Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 40469, 13 pages
doi:10.1155/2007/40469

Research Article

Contra-ω-Continuous and Almost Contra-ω-Continuous

Ahmad Al-Omari and Mohd Salmi Md Noorani

Received 29 May 2007; Accepted 31 July 2007

Recommended by Sehie Park

The notion of contra continuous functions was introduced and investigated by Dontchev. In this paper, we apply the notion of ω-open sets in topological space to present and study a new class of functions called almost contra ω-continuous functions as a new generalization of contra continuity.

1. Introduction

Dontchev [1] introduced the notions of contra continuity and strong S-closedness in topological spaces. He defined a function $f : X \to Y$ is contra continuous if the preimage of every open set of Y is closed in X. A new weaker form of this class of functions called contra semicontinuous function is introduced and investigated by Dontchev and Noiri [2]. Caldas and Jafari [3] have introduced and studied contra β-continuous function. Jafri and Noiri [4, 5] introduced and investigated the notions of contra super continuous, contra precontinuous, and contra α-continuous functions. Almost contra precontinuous functions were introduced by Ekici [6] and recently have been investigated further by Noiri and Popa [7]. