tag{3.51}$$

where $k^2(c) = (\alpha_3(c) - \alpha_2(c))/(\alpha_3(c) - \alpha_1(c))$;

(2) *the cnoidal wave* $\varphi_{c_0}(\cdot, \alpha_1(c), \alpha_2(c), \alpha_3(c))$ *has fundamental period* $T_c = \lambda(c) T_{c_0}$, *mean zero over* $[0, T_c]$, *and satisfies the equation*

$$\varphi_{c_0}'' + \frac{1}{2}\varphi_{c_0}^2 - (1 - c_0^2)\varphi_{c_0} = A_{\varphi_{c_0}(\cdot, \alpha_i(c))}, \tag{3.52}$$

where

$$A_{\varphi_{c_0}(\cdot, \alpha_i(c))} = \frac{1}{2T_c}\int_0^{T_c} \varphi_{c_0}^2(x, \alpha_i(c))\, dx = -\frac{1}{6}\sum_{i<j} \alpha_i(c)\alpha_j(c), \tag{3.53}$$

for all $c \in I(c_0)$.

Proof. We proceed as by Angulo et al. in (see [3]). Let $\Omega \subset \mathbb{R}^4$ be the set defined by

$$\Omega = \left\{ (\alpha_1, \alpha_2, \alpha_3, c) : \alpha_1 < \alpha_2 < 0 < \alpha_3,\ \alpha_3 > \alpha_0,\ 0 < c^2 < \min\left\{1, \frac{a}{b}\right\} \right\}, \tag{3.54}$$

let $k^2 \equiv (\alpha_3 - \alpha_2)/(\alpha_3 - \alpha_1)$, and let $\Phi : \Omega \to \mathbb{R}^3$ be the function defined by

$$\Phi(\alpha_1, \alpha_2, \alpha_3, c) = (\Phi_1(\alpha_1, \alpha_2, \alpha_3, c),\ \Phi_2(\alpha_1, \alpha_2, \alpha_3, c),\ \Phi_3(\alpha_1, \alpha_2, \alpha_3, c)), \tag{3.55}$$

where

$$\begin{aligned} \Phi_1(\alpha_1, \alpha_2, \alpha_3, c) &= \frac{4\sqrt{3}}{\sqrt{\alpha_3 - \alpha_1}} K(k) - \lambda(c) T_{c_0}, \\ \Phi_2(\alpha_1, \alpha_2, \alpha_3, c) &= \alpha_1 + \alpha_2 + \alpha_3 - \alpha_0, \\ \Phi_3(\alpha_1, \alpha_2, \alpha_3, c) &= \alpha_1 K(k) + (\alpha_3 - \alpha_1) E(k). \end{aligned} \tag{3.56}$$

From Theorem 3.2, $\Phi(\beta_1, \beta_2, \beta_3, c_0) = 0$. The first observation is that

$$\nabla_{(\alpha_1, \alpha_2, \alpha_3)} \Phi_2(\vec{\alpha}, c) = (1, 1, 1). \tag{3.57}$$

On the other hand, a direct computation shows that

$$\begin{aligned}
\partial_{\alpha_1}\Phi_1(\vec{\alpha},c) &= \frac{2\sqrt{3}K(k)}{(\alpha_3-\alpha_1)^{3/2}} + \frac{4\sqrt{3}\partial_{\alpha_1}K(k)}{(\alpha_3-\alpha_1)^{1/2}},\\
\partial_{\alpha_2}\Phi_1(\vec{\alpha},c) &= \frac{4\sqrt{3}\partial_{\alpha_2}K(k)}{(\alpha_3-\alpha_1)^{1/2}},\\
\partial_{\alpha_3}\Phi_1(\vec{\alpha},c) &= -\frac{2\sqrt{3}K(k)}{(\alpha_3-\alpha_1)^{3/2}} + \frac{4\sqrt{3}\partial_{\alpha_3}K(k)}{(\alpha_3-\alpha_1)^{1/2}},
\end{aligned} \tag{3.58}$$

and that

$$\partial_{\alpha_1}k = \frac{\alpha_3-\alpha_2}{2k(\alpha_3-\alpha_1)^2}, \qquad \partial_{\alpha_2}k = \frac{-1}{2k(\alpha_3-\alpha_1)}, \qquad \partial_{\alpha_3}k = \frac{\alpha_2-\alpha_1}{2k(\alpha_3-\alpha_1)^2}. \tag{3.59}$$

If we assume that $\alpha_1K(k)+(\alpha_3-\alpha_1)E(k)=0$, then using that

$$K'(k) = \frac{E(k)-k'^2K(k)}{kk'^2}, \qquad E'(k) = \frac{E-K}{k}, \tag{3.60}$$

we obtain the following formulas:

$$\begin{gathered}
\partial_{\alpha_1}K(k) = -\frac{\alpha_2K(k)}{2(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)}, \qquad \partial_{\alpha_2}K(k) = \frac{\alpha_2K(k)}{2(\alpha_2-\alpha_1)(\alpha_3-\alpha_2)},\\
\partial_{\alpha_3}K(k) = -\frac{\alpha_2K(k)}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}.
\end{gathered} \tag{3.61}$$

Similarly, under such assumptions, we conclude that

$$\begin{gathered}
\partial_{\alpha_1}E(k) = -\frac{\alpha_3K(k)}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_1)}, \qquad \partial_{\alpha_2}E(k) = \frac{\alpha_3K(k)}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},\\
\partial_{\alpha_3}E(k) = -\frac{\alpha_3(\alpha_2-\alpha_1)K(k)}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}.
\end{gathered} \tag{3.62}$$

Replacing these in previous equalities, we have, for $(\vec{\alpha},c)$ satisfying

$$\frac{4\sqrt{3}}{\sqrt{\alpha_3-\alpha_1}}K(k) - \lambda(c)T_{c_0} = 0, \tag{3.63}$$

that

$$\begin{aligned}
\partial_{\alpha_1}\Phi_1(\vec{\alpha},c) &= \frac{4\sqrt{3}K(k)}{(\alpha_3-\alpha_1)^{1/2}}\left(\frac{-\alpha_1}{2(\alpha_3-\alpha_1)(\alpha_2-\alpha_1)}\right) = \frac{-\alpha_1\lambda(c)T_{c_0}}{2(\alpha_3-\alpha_1)(\alpha_2-\alpha_1)},\\
\partial_{\alpha_2}\Phi_1(\vec{\alpha},c) &= \frac{4\sqrt{3}K(k)}{(\alpha_3-\alpha_1)^{1/2}}\left(\frac{\alpha_2}{2(\alpha_2-\alpha_1)(\alpha_3-\alpha_2)}\right) = \frac{\alpha_2\lambda(c)T_{c_0}}{2(\alpha_2-\alpha_1)(\alpha_3-\alpha_2)},\\
\partial_{\alpha_3}\Phi_1(\vec{\alpha},c) &= \frac{4\sqrt{3}K(k)}{(\alpha_3-\alpha_1)^{1/2}}\left(\frac{-\alpha_3}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)}\right) = \frac{-\alpha_3\lambda(c)T_{c_0}}{2(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},
\end{aligned} \tag{3.64}$$

and that

$$\begin{aligned}&\nabla_{(\alpha_1,\alpha_2,\alpha_3)}\Phi_3(\vec{\alpha},c_0)\\&\quad=K(k)\left(\frac{\alpha_2\alpha_3-\alpha_1\alpha_2-\alpha_1\alpha_3}{2(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)},\frac{\alpha_2\alpha_3-\alpha_1\alpha_3+\alpha_1\alpha_2}{2(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)},\frac{\alpha_2\alpha_1-\alpha_1\alpha_3-\alpha_2\alpha_3}{2(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}\right).\end{aligned}\tag{3.65}$$

In particular, for $(\vec{\beta},c_0)$, we obtain that $\lambda(c_0)=1$, so it follows that

$$\begin{aligned}\partial_{\alpha_1}\Phi_1(\vec{\beta},c_0)&=\frac{-\beta_1T_{c_0}}{2(\beta_3-\beta_1)(\beta_2-\beta_1)},\\\partial_{\alpha_2}\Phi_1(\vec{\beta},c_0)&=\frac{\beta_2T_{c_0}}{2(\beta_2-\beta_1)(\beta_3-\beta_2)},\\\partial_{\alpha_3}\Phi_1(\vec{\beta},c_0)&=\frac{-\beta_3T_{c_0}}{2(\beta_3-\beta_1)(\beta_3-\beta_2)},\end{aligned}\tag{3.66}$$

and that

$$\begin{aligned}&\nabla_{(\alpha_1,\alpha_2,\alpha_3)}\Phi_3(\vec{\beta},c_0)\\&\quad=K(k_1)\left(\frac{\beta_2\beta_3-\beta_1\beta_2-\beta_1\beta_3}{2(\beta_2-\beta_1)(\beta_3-\beta_1)},\frac{\beta_2\beta_3-\beta_1\beta_3+\beta_1\beta_2}{2(\beta_3-\beta_2)(\beta_2-\beta_1)},\frac{\beta_2\beta_1-\beta_1\beta_3-\beta_2\beta_3}{2(\beta_3-\beta_2)(\beta_3-\beta_1)}\right),\end{aligned}\tag{3.67}$$

with $k_1^2=(\beta_3-\beta_2)/(\beta_3-\beta_1)$. The properties of the cnoidal wave $\varphi_{c_0}(\cdot,\vec{\beta})$ lead to

$$\frac{1}{2T_{c_0}}\int_0^{T_{c_0}}\varphi_{c_0}^2(\xi,\vec{\beta})d\xi=A_{\varphi_{c_0}(\cdot,\vec{\beta})}.\tag{3.68}$$

Using previous calculation, the Jacobian determinant of $\Phi(\cdot,\cdot,\cdot,c)$ at $(\vec{\beta},c_0)$ is given by

$$\begin{aligned}\det\begin{pmatrix}\nabla_\alpha\Phi_1\\\nabla_\alpha\Phi_2\\\nabla_\alpha\Phi_3\end{pmatrix}_{(\vec{\beta},c_0)}&=\frac{T_{c_0}K(k)(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)}{4(\beta_2-\beta_1)(\beta_3-\beta_2)(\beta_3-\beta_1)}\\&=\frac{-3K(k)\int_0^{T_{c_0}}\varphi_{c_0}^2(\xi,\vec{\beta})d\xi}{4(\beta_2-\beta_1)(\beta_3-\beta_2)(\beta_3-\beta_1)}\neq0.\end{aligned}\tag{3.69}$$

As a consequence of this, the implicit function theorem implies the existence of a function Π from a neighborhood of $I(c_0)$ of c_0 to a neighborhood of $(\beta_1,\beta_2,\beta_3)$ satisfying the first part of the lemma. The second part of the lemma is immediate. □

3.4. Existence of curve of solutions. In this subsection as a consequence of the previous results, we establish the proof of Theorem 3.1.

Proof of Theorem 3.1. We start by proving the existence of a smooth curve of cnoidal waves solutions for (3.7) with a fixed period $\sqrt{a-bc_0^2}T_{c_0}$ and with mean zero on $[0,\sqrt{a-bc_0^2}T_{c_0}]$. In fact, let $\varphi_{c_0}(\cdot,\alpha_i)$ be the cnoidal wave determined by Lemma 3.3 with $\alpha_i=\alpha_i(c)$ for $c\in I(c_0)$. Define

$$\varphi_c(\zeta,\alpha_i)\equiv\frac{1-c^2}{1-c_0^2}\varphi_{c_0}\left(\sqrt{\frac{1-c^2}{1-c_0^2}}\zeta,\alpha_i\right),\quad \zeta\in\mathbb{R}. \tag{3.70}$$

Then $\varphi_c(\cdot,\alpha_i)$ has period $T_{\varphi_c}\equiv\theta(c)T_{c_0}$, with $\theta(c)=\sqrt{a-bc_0^2}/\sqrt{a-bc^2}$, and mean zero on $[0,T_{\varphi_c}]$. Moreover, it is not hard to see that $\varphi_c(\cdot,\alpha_i)$ satisfies the differential equation

$$\varphi_c''+\frac{1}{2}\varphi_c^2-(1-c^2)\varphi_c=A_{\varphi_c}, \tag{3.71}$$

where

$$\begin{aligned}A_{\varphi_c}&=\frac{1}{2T_{\varphi_c}}\int_0^{T_{\varphi_c}}\varphi_c^2(\xi,\alpha_i(c))\,d\xi\\&=\left(\frac{1-c^2}{1-c_0^2}\right)^2A_{\varphi_{c_0}(\cdot,\alpha_i)}=-\frac{1}{6}\left(\frac{1-c^2}{1-c_0^2}\right)^2\sum_{i<j}\alpha_i(c)\alpha_j(c).\end{aligned} \tag{3.72}$$

Next, we obtain a smooth curve of solution for (3.7). From (3.8), define

$$\psi_c(x)\equiv-\frac{1}{3c}\varphi_c\left(\frac{x}{\sqrt{a-bc^2}},\alpha_i(c)\right). \tag{3.73}$$

Then it is easy to see that ψ_c has period $T_0=\sqrt{a-bc_0^2}T_{c_0}$ and mean zero on $[0,T_0]$. On the other hand, from (3.71) and (3.72), it follows that ψ_c satisfies the differential equation (3.2) with A_{ψ_c} given by (3.3).

Now, the regularity of the map $c\to\psi_c$ follows from the properties of φ_{c_0} and α_i. Moreover, from Theorem 3.2, we obtain that $T_{c_0}=T_{c_0}(\beta_3)$ can be taken arbitrarily in the interval $(0,+\infty)$, and so the solution ψ_c can be taken with an arbitrary period T_0 with $T_0\in(0,+\infty)$. Finally, by uniqueness of the map Π in Lemma 3.3 and by c_0 being arbitrary with $0<c_0^2<\min\{1,a/b\}$, we can conclude that the map Π can be extended such that we obtain the following smooth curves of cnoidal wave solutions to (3.7):

$$c\in\left(-\sqrt{\min\left\{1,\frac{a}{b}\right\}},\sqrt{\min\left\{1,\frac{a}{b}\right\}}\right)\setminus\{0\}\longrightarrow\psi_c\in H^1_{\mathrm{per}}([0,T_{c_0}]), \tag{3.74}$$

with an arbitrary period T_{c_0} and mean zero on $[0,T_{c_0}]$. This finishes the proof of Theorem 3.1. □

3.5. Monotonicity of the modulus k. In this subsection, we show some properties of the modulus k determined in Lemma 3.3. We start by recalling that for every $c\in I(c_0)$,

$\Phi(\Pi(c),c) = (0,0,0)$, where $\Pi(c) = (\alpha_1(c),\alpha_2(c),\alpha_3(c))$. As done above, from formulas (3.57), (3.64), and (3.65) it is not hard to see that

$$\det \begin{pmatrix} \nabla_\alpha \Phi_1 \\ \nabla_\alpha \Phi_2 \\ \nabla_\alpha \Phi_3 \end{pmatrix}_{(\vec{\alpha},c)} = \frac{\lambda(c) T_{c_0} K(k)(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)}{4(\alpha_2-\alpha_1)(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)}. \tag{3.75}$$

As a consequence of this, we conclude that

$$\frac{d}{dc}\Pi(c) = -\begin{pmatrix} \nabla_\alpha \Phi_1 \\ \nabla_\alpha \Phi_2 \\ \nabla_\alpha \Phi_3 \end{pmatrix}^{-1} \begin{pmatrix} -\lambda'(c) T_{c_0} \\ 0 \\ 0 \end{pmatrix}, \tag{3.76}$$

with $\alpha = (\alpha_1,\alpha_2,\alpha_3)$. Next by finding the inverse matrix in (3.76), we obtain that

$$\frac{d}{dc}\Pi(c) = \frac{2\lambda'(c)}{\lambda(c)\Sigma\alpha_i\alpha_j} \begin{pmatrix} \alpha_3^2(\alpha_1-\alpha_2)+\alpha_2^2(\alpha_1-\alpha_3) \\ \alpha_1^2(\alpha_2-\alpha_3)+\alpha_3^2(\alpha_2-\alpha_1) \\ \alpha_1^2(\alpha_3-\alpha_2)+\alpha_2^2(\alpha_3-\alpha_1) \end{pmatrix}. \tag{3.77}$$

Using this fact, we are able to establish that k is a monotone function, depending on the wave speed.

Theorem 3.4. *Consider c with $0 < c^2 < \min\{1, a/b\}$. Define the modulus function*

$$k(c) = \sqrt{\frac{\alpha_3(c)-\alpha_2(c)}{\alpha_3(c)-\alpha_1(c)}}. \tag{3.78}$$

Then,

(1) *for* $0 < |c| < \sqrt{a/b} < 1$, $\Rightarrow c(d/dc)k(c) > 0$,
(2) *for* $0 < |c| < 1 < \sqrt{a/b}$, $\Rightarrow c(d/dc)k(c) < 0$.

Proof. Denoting $A(c) = \Sigma\alpha_i\alpha_j$ and $B(c) = 2(\alpha_3-\alpha_1)^{3/2}\sqrt{\alpha_3-\alpha_2}$, we obtain from (3.77) and Lemma 3.3 that

$$\begin{aligned} \frac{d}{dc}k(c) &= \frac{1}{B(c)}\left[(\alpha_3-\alpha_2)\frac{d\alpha_1(c)}{dc} - (\alpha_3-\alpha_1)\frac{d\alpha_2(c)}{dc} + (\alpha_2-\alpha_1)\frac{d\alpha_3(c)}{dc}\right] \\ &= -\frac{4\lambda'(c)}{\lambda(c)A(c)B(c)}(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\Sigma\alpha_i \\ &= -\frac{6(1-c_0^2)(\alpha_2-\alpha_1)}{\lambda(c)A(c)}\lambda'(c)k(c). \end{aligned} \tag{3.79}$$

Next, since $A(c) < 0$ and

$$\lambda'(c) = \sqrt{\frac{a-bc_0^2}{1-c_0^2}} \frac{c(b-a)}{(a-bc^2)^{3/2}\sqrt{1-c^2}}, \tag{3.80}$$

we obtain immediately our theorem. □

4. Spectral analysis and convexity

In this section, attention is turned to set the main tools to be used in order to establish stability of the cnoidal-wave solutions $(\psi_c, -c\psi_c)$ determined by Theorem 3.1 for system (1.10).

4.1. Spectral analysis of the operator $\mathcal{L}_{cn} = -(a - bc^2)(d^2/dx^2) + (1 - c^2) + 3c\psi_c$**.** By Theorem 3.1, we consider for $L = T_0 > 0$ the smooth curve of cnoidal wave $c \to \psi_c \in H^1_{per}([0,L])$ with fundamental period L. As it is well known, the study of the periodic eigenvalue problem for the linear operator $\mathcal{L}_{cn}$ considered on $[0,L]$ is required in the stability theory. The spectral problem in question is

$$\mathcal{L}_{cn}\chi \equiv \left[-(a - bc^2)\frac{d^2}{dx^2} + (1 - c^2) + 3c\psi_c \right]\chi = \lambda\chi, \qquad \chi(0) = \chi(L), \quad \chi'(0) = \chi'(L), \tag{4.1}$$

where c is fixed such that $0 < c^2 < \min\{1, a/b\}$. The following result is obtained in this context.

Theorem 4.1. *Let ψ_c be the cnoidal wave solution given by Theorem 3.1. Then the linear operator*

$$\mathcal{L}_{cn} = -(a - bc^2)\frac{d^2}{dx^2} + (1 - c^2) + 3c\psi_c \tag{4.2}$$

defined on $H^2_{per}([0,L])$ has exactly its three first eigenvalues $\lambda_0 < \lambda_1 < \lambda_2$ simple. Moreover, ψ_c' is an eigenfunction with eigenvalue $\lambda_1 = 0$. The rest of the spectrum is a discrete set of eigenvalues which is double. The eigenvalues only accumulate at $+\infty$.

Proof. From the theory of compact symmetric operators applied to the periodic eigenvalue problem (4.1), it is known that the spectrum of $\mathcal{L}_{cn}$ is a countable infinity set of eigenvalues with

$$\lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots, \tag{4.3}$$

where double eigenvalue is counted twice and $\lambda_n \to \infty$ as $n \to \infty$. Now, from the Floquet theory [15], with the eigenvalue periodic problem, there is an associated eigenvalue problem, named semiperiodic problem in $[0,L]$,

$$\mathcal{L}_{cn}\psi = \mu\psi, \qquad \psi(0) = -\psi(L), \quad \psi'(0) = -\psi'(L). \tag{4.4}$$

As in the periodic case, there is a sequence of eigenvalues

$$\mu_0 \le \mu_1 \le \mu_2 \le \mu_3 \le \cdots, \tag{4.5}$$

where double eigenvalue is counted twice and $\mu_n \to \infty$ as $n \to \infty$. So, for the equation

$$\mathcal{L}_{cn} f = \gamma f, \tag{4.6}$$

we have that the only periodic solutions, f, of period L correspond to $\gamma = \lambda_j$ for some j whilst the only periodic solutions of period $2L$ are either those associated with $\gamma = \lambda_j$, but viewed on $[0,2L]$, or those corresponding $\gamma = \mu_j$, but extended as $f(L+x) = f(L-x)$ for $0 \le x \le L$.

Next, from oscillation theory [15], we have that the sequences of eigenvalues (4.3) and (4.5) have the following property:

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \le \cdots. \tag{4.7}$$

Now, for a given value γ, if all solutions of (4.6) are bounded, then γ is called a *stable* value, whereas if there is an unbounded solution, γ is called *unstable*. The open intervals $(\lambda_0,\mu_0),(\mu_1,\lambda_1),(\lambda_2,\mu_2),(\mu_3,\lambda_3),\ldots$ are called *intervals of stability*. The endpoints of these intervals are generally unstable. This is always so for $\gamma = \lambda_0$ as λ_0 is always simple. The intervals $(-\infty,\lambda_0),(\mu_0,\mu_1),(\lambda_1,\lambda_2),(\mu_2,\mu_3),\ldots$, and so on are called *intervals of instability*. Of course, at a double eigenvalue, the interval is empty and omitted from the discussion. Absence of an interval of instability means that there is a value of γ for which all solutions of (4.6) are either periodic of period L or periodic with basic period $2L$.

We also have the following characterization of the zeros of the eigenfunctions associated with the problems (4.1) and (4.4), χ_n and ψ_n, respectively,

(1) χ_0 has no zeros in $[0,L]$,

(2) χ_{2n+1} and χ_{2n+2} have exactly $2n+2$ zeros in $[0,L)$,

(3) ψ_{2n+1} and ψ_{2n+2} have exactly $2n+1$ zeros in $[0,L)$.

Using this fact and the relationship between the sequence of eigenvalues (4.7) and $\mathcal{L}_{\text{cn}}\psi_c' = 0$, we conclude that $\lambda_0 < \lambda_1 \le \lambda_2$ with $\lambda_1 = 0$ or $\lambda_2 = 0$.

We will show that $\lambda_2 > \lambda_1 = 0$. First define the transformation $R_\eta(x) = \chi(\eta x)$, where $\eta^2 = 12(a-bc^2)/(\beta_3-\beta_1)$. It is not hard to see using the explicit form for ψ_c in (3.4) that the problem (4.1) is equivalent to the eigenvalue problem for $\Lambda \equiv R_\eta$,

$$\begin{gathered} \frac{d^2}{dx^2}\Lambda + [\gamma - 12k^2 \operatorname{sn}^2(x;k)]\Lambda = 0, \\ \Lambda(0) = \Lambda(2K), \qquad \Lambda'(0) = \Lambda'(2K), \end{gathered} \tag{4.8}$$

where $K = K(k)$ is defined by (3.31) and

$$\gamma = -\frac{12}{\beta_3-\beta_1}[1-c^2-\beta_3-\lambda]. \tag{4.9}$$

The differential equation in (4.8) is called the *Jacobian form of Lame's equation*. Now, from [15, 16], we obtain that Lame's equation has four intervals of instability which are

$$(-\infty,\gamma_0),(\mu_0',\mu_1'),(\gamma_1,\gamma_2),(\mu_2',\mu_3'), \tag{4.10}$$

where $\mu_i' \ge 0$ are the eigenvalues associated to the semiperiodic problem determined by Lame's equation (see [1, 15]). So, we have that the first three eigenvalues γ_0, γ_1, γ_2 associated with (4.8) are simple and the rest of eigenvalues are double, namely, $\gamma_3 = \gamma_4$, $\gamma_5 = \gamma_6,\ldots$.

We present the first three eigenvalues γ_0, γ_1, γ_2 and their corresponding eigenfunctions Λ_0, Λ_1, Λ_2. Since $\gamma_1 = 4+4k^2$ is an eigenvalue of (4.8) with eigenfunction $\Lambda_1(x) = \mathrm{cn}(x)\,\mathrm{sn}(x)\,\mathrm{dn}(x) = CR_\eta(\psi_c'(x))$, it follows from (4.9) that $\lambda = 0$ is a simple eigenvalue of problem (4.1) with eigenfunction ψ_c'. It was shown by Ince (see [16]) that the eigenfunctions of (4.8) Λ_0 and Λ_2 have the forms

$$\begin{aligned}\Lambda_0(x) &= \left[1-\left(1+2k^2-\sqrt{1-k^2+4k^4}\right)\mathrm{sn}^2(x)\right]\mathrm{dn}(x),\\ \Lambda_2(x) &= \left[1-\left(1+2k^2+\sqrt{1-k^2+4k^4}\right)\mathrm{sn}^2(x)\right]\mathrm{dn}(x).\end{aligned} \tag{4.11}$$

In this case, the associated eigenvalues γ_0 and γ_2 have to satisfy the equation

$$\gamma = k^2 + \frac{5k^2}{1+(9/4)k^2-(1/4)\gamma}, \tag{4.12}$$

which has two roots

$$\gamma_0 = 2+5k^2-2\sqrt{1-k^2+4k^4}, \qquad \gamma_2 = 2+5k^2+2\sqrt{1-k^2+4k^4}. \tag{4.13}$$

Now note that Λ_0 has no zeros in $[0,2K]$ and Λ_2 has exactly 2 zeros in $[0,2K]$, then Λ_0 corresponds to eigenfunction associated to γ_0, which must be the first eigenvalue of (4.8). On the other hand, $(\beta_3-\beta_1)k^2 = \beta_3-\beta_2$ and $\beta_1+\beta_2+\beta_3 = 3(1-c^2)$, then

$$-\beta_1(k^2+1) = (2-k^2)\beta_3 - 3(1-c^2), \qquad \lambda_0 = \left[1-\frac{1}{4(1+k^2)}\gamma_0\right](1-c^2-\beta_3). \tag{4.14}$$

Since $\gamma_0 < \gamma_1 = 4(1+k^2)$, we have that $\lambda_0 < 0$ and it is the first negative eigenvalue of $\mathcal{L}_{\mathrm{cn}}$ with eigenfunction $\chi_0(x) = \Lambda_0(x/\eta)$ which has no zeros. We also have that $\gamma_1 < \gamma_2$ for any $k \in (0,1)$, then we get from (4.9) that

$$\lambda_2 = \left[1-\frac{1}{4(1+k^2)}\gamma_2\right](1-c^2-\beta_3) > 0. \tag{4.15}$$

This implies that λ_2 is the third eigenvalue of $\mathcal{L}_{\mathrm{cn}}$ with eigenfunction $\chi_2(x) = \Lambda_2(x/\eta)$, which has exactly 2 zeros in $[0,L)$. On the other hand, it can be shown that the first two eigenvalues of Lame's equation in the semiperiodic case are

$$\mu_0' = 5+2k^2-2\sqrt{4-k^2+k^4}, \qquad \mu_1' = 5+5k^2-2\sqrt{4-7k^2+4k^4}, \tag{4.16}$$

with corresponding eigenfunctions with exactly one zero in $[0,2K)$,

$$\begin{aligned}\zeta_{0,\mathrm{sp}} &= \mathrm{cn}(x)\left[1-\left(2+k^2-\sqrt{4-k^2+k^4}\right)\mathrm{sn}^2(x)\right],\\ \zeta_{1,\mathrm{sp}} &= 3\,\mathrm{sn}(x)-\left(2+2k^2-\sqrt{4-7k^2+4k^4}\right)\mathrm{sn}^3(x).\end{aligned} \tag{4.17}$$

But we have that $\mu_0' < \mu_1' < \gamma_1 = 4+4k^2$ and that

$$\mu' = -\frac{12}{\beta_3-\beta_1}\left[1-c^2-\beta_3-\mu\right]. \tag{4.18}$$

As a consequence of this, we found that the first three intervals of instability of $\mathcal{L}_{\text{cn}}$ are

$$(-\infty,\lambda_0), \qquad (\mu_0,\mu_1), \qquad (\lambda_1,\lambda_2). \tag{4.19}$$

We also have,

$$\mu_2' = 5+2k^2+2\sqrt{4-k^2+k^4}, \qquad \mu_3' = 5+5k^2+2\sqrt{4-7k^2+4k^4} \tag{4.20}$$

eigenvalues with corresponding eigenfunctions

$$\begin{aligned}\zeta_{2,\text{sp}} &= \text{cn}(x)\left[1-\left(2+k^2+\sqrt{4-k^2+k^4}\right)\text{sn}^2(x)\right],\\ \zeta_{3,\text{sp}} &= 3\,\text{sn}(x)-\left(2+2k^2+\sqrt{4-7k^2+4k^4}\right)\text{sn}^3(x),\end{aligned} \tag{4.21}$$

respectively, with exactly three zeros in $[0,2K)$. Finally, we conclude from (4.18) that the last interval of instability for $\mathcal{L}_{\text{cn}}$ is (μ_2,μ_3). □

4.2. Convexity of $d(c)$**.** As we showed in Section 2, cnoidal wave solutions ψ_c are characterized in such a way that the couple $V_c = \binom{\psi_c}{-c\psi_c}$ is a solution of

$$\delta\mathcal{F}(V_c) = \delta(\mathcal{H}+c\mathcal{N})(V_c) = \begin{pmatrix}\mathcal{A}\\ 0\end{pmatrix}, \tag{4.22}$$

with $\mathcal{A}$ being a nonzero number. Now we consider the study of the convexity of the function d defined by

$$d(c) = \mathcal{F}(V_c), \tag{4.23}$$

where the cnoidal wave solution ψ_c is given by Theorem 3.1. If we differentiate (4.23) with respect to c, we get that

$$d'(c) = \delta\mathcal{F}(V_c)\left(\frac{d}{dc}V_c\right)+\mathcal{N}(V_c) = \left\langle\mathcal{A},\frac{d}{dc}\psi_c\right\rangle+\mathcal{N}(V_c), \tag{4.24}$$

where $\langle\cdot,\cdot\rangle$ represents the pairing between $H^1_{\text{per}}([0,T_0])$ and $H^{-1}_{\text{per}}([0,T_0])$. Using that $\int_0^{T_0}(d/dc)\psi_c(x)dx = 0$, we have $\langle\mathcal{A},(d/dc)\psi_c\rangle = 0$. As a consequence, we obtain

$$d'(c) = \mathcal{N}(V_c) = -\int_0^{T_0}\left(c\psi_c^2+bc(\psi_c')^2-\frac{1}{2}\psi_c^3\right)dx. \tag{4.25}$$

Next we obtain the following expression to d' in (4.25):

$$d'(c) = -\frac{2T_{c_0}\sqrt{a-bc_0^2}}{45}\left[\frac{a(1-c^2)+\theta_0^2(4c^2+1)}{c^3\theta_0^2}A_{\varphi_c}+\frac{a+2c^2b}{c^3\theta_0^2}B_{\varphi_c}\right], \tag{4.26}$$

where $\theta_0 = \sqrt{a-bc^2}$. Indeed, since $T_0 = \sqrt{a-bc_0^2}\,T_{c_0}$ and from formula (3.73), we obtain from (4.25) that

$$d'(c) = -\theta_0\int_0^{T_{\varphi_c}}\frac{1}{9c}\varphi_c^2(x)+\frac{b}{9c\theta_0^2}\left[\varphi_c'(x)\right]^2+\frac{1}{54c^3}\varphi_c^3(x)dx, \tag{4.27}$$

where φ_c satisfies (3.71) and $T_{\varphi_c} \equiv \theta(c)T_{c_0}$, with $\theta(c) = \sqrt{a-bc_0^2}/\sqrt{a-bc^2}$. Next from (3.71), we obtain the following formula:

$$(\varphi_c')^2 = \frac{1}{3}\left[-\varphi_c^3 + 3(1-c^2)\varphi_c^2 + 6A_{\varphi_c}\varphi_c + 6B_{\varphi_c}\right], \tag{4.28}$$

where A_{φ_c} is given by (3.72) and from (3.52), (3.12), (3.15), and (3.70), we have that

$$B_{\varphi_c} = \left(\frac{1-c^2}{1-c_0^2}\right)^3 B_{\varphi_{c_0}}(c), \quad \text{with } B_{\varphi_{c_0}}(c) = \frac{1}{6}\Pi\alpha_i(c). \tag{4.29}$$

Moreover, from (3.71) we obtain

$$\frac{2}{3}\int_0^{T_{\varphi_c}} (\varphi_c')^2 dx = \frac{1}{3}\int_0^{T_{\varphi_c}} \varphi_c^3 dx - \frac{2(1-c^2)}{3}\int_0^{T_{\varphi_c}} \varphi_c^2 dx. \tag{4.30}$$

So integrating (4.28), we obtain from (4.30) that

$$\int_0^{T_{\varphi_c}} (\varphi_c')^2 dx = \frac{1}{5}(1-c^2)\int_0^{T_{\varphi_c}} \varphi_c^2 dx + \frac{6}{5}B_{\varphi_c}T_{\varphi_c}. \tag{4.31}$$

Similarly, by integrating (4.28) and from (4.31), it follows that

$$\int_0^{T_{\varphi_c}} \varphi_c^3 dx = \frac{12}{5}(1-c^2)\int_0^{T_{\varphi_c}} \varphi_c^2 dx + \frac{12}{5}B_{\varphi_c}T_{\varphi_c}. \tag{4.32}$$

Hence by substituting formulas (4.31) and (4.32) in (4.27), we obtain (4.26) after some manipulations.

Next, by using the formula for d'-(4.26), formula for A_{φ_c}-(3.72), and formula for B_{φ_c}-(4.29), we find that

$$d'(c) = \frac{T_{c_0}\sqrt{a-bc_0^2}}{405(1-c_0^2)^3}\left(\alpha_0 g_1(c)\sum_{i<j}\alpha_i\alpha_j - g_2(c)\alpha_1\alpha_2\alpha_3\right), \tag{4.33}$$

where

$$\begin{aligned} g_1(c) &= \left[-4bc^4 + (3a-b)c^2 + 2a\right](1-c^2)^2 c^{-3}(a-bc^2)^{-1}, \\ g_2(c) &= 3(2bc^2+a)(1-c^2)^3 c^{-3}(a-bc^2)^{-1}. \end{aligned} \tag{4.34}$$

In order to establish the convexity of d, we have to compute

$$\frac{d}{dc}\left(\sum_{i<j}\alpha_i\alpha_j\right), \qquad \frac{d}{dc}(\alpha_1\alpha_2\alpha_3). \tag{4.35}$$

The following result is obtained in this context.

Lemma 4.2. *Let α_1, α_2, and α_3 be as in Lemma 3.3. Then,*

$$\frac{d}{dc}\sum_{i<j}\alpha_i\alpha_j = 6F\alpha_0\alpha_1\alpha_2\alpha_3 - 2M\left(\sum_{i<j}\alpha_i\alpha_j\right),$$
$$\frac{d}{dc}(\alpha_1\alpha_2\alpha_3) = 4F\alpha_0^2\alpha_1\alpha_2\alpha_3 - \alpha_0 M\left(\sum_{i<j}\alpha_i\alpha_j\right) - 3M\alpha_1\alpha_2\alpha_3, \tag{4.36}$$

where $\alpha_0 = 3(1-c_0^2)$, $M(a,b,c) \equiv 2\lambda'(c)/\lambda(c)$, and $F = M/\sum_{i<j}\alpha_i\alpha_j$.

Proof. First we note that for $d/dc =$ "$'$" that

$$\begin{aligned}\left(\sum_{i<j}\alpha_i\alpha_j\right)' &= \alpha_1'(\alpha_2+\alpha_3)+\alpha_2'(\alpha_1+\alpha_3)+\alpha_3'(\alpha_2+\alpha_1)\\ &= \alpha_1'(\alpha_0-\alpha_1)+\alpha_2'(\alpha_0-\alpha_2)+\alpha_3'(\alpha_0-\alpha_3)\\ &= -(\alpha_1\alpha_1'+\alpha_2\alpha_2'+\alpha_3\alpha_3')+\alpha_0(\alpha_0)' = -(\alpha_1\alpha_1'+\alpha_2\alpha_2'+\alpha_3\alpha_3').\end{aligned} \tag{4.37}$$

Using expressions for α_i' obtained in Lemma 3.3, we have that for $i,j,k \in \{1,2,3\}$ with $j \neq i$, $j \neq k$, and $k \neq i$ that

$$\alpha_i\alpha_i' = F[\alpha_i^2(\alpha_j^2+\alpha_k^2) - \alpha_i\alpha_j\alpha_k(\alpha_j+\alpha_k)]. \tag{4.38}$$

Then summation over i gives us that

$$\left(\sum_{i<j}\alpha_i\alpha_j\right)' = -F\left[-2\alpha_0\alpha_1\alpha_2\alpha_3 + 2\sum_{i<j}\alpha_i^2\alpha_j^2\right]. \tag{4.39}$$

But a direct computation shows that

$$\left(\sum_{i<j}\alpha_i\alpha_j\right)^2 = \left(\sum_{i<j}\alpha_i^2\alpha_j^2\right) + 2\alpha_0\alpha_1\alpha_2\alpha_3. \tag{4.40}$$

Using these formulas in previous equation lets us conclude that

$$\left(\sum_{i<j}\alpha_i\alpha_j\right)' = -F\left[-6\alpha_0\alpha_1\alpha_2\alpha_3 + 2\left(\sum_{i<j}\alpha_i\alpha_j\right)^2\right] = 6F\alpha_0\alpha_1\alpha_2\alpha_3 - 2M\sum_{i<j}\alpha_i\alpha_j. \tag{4.41}$$

To get the second part, we note that

$$(\alpha_1\alpha_2\alpha_3)' = \alpha_1'\alpha_2\alpha_3 + \alpha_1\alpha_2'\alpha_3 + \alpha_1\alpha_2\alpha_3' \tag{4.42}$$

and for $i,j,k \in \{1,2,3\}$ with $j \neq i$, $j \neq k$, and $k \neq i$,

$$\alpha_i\alpha_j\alpha_k' = F[(\alpha_0-\alpha_k)^2\alpha_1\alpha_2\alpha_3 - (\alpha_0-\alpha_k)\alpha_i^2\alpha_j^2 - 2\alpha_i^2\alpha_j^2\alpha_k]. \tag{4.43}$$

Thus, we get

$$\begin{aligned}(\alpha_1\alpha_2\alpha_3)' &= F\left[\alpha_1\alpha_2\alpha_3\left(\sum(\alpha_0-\alpha_i)^2\right)-\alpha_0\left(\sum_{i<j}\alpha_i^2\alpha_j^2\right)-\alpha_1\alpha_2\alpha_3\left(\sum_{i<j}\alpha_i\alpha_i\right)\right]\\ &= F\left[\alpha_1\alpha_2\alpha_3\left(2\alpha_0^2-2\sum_{i<j}\alpha_i\alpha_j\right)-\alpha_0\left(\sum_{i<j}\alpha_i^2\alpha_j^2\right)-\alpha_1\alpha_2\alpha_3\left(\sum_{i<j}\alpha_i\alpha_i\right)\right]\\ &= F\left[\alpha_1\alpha_2\alpha_3\left(4\alpha_0^2-3\sum_{i<j}\alpha_i\alpha_j\right)-\alpha_0\left(\sum_{i<j}\alpha_i\alpha_j\right)^2\right]\\ &= 4F\alpha_0^2\alpha_1\alpha_2\alpha_3-\alpha_0M\left(\sum_{i<j}\alpha_i\alpha_j\right)-3M\alpha_1\alpha_2\alpha_3.\end{aligned} \tag{4.44}$$

□

Now we are in position to prove the convexity of function d.

THEOREM 4.3. *d is a strictly convex function for $0<|c|<1<\sqrt{a/b}$, and for $0<|c|<c_*+\theta<\sqrt{a/b}<1$, where θ is small and c_* is the unique positive root of the polynomial*

$$P(c)=12b^2c^6+(13b^2-19ab)c^4+(9ab-9a^2)c^2-6a^2. \tag{4.45}$$

Proof. From our previous computations and using formulas in Lemma 4.2 for the derivatives of $\sum_{i<j}\alpha_i\alpha_j$ and $\alpha_1\alpha_2\alpha_3$ with respect to c, we obtain that

$$\begin{aligned}d''(c)=\frac{T_{c_0}\sqrt{a-bc_0^2}}{405(1-c_0^2)^3}\Bigg\{&\alpha_0\left(g_1'(c)+M[g_2(c)-2g_1(c)]\right)\sum_{i<j}\alpha_i\alpha_j\\ &+\left[\frac{2\alpha_0^2M}{\sum_{i<j}\alpha_i\alpha_j}(3g_1(c)-2g_2(c))+(3g_2(c)M-g_2'(c))\right]\alpha_1\alpha_2\alpha_3\Bigg\}.\end{aligned} \tag{4.46}$$

Using that

$$M=\frac{2\lambda'(c)}{\lambda(c)}=\frac{2c(b-a)}{(1-c^2)(a-bc^2)}, \tag{4.47}$$

we obtain by a direct computation that

$$\begin{aligned}&g_1'(c)+M[g_2(c)-2g_1(c)]\\ &\quad=\frac{(1-c^2)^2(12b^2c^6+(13b^2-19ab)c^4+(9ab-9a^2)c^2-6a^2)}{c^4(a-bc^2)^2},\end{aligned} \tag{4.48}$$

$$\begin{gathered}[3g_1(c)-2g_2(c)]M=\frac{-30(a-b)^2(1-c^2)}{c^2(a-bc^2)^2}<0,\\ 3g_2(c)M-g_2'(c)=\frac{9(1-c^2)^3(2b^2c^4+abc^2+a^2)}{5c^4(a-bc^2)^2}>0.\end{gathered} \tag{4.49}$$

Now, let us define the following polynomial:

$$P(c) = 12b^2c^6 + (13b^2 - 19ab)c^4 + (9ab - 9a^2)c^2 - 6a^2. \tag{4.50}$$

The first observation is that

$$P(c) = 6b^2c^4(c^2 - 1) + 6(b^2c^6 - a^2) + 19bc^4(b - a) + 9ac^2(b - a). \tag{4.51}$$

If we assume that $a > b$ and $c^2 < 1$, we have that $b^2c^6 - a^2 < 0$. Thus we conclude that $P(c) < 0$, for $a > b$ and $c^2 < 1$. On the other hand, if we assume that $a < b$ and $0 < |c| < \sqrt{a/b}$, we have that $P(\pm\sqrt{a/b}) = 16a^2(b - a)/b > 0$, but $P(0) = -6a^2 < 0$. As a consequence of this, there exists a unique c_* with $P(c_*) = 0$, $0 < c_* < \sqrt{a/b} < 1$, such that $P(c) < 0$ for $0 < |c| < c_*$ in case of having $b > a$. Note that $P'(c) > 0$ for $c > 0$.

Therefore we have from Lemma 3.3 that $\alpha_1\alpha_2\alpha_3 > 0$ and from (3.72) that $\sum_{i<j}\alpha_i\alpha_j < 0$, so we can conclude initially from (4.46) and (4.49) that

(1) d is a strictly convex function in $0 < |c| < 1$ for $a > b$,

(1*) d is a strictly convex function in $0 < |c| < c_*$ for $0 < c_* < \sqrt{a/b} < 1$.

Next, note that $d''(c_*) > 0$, and so from continuity we can choose θ small such that

(2) d is a strictly convex function in $0 < |c| < c_* + \theta$ for $0 < c_* + \theta < \sqrt{a/b} < 1$,

as desired. □

5. Stability theory for the Boussinesq-type system (1.10)

In this section, we establish a theory of stability for the branch of cnoidal waves solutions

$$c \longrightarrow (\psi_c(x - ct), -c\psi_c(x - ct)), \tag{5.1}$$

associated to system (1.10) determined by Theorem 3.1. These smooth curves of solutions to (3.7), $c \in (-\sqrt{\min\{1, a/b\}}, \sqrt{\min\{1, a/b\}}) \setminus \{0\} \to \psi_c \in H^1_{\text{per}}([0, T_0])$, have an arbitrary period T_0 and mean zero on $[0, T_0]$.

We first note from (2.12) that $\vec{\psi_c} = (\psi_c, -c\psi_c)^t$ is not a critical point to the action functional $\mathcal{F}$ in (2.13) indicating that the general theory of Grillakis, Shatah, and Strauss cannot be applied directly to the problem at hand over all $H^1_{\text{per}}([0, T_0]) \times H^1_{\text{per}}([0, T_0])$. To overcome this, we will proceed as in the proof of orbital stability of cnoidal wave solutions with respect to the periodic flow of solutions with mean zero for the initial value problem associated with the KdV equation (see [3]). In other words, we consider the following spaces:

$$\mathcal{W}^1 = \left\{q \in H^1_{\text{per}}([0, T_0]) : \int_0^{T_0} q(y)dy = 0\right\}, \qquad \mathcal{X} \equiv \mathcal{W}^1 \times H^1_{\text{per}}([0, T_0]), \tag{5.2}$$

where $\|\cdot\|_{\mathcal{X}} := \|\cdot\|_{H^1_{\text{per}} \times H^1_{\text{per}}}$. We will see below that Grillakis et al.'s approach in [4] can be used to obtain the stability of $\vec{\psi_c}$ by perturbations belonging to $\mathcal{X}$. In fact, in this case we consider $\mathcal{F}$ defined on the space $\mathcal{X}$ and so the cnoidal waves $\vec{\psi_c}$ is a critical point, namely, for $\vec{v} = (f,g) \in \mathscr{X}$, we obtain

$$\mathscr{F}'(\vec{\psi_c})\vec{v} = \langle(-A_{\psi_c},0),\vec{v}\rangle = -A_{\psi_c}\int_0^{T_0} f\,dx = 0. \tag{5.3}$$

More exactly, we obtain the following stability result associated to system (1.10).

Theorem 5.1. *Consider c with* $0 < c^2 < \min\{1, a/b\}$ *and let* $\{\psi_c\}$ *be the cnoidal wave branch of period* T_0 *given in Theorem 3.1. Then, for c satisfying the conditions in Theorem 4.3, the orbit* $\{(\psi_c(\cdot+s), -c\psi_c(\cdot+s))\}_{s\in\mathbb{R}}$ *is stable in* $\mathscr{X}$ *with regard to* T_0*-periodic perturbations and the flow generated by system (1.10). More precisely, given any* $\epsilon > 0$*, there is a* $\delta = \delta(\epsilon)$ *such that if* $(q_0,r_0) \in \mathscr{X}$ *and* $\|(q_0,r_0) - (\psi_c, -c\psi_c)\|_{\mathscr{X}} < \delta$*, then*

$$\inf_{s\in\mathbb{R}} \|(q(t),r(t)) - (\psi_c(\cdot+s), -c\psi_c(\cdot+s))\|_{\mathscr{X}} < \epsilon \tag{5.4}$$

for all t, where $(q(t),r(t))$ *is the solution of system (1.10) with initial value* (q_0,r_0)*.*

The proof of Theorem 5.1 needs some preliminary results. First of all, we have to establish the existence and uniqueness of global mild periodic solutions for the periodic Cauchy problem associated with system (1.10), and second, we need to study the periodic eigenvalue problem associated with the operator $\mathscr{F}''(\psi_c, -c\psi_c)$.

5.1. Global existence and uniqueness of mild solutions. The proof of global mild solution for system (1.10) follows by the use of classical theory of semigroups. We will use that $\mathscr{H}$ in (2.8) is conserved in time along classical solutions of system (1.10) to prove that local mild solutions are already global mild solutions. To do this, we will use a density argument and the fact that the nonlinear part has a nice behavior (see [17]). We start by rewriting the first-order system (1.10) as

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = M\begin{pmatrix} q \\ r \end{pmatrix} + \mathscr{G}_1\begin{pmatrix} q \\ r \end{pmatrix}, \tag{5.5}$$

where

$$M = \begin{pmatrix} 0 & \partial_x \\ \partial_x B^{-1}A & 0 \end{pmatrix}, \qquad \mathscr{G}_1\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -B^{-1}(rq_x + 2qr_x) \end{pmatrix}. \tag{5.6}$$

In order to study the initial value problem for system (5.5), we have to consider the natural spaces given by the Hamiltonian $\mathscr{H}$. In other words, we will seek for solutions $(q(t,\cdot), r(t,\cdot)) \in H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$. We start discussing some properties of the operator M which is defined in the Hilbert space $H^2_{\text{per}}([0,T_0]) \times H^2_{\text{per}}([0,T_0])$. In this space, we have that $M \in \mathscr{L}(H^2_{\text{per}}([0,T_0]) \times H^2_{\text{per}}([0,T_0]), H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0]))$. Moreover, M is the infinitesimal generator of a bounded C_0-group $\mathscr{S}^1(t)$ on $H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$. If we define $\mathscr{S}^1(t) := \sum_{-\infty}^{\infty} \widehat{\mathscr{S}^1}_n(t)e^{2\pi i n x/T_0}$, it can be shown by using

Fourier series that the n-symbol for $\mathcal{S}^1(t)$ is

$$\widehat{\mathcal{S}^1}_n(t) = \begin{pmatrix} \cos(n\Lambda(n)t) & i\dfrac{\sin(n\Lambda(n)t)}{\Lambda(n)} \\ i\Lambda(n)\sin(n\Lambda(n)t) & \cos(n\Lambda(n)t) \end{pmatrix}, \quad \text{with } \Lambda^2(n) = \frac{1+4\pi^2 a|n|^2/T_0^2}{1+4\pi b|n|^2/T_0^2}. \tag{5.7}$$

Now note that B^{-1} is a bounded linear operator from $L^2_{\text{per}}([0,T_0])$ to $H^2_{\text{per}}([0,T_0])$ since it is defined as

$$B^{-1}f = \sum_{-\infty}^{\infty}\left(\frac{1}{1+4b\pi^2n^2/T_0^2}\right)\hat{f}_n e^{2\pi inx/T_0}, \quad \text{for } f = \sum_{-\infty}^{\infty}\hat{f}_n e^{2\pi inx/T_0} \in L^2_{\text{per}}([0,T_0]). \tag{5.8}$$

On the other hand, if $f \in H^1_{\text{per}}([0,T_0])$, then $f \in L^\infty(\mathbb{R})$. Thus, if we assume that $(q,r) \in H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$, then q and r are bounded functions. Since $q_x, r_x \in L^2_{\text{per}}$, we conclude that $rq_x + 2qr_x \in L^2_{\text{per}}([0,T_0])$, and so we have that $B^{-1}(rq_x+2qr_x) \in H^2_{\text{per}}([0,T_0])$. In other words, we have shown that $\mathcal{G}_1$ maps $H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$ into $H^2_{\text{per}}([0,T_0]) \times H^2_{\text{per}}([0,T_0])$, meaning that $\mathcal{G}_1$ gains some regularity. Moreover, inequality

$$\left\|B^{-1}(r_1(q_1)_x + 2q_1(r_1)_x) - B^{-1}(r_2(q_2)_x + 2q_2(r_2)_x)\right\|_{H^2_{\text{per}}} \le \left\|(q_1,r_1) - (q_2,r_2)\right\|_{H^1_{\text{per}}\times H^1_{\text{per}}} \tag{5.9}$$

implies that $\mathcal{G}_1$ is locally Lipschitz from $H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$ into $H^2_{\text{per}}([0,T_0]) \times H^2_{\text{per}}([0,T_0])$. Using this fact, it is easy to prove the following existence and uniqueness result.

Theorem 5.2. *Let $(q_0,r_0) \in H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$. The initial value problem*

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = M\begin{pmatrix} q \\ r \end{pmatrix} + \mathcal{G}_1\begin{pmatrix} q \\ r \end{pmatrix}, \tag{5.10}$$

$$\begin{pmatrix} q \\ r \end{pmatrix}(0,\cdot) = \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \tag{5.11}$$

has a unique global mild solution $(q(t,\cdot), r(t,\cdot)) \in H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0])$.

Corollary 5.3. *Let $(q_0,r_0) \in \mathcal{X}$. Then $(q(t,\cdot), r(t,\cdot))$ solution of (5.10) with initial data (q_0,r_0) belongs to $\mathcal{X}$ for all $t \in \mathbb{R}$.*

Corollary 5.3 is a direct consequence of Theorem 5.2 and the conservation property of the functional

$$\mathcal{M}(q,r) = \int_0^{T_0} q(x)dx \tag{5.12}$$

by the flow of (5.10).

Proof of Theorem 5.2. The first step to prove this result is to show local (in-time) existence and uniqueness of T_0-periodic classical solutions for initial data $(q_0,r_0) \in H^2_{per}([0,T_0]) \times H^2_{per}([0,T_0])$, which follows because M is the infinitesimal generator of a bounded C_0-group $S(t)$ in $H^1_{per}([0,T_0]) \times H^1_{per}([0,T_0])$ and $\mathcal{G}_1$ is locally Lipschitz form $H^1_{per}([0,T_0]) \times H^1_{per}([0,T_0])$ to $H^2_{per}([0,T_0]) \times H^2_{per}([0,T_0])$. Now, from the variation of constants formula, we can obtain for a $(q,r) - H^2_{per} \times H^2_{per}$ solution the *a priori* bound

$$\left\| \partial_{xx} \begin{pmatrix} q \\ r \end{pmatrix} (t,\cdot) \right\|_{L^2\times L^2} \leqq C \left(\left\| \partial_{xx} \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} (\cdot) \right\|_{L^2\times L^2} + t \left\| \begin{pmatrix} q \\ r \end{pmatrix} (t,\cdot) \right\|_{H^1_{per}\times H^1_{per}} \right). \tag{5.13}$$

Since the Hamiltonian $\mathcal{H}$ (equivalent norm in $H^1_{per} \times H^1_{per}$) is a conserved quantity in time on classical solutions for (5.5), it follows that any local T_0-periodic classical solution in $H^2_{per}([0,T_0]) \times H^2_{per}([0,T_0])$ can be extended to $[0,\infty)$.

The second step is to prove that a local mild T_0-periodic solution $(q,r) \in H^1_{per}([0,T_0]) \times H^1_{per}([0,T_0])$ exists with initial data $(q_0,r_0) \in H^1_{per}([0,T_0]) \times H^1_{per}([0,T_0])$, which follows from classical semigroup theory.

Finally, by using that the embedding $H^2_{per}([0,T_0]) \hookrightarrow H^1_{per}([0,T_0])$ is dense and we have classical solutions in $H^2_{per}([0,T_0]) \times H^2_{per}([0,T_0])$, we prove that the Hamiltonian $\mathcal{H}$ is also conserved in time on mild T_0-periodic solutions, and therefore for any $t_0 > 0$ we have

$$\lim_{t\uparrow t_0} \left\| \begin{pmatrix} q \\ r \end{pmatrix} (t,\cdot) \right\|_{H^1_{per}\times H^1_{per}} < \infty, \tag{5.14}$$

which implies global existence. □

5.2. Spectral analysis for $\mathcal{F}''(\psi_c,-c\psi_c) = \mathcal{H}''(\psi_c,-c\psi_c) + c\mathcal{N}''(\psi_c,-c\psi_c)$. As already known, the study of the periodic eigenvalue problem considered on $[0,T_0]$ is required to use the stability theory outlined in [3, 4]. The spectral problem in question is given for

$$\begin{gathered} \mathcal{L}\chi = \lambda\chi, \\ \chi(0) = \chi(T_0), \qquad \chi'(0) = \chi'(T_0), \end{gathered} \tag{5.15}$$

where

$$\mathcal{L} \equiv \mathcal{F}''(\psi_c,-c\psi_c) = \begin{pmatrix} 1 - a\partial_x^2 + 3c\psi_c & c(1-b\partial_x^2) \\ c(1-b\partial_x^2) & 1-b\partial_x^2 \end{pmatrix}, \tag{5.16}$$

ψ_c being the cnoidal wave solution given in Theorem 3.1 for $0 < c^2 < \min\{1, a/b\}$. The following result is obtained in this context.

Theorem 5.4. *Let $0 < c^2 < \min\{1, a/b\}$ and let ψ_c be the cnoidal wave given in Theorem 3.1. Then the periodic eigenvalue problem (5.15) on $H^2_{per}([0,T_0]) \times H^2_{per}([0,T_0])$ has exactly a negative eigenvalue which is simple. $\lambda = 0$ is a simple eigenvalue with eigenfunction $(\psi_c', -c\psi_c')^t$ and the rest of the spectrum is bounded away from zero.*

Proof. The proof is based on the min-max principle and Theorem 4.1. Indeed, for $\zeta = (\psi_c', -c\psi_c')^t$, it follows that $\mathcal{L}\zeta = (\mathcal{L}_{\text{cn}}\psi_c', 0)^t = (0,0)^t$. In other words, $\lambda = 0$ is an eigenvalue of $\mathcal{L}$ with eigenvector $\zeta = (\psi_c', -c\psi_c')^t$. Moreover, $\lambda = 0$ is simple. In fact, consider $\chi = (f,g)^t$ such that $\mathcal{L}\chi = (0,0)^t$. Then we obtain that $g = -cf$ and $\mathcal{L}_{\text{cn}} f = 0$. So, $f = \alpha\psi_c'$, and therefore $(f,g) = \alpha(\psi_c', -c\psi_c')$, showing that $\lambda = 0$ is simple. Now, for $\zeta = (f,g)^t$, it is easy to see that

$$\langle \mathcal{L}\zeta, \zeta\rangle = \langle \mathcal{L}_{\text{cn}} f, f\rangle + \|cB^{1/2}f + B^{1/2}g\|^2, \tag{5.17}$$

where $B^{1/2}$ is the square root of the positive linear operator B and $\langle\cdot,\cdot\rangle$, $\langle\cdot,\cdot\rangle$ represent the scalar products in $L^2 \times L^2$ and L^2, respectively. So, by taking χ_0 such that $\mathcal{L}_{\text{cn}}\chi_0 = \lambda_0\chi_0$ for $\lambda_0 < 0$, we obtain from (5.17) that $\langle \mathcal{L}\zeta_0, \zeta_0\rangle = \langle \mathcal{L}_{\text{cn}}\chi_0, \chi_0\rangle < 0$ for $\zeta_0 = (\chi_0, -c\chi_0)^t$. Therefore, $\mathcal{L}$ has a negative eigenvalue. Now, we show that this negative eigenvalue is unique, and so will be simple. In fact, let $\zeta_1 = (\chi_0, 0)^t$ and $\zeta_2 = (\psi_c', 0)^t$. Then for $\varphi = (f,g)^t$, $\|\varphi\| = 1$, it follows from min-max principle that the third eigenvalue for $\mathcal{L}$, η_3 satisfies

$$\begin{aligned} \eta_3 = \sup_{[\xi_1,\xi_2]} \inf_{\varphi\perp\xi_1,\varphi\perp\xi_2} \langle \mathcal{L}\varphi,\varphi\rangle &\geqq \inf_{\varphi\perp\zeta_1,\varphi\perp\zeta_2} \langle \mathcal{L}\varphi,\varphi\rangle \\ &\geqq \inf_{f\perp\chi_0, f\perp\psi_c'} \left[\langle \mathcal{L}_{\text{cn}} f, f\rangle + \|cB^{1/2}f + B^{1/2}g\|^2\right] \geqq \delta_0 > 0, \end{aligned} \tag{5.18}$$

where in the last inequality we have used Theorem 4.1. So, we finish the theorem. □

Now we focus on a mean-zero branch $\{\psi_c\}$ of cnoidal waves as was guaranteed by Theorem 3.1. As we saw in (4.25),

$$d'(c) = -\int_0^{T_0} \left[c\psi_c^2 + bc(\psi_c')^2 - \frac{1}{2}\psi_c^3\right] dx. \tag{5.19}$$

Next, differentiating (3.2) with respect to c, we also have for $\zeta = (d/dc)(\psi_c, -c\psi_c)$ that

$$\mathcal{L}\zeta^t = \left(cB\psi_c - \frac{3}{2}\psi_c^2 - \frac{d}{dc}A_{\psi_c}, -B\psi_c\right)^t. \tag{5.20}$$

Thus we obtain the basic relation

$$-\langle \mathcal{L}\zeta^t, \zeta^t\rangle = d''(c) = \frac{d}{dc}\mathcal{N}(\psi_c, -c\psi_c). \tag{5.21}$$

We note in this point that even if we have that $\zeta \in \mathcal{X}$ and $\langle \mathcal{L}\zeta^t, \zeta^t\rangle < 0$, we cannot assure that the restriction of $\mathcal{L}$ to $\mathcal{X}$, $\mathcal{L}|_{\mathcal{X}}$, will have a negative eigenvalue. In fact, since $\mathcal{L}$ in general does not map vectors with first component having mean zero in vectors with this same property, we cannot perform a min-max principle's argument for $\mathcal{L}|_{\mathcal{X}}$.

Proof of Theorem 5.1. Next, for convenience of the readers, we will establish the basic changes in the abstract theory by Grillakis et al. such that we can apply it to the solutions $\vec{\psi_c} = (\psi_c, -c\psi_c)$. In fact, from [4, Lemma 3.2], there exist $\epsilon > 0$ and a C^1 map

$\alpha : U_\epsilon \to \mathbb{R}/T_0$, where

$$U_\epsilon = \Big\{ \vec{p} \in H^1_{\text{per}}([0,T_0]) \times H^1_{\text{per}}([0,T_0]) : \inf_{s\in\mathbb{R}} \| \vec{p} - (\psi_c(\cdot+s), -c\psi_c(\cdot+s)) \|_{H^1_{\text{per}}\times H^1_{\text{per}}} \leqq \epsilon \Big\} \tag{5.22}$$

such that for all $\vec{p} \in U_\epsilon$,

$$\langle \tau_{\alpha(\vec{p})}\vec{p}, \vec{\psi_c}' \rangle = 0, \quad \tau_\alpha \vec{p} = \vec{p}(\cdot+\alpha). \tag{5.23}$$

Now, from (5.20), (5.21), and Theorem 4.3 which assure that $d''(c) > 0$, we can use the ideas in [4, Theorem 3.3] and [18] to show that

$$\eta = \inf\{ \langle \mathcal{L}\varphi, \varphi \rangle : \langle \varphi, \xi_c \rangle = \langle \varphi, \vec{\psi_c}' \rangle = 0,\ \|\varphi\|^2 = 1 \} > 0, \tag{5.24}$$

where $\xi_c = (cB\psi_c - (3/2)\psi_c^2 - (d/dc)A_{\psi_c}, -B\psi_c)$ has the main property that $\mathcal{L}\zeta^t = \xi_c^t$. Moreover, from (5.24) and from the specific form of $\mathcal{L}$, there is a positive constant β such that if $\langle \varphi, \xi_c \rangle = \langle \varphi, \vec{\psi_c}' \rangle = 0$, then

$$\langle \mathcal{L}\varphi, \varphi \rangle \geqq \beta \|\varphi\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}}. \tag{5.25}$$

Now for $\vec{p} = (p,r)$ with $\int_0^{T_0} p\,dx = 0$, write $\tau_{\alpha(\vec{p})}\vec{p} - \vec{\psi_c} = \mu\xi_c + \varphi$, where $\langle \varphi, \xi_c \rangle = 0$. Then by taking $\mathcal{N}(\vec{p}) = \mathcal{N}(\vec{\psi_c})$ and Taylor's theorem, we have

$$\begin{aligned} \mathcal{N}(\vec{\psi_c}) &= \mathcal{N}(\vec{p}) = \mathcal{N}(\tau_{\alpha(\vec{p})}\vec{p}) \\ &= \mathcal{N}(\vec{\psi_c}) + (\mathcal{N}'(\vec{\psi_c}), \tau_{\alpha(\vec{p})}\vec{p} - \vec{\psi_c}) + O\Big(\| \tau_{\alpha(\vec{p})}\vec{p} - \vec{\psi_c} \|^2_{H^1_{\text{per}}\times H^1_{\text{per}}} \Big). \end{aligned} \tag{5.26}$$

So, we obtain that $\mu = O(\|\tau_{\alpha(\vec{p})}\vec{p} - \vec{\psi_c}\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}})$.

Considering $L(\vec{p}) = \mathcal{H}(\vec{p}) + c\mathcal{N}(\vec{p})$, then another Taylor expansion gives

$$L(\vec{p}) = L(\tau_{\alpha(\vec{p})}\vec{p}) = L(\vec{\psi_c}) + \langle L'(\vec{\psi_c}), v \rangle + \frac{1}{2}\langle L''(\vec{\psi_c})v, v \rangle + o\Big(\|v\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}} \Big), \tag{5.27}$$

where $v \equiv \tau_{\alpha(\vec{p})}\vec{p} - \vec{\psi_c}$. Now, for $v = (f,g)$, we have $\int_0^{T_0} f\,dx = 0$. Therefore, since

$$L'(\vec{\psi_c})v = -\langle (A_{\psi_c}, 0), v \rangle = -A_{\psi_c}\int_0^{T_0} f\,dx = 0, \tag{5.28}$$

$\mathcal{N}(\vec{p}) = \mathcal{N}(\vec{\psi_c})$ and $L''(\vec{\psi_c}) = \mathcal{L}$, we get

$$\mathcal{H}(\vec{p}) - \mathcal{H}(\vec{\psi_c}) = \frac{1}{2}\langle \mathcal{L}v, v\rangle + o\Big(\|v\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}}\Big) = \frac{1}{2}\langle \mathcal{L}\varphi, \varphi\rangle + o\Big(\|v\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}}\Big). \tag{5.29}$$

Now, since $\langle \vec{\psi_c}', \xi_c\rangle = 0$, it follows from (5.23) that $\langle \varphi, \vec{\psi_c}'\rangle = 0$. Therefore, (5.25) implies that

$$\mathcal{H}(\vec{p}) - \mathcal{H}(\vec{\psi_c}) \geqq \frac{1}{2}D\|\varphi\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}} + o\Big(\|v\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}}\Big) \tag{5.30}$$

for some constant $D > 0$. Finally, since $\|\varphi\|_{H^1_{\text{per}}\times H^1_{\text{per}}} \geqq \|v\|_{H^1_{\text{per}}\times H^1_{\text{per}}} - O(\|v\|^2_{H^1_{\text{per}}\times H^1_{\text{per}}})$, we obtain that for $\|v\|_{H^1_{\text{per}}\times H^1_{\text{per}}}$ small enough,

$$\mathcal{H}(\vec{p}) - \mathcal{H}(\vec{\psi_c}) \geqq D_1 \big|\big|\tau_{\alpha(\vec{p})}\vec{p} - \vec{\psi_c}\big|\big|^2_{H^1_{\text{per}}\times H^1_{\text{per}}} \tag{5.31}$$

for all $\vec{p} = (p, r) \in U_\epsilon$ which satisfy $\mathcal{N}(\vec{p}) = \mathcal{N}(\vec{\psi_c})$ and $\int_0^{T_0} p\,dx = 0$.

Thus by using standard arguments, we show from inequality (5.31) and from the invariance of the functional $\mathcal{M}(p, r) = \int_0^{T_0} p(x)dx$ with regard to system (1.10) that the orbit $\{(\psi_c(\cdot + s), -c\psi_c(\cdot + s))\}_{s\in\mathbb{R}}$ is stable in $\mathcal{X}$. □

6. Orbital stability for the Benney-Luke equation

In order to prove the stability of periodic solutions for the Benney-Luke equation, we have to establish existence and uniqueness of mild solutions in an appropriate space for the Cauchy problem associated with (1.1). We start by defining the natural space to consider the existence result. Let

$$\mathcal{Z} = \{w \in C^\infty_{\text{per}}([0, T_0]) : w_x \in H^1_{\text{per}}([0, T_0])\}, \tag{6.1}$$

and define the equivalence relation on $\mathcal{Z}$ given by: $u \sim v$ if and only if $u(x) - v(x) = \theta$ for $x \in \mathbb{R}$ and θ a real constant. Now define the quotient space $\mathcal{Y} = \mathcal{Z}/\sim$ with the norm

$$\big|\big|[y]\big|\big|_{\mathcal{Y}} = \big|\big|y_x\big|\big|_{H^1_{\text{per}}} = \big|\big|u_x\big|\big|_{H^1_{\text{per}}}, \quad \forall u \in [y]. \tag{6.2}$$

Let $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ be the closure of $\mathcal{Y}$ with respect to $\|\cdot\|_{\mathcal{Y}}$. In particular, if $[\Phi] \in \mathcal{Y}$,

$$\big|\big|[\Phi]\big|\big|_{\mathcal{V}} = \big|\big|[\Phi]\big|\big|_{\mathcal{Y}} = \big|\big|\Phi_x\big|\big|_{H^1_{\text{per}}}. \tag{6.3}$$

Hereafter we will identify any equivalence class with a representative. Roughly speaking, the space $\mathcal{V}$ can be viewed as the closure of $\mathcal{Z}$ with respect to the "norm" $\|\cdot\|_{\mathcal{V}}$. Moreover, as we will see below, $\mathcal{V}$ can be identified with the space $\mathcal{W}^1$ through the linear operator ∂_x. So, it will give us an easy way to recover all the stability theories associated to system (1.10) in $\mathcal{W}^1$ to $\mathcal{V}$. Finally, we note that as (1.1) is invariant by the translation $\Phi \to \Phi + \text{const}$, $\mathcal{V}$ is a natural space to be considered in our stability study.

Now we have to note that the Benney-Luke equation can be written as the system in the variables Φ and $\Phi_t = r$,

$$\begin{pmatrix}\Phi\\ r\end{pmatrix}_t = M_0\begin{pmatrix}\Phi\\ r\end{pmatrix} + \mathcal{G}_0\begin{pmatrix}\Phi\\ r\end{pmatrix}, \tag{6.4}$$

where operators M_0 and $\mathcal{G}_0$ are given by

$$M_0 = \begin{pmatrix}0 & I\\ \partial_x^2 B^{-1}A & 0\end{pmatrix}, \qquad \mathcal{G}_0\begin{pmatrix}\Phi\\ r\end{pmatrix} = \begin{pmatrix}0\\ -B^{-1}(r\Phi_{xx} + 2\Phi_x r_x)\end{pmatrix}. \tag{6.5}$$

It is not hard to verify that associated with the linear operator M_0, there exists a $\mathcal{S}_0$ group defined in $\mathcal{V} \times H^1_{\text{per}}$, whose Fourier symbols are given by

$$\widehat{\mathcal{S}^0}_n(t) = \begin{pmatrix}\cos(n\Lambda(n)t) & \dfrac{\sin(n\Lambda(n)t)}{n\Lambda(n)}\\ -n\Lambda(n)\sin(n\Lambda(n)t) & \cos(n\Lambda(n)t)\end{pmatrix}. \tag{6.6}$$

In this paper, we are going to say that a mild solution of the Benney-Luke equation (1.1) with initial data (u_0, u_1) is a couple (Φ, r) such that

$$(\Phi, r) \in C(\mathbb{R}_t; \mathcal{V} \times H^1_{\text{per}}([0, T_0])), \tag{6.7}$$

and it satisfies the integral equation

$$\begin{pmatrix}\Phi\\ r\end{pmatrix}(t) = \mathcal{S}_0(t)\begin{pmatrix}u_0\\ u_1\end{pmatrix} + \int_0^t \mathcal{S}_0(t-y)\mathcal{G}_0\begin{pmatrix}\Phi\\ r\end{pmatrix}(y)\,dy. \tag{6.8}$$

The first observation is that we have the following relationship between $\mathcal{S}_0$ and $\mathcal{S}_1$:

$$\begin{pmatrix}\partial_x & 0\\ 0 & I\end{pmatrix}\mathcal{S}_0 = \mathcal{S}_1\begin{pmatrix}\partial_x & 0\\ 0 & I\end{pmatrix}, \tag{6.9}$$

where ∂_x is the bounded linear map defined from $\mathcal{V}$ to $\mathcal{W}^1$ such that $\partial_x(\Phi) = \Phi'$ for $\Phi \in \mathcal{Y}$. Moreover, each element in $\mathcal{W}^1$ induces an element (*antiderivative*) in $\mathcal{V}$. More precisely, there is an operator $\partial_x^{-1} : \mathcal{W}^1 \to \mathcal{V}$ defined by

$$\partial_x^{-1}(q)(x) = \frac{1}{2\pi}\sum_{n\neq 0}\frac{q_n}{in}e^{2i\pi nx/T_0}, \quad \text{for } q(x) = \sum_{n\neq 0} q_n e^{2i\pi nx/T_0} \in \mathcal{W}^1. \tag{6.10}$$

Note that $\partial_x^{-1}(q)$ is a well-defined element in $\mathcal{V}$ since the sequence $\{\Phi_k\} \subseteq C^\infty_{\text{per}}([0, T_0])$ defined by

$$\Phi_k(x) = \frac{1}{2\pi}\sum_{n\neq 0,\, n=-k}^{n=k}\frac{q_n}{in}e^{2i\pi nx/T_0} \tag{6.11}$$

is a Cauchy sequence in $\mathcal{Y}$. So, we have that

$$\partial_x(\partial_x^{-1}(q)) \equiv \lim_{k\to\infty} \Phi_k' = \lim_{k\to\infty} \sum_{n\neq 0, n=-k}^{n=k} q_n e^{2i\pi n(\cdot)/T_0} = q. \tag{6.12}$$

Moreover, it is easy to see that ∂_x^{-1} is a bounded linear operator. As a consequence of this fact, we obtain that $\partial_x^{-1}\partial_x\Phi = \Phi$ for all $\Phi \in \mathcal{V}$.

We will see that the existence and uniqueness associated with the Benney-Luke equation (1.1) follow directly from the existence and uniqueness of mild periodic solutions for the Cauchy problem associated with the Boussinesq system (1.10) (see Theorem 5.2). In fact, let $(u_0, r_0) \in \mathcal{V} \times H^1_{\text{per}}([0,T_0])$. Define $(q_0, r_0) \in \mathcal{X} = \mathcal{W}^1 \times H^1_{\text{per}}([0,T_0])$ by $q_0 = \partial_x u_0$. Then from Theorem 5.2, there exists a unique global mild solution for (1.10),

$$(q, r) \in C(\mathbb{R}_t; \mathcal{W}^1 \times H^1_{\text{per}}([0,T_0])), \tag{6.13}$$

such that $(q(0), r(0)) = (q_0, r_0)$ and

$$\begin{pmatrix} q \\ r \end{pmatrix}(t) = \mathcal{S}_1(t)\begin{pmatrix} q_0 \\ r_0 \end{pmatrix} + \int_0^t \mathcal{S}_1(t-y)\mathcal{G}_1\begin{pmatrix} q \\ r \end{pmatrix}(y)dy. \tag{6.14}$$

Next, define $\Phi(t) \in \mathcal{V}$, $t \in \mathbb{R}$, in such a way that $\Phi(t) \equiv \partial_x^{-1}q(t)$ or $\partial_x\Phi(t) = q(t)$. As a consequence of this, we conclude that

$$\mathcal{G}_1\begin{pmatrix} q \\ r \end{pmatrix} = \mathcal{G}_0\begin{pmatrix} \Phi \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ -B^{-1}(r\Phi_{xx} + 2\Phi_x r_x) \end{pmatrix}. \tag{6.15}$$

Again, applying the operator $\left(\begin{smallmatrix} \partial_x^{-1} & 0 \\ 0 & I \end{smallmatrix}\right)$ to (6.14) and using (6.9), one can see that

$$\begin{pmatrix} \Phi \\ r \end{pmatrix}(t) = \mathcal{S}_0(t)\begin{pmatrix} u_0 \\ r_0 \end{pmatrix} + \int_0^t \mathcal{S}_0(t-y)\mathcal{G}\begin{pmatrix} \Phi \\ r \end{pmatrix}(y)dy. \tag{6.16}$$

Therefore, we conclude that (Φ, r) is a mild solution of the Benney-Luke equation (1.1). So, from Theorem 5.2 we obtain existence and uniqueness of mild solutions for (1.1) in the space $\mathcal{V} \times H^1_{\text{per}}([0,T_0])$.

Finally, due to the equivalence between the Cauchy problems associated with the Benney-Luke equation (1.1) and the Boussinesq system (1.10), we easily derived from Theorem 5.1 the following stability result.

Theorem 6.1. *Consider c with $0 < c^2 < \min\{1, a/b\}$ and let $\{\phi_c\}$ be such that $\phi_c' = \psi_c$, where ψ_c is the cnoidal wave of period T_0 given in Theorem 3.1. Then, for c satisfying the conditions in Theorem 4.3, the orbit $\{(\phi_c(\cdot+s), -c\phi_c'(\cdot+s))\}_{s\in\mathbb{R}}$ is stable in $\mathcal{V} \times H^1_{\text{per}}([0,T_0])$ with regard to T_0-periodic perturbations and the flow generated by the Benney-Luke equation (1.1). More precisely, given any $\epsilon > 0$, there is a $\delta = \delta(\epsilon)$ such that if $(u_0, r_0) \in \mathcal{V} \times H^1_{\text{per}}([0,T_0])$ and $\|(u_0, r_0) - (\phi_c, -c\phi_c')\|_{\mathcal{V}\times H^1_{\text{per}}} < \delta$, then*

$$\inf_{s\in\mathbb{R}} \left|\left|(\Phi(t), r(t)) - (\phi_c(\cdot+s), -c\phi_c'(\cdot+s))\right|\right|_{\mathcal{V}\times H^1_{\text{per}}} < \epsilon \tag{6.17}$$

for all t, where $(\Phi(t), r(t))$ is the mild solution (1.1) with initial value (u_0, r_0).

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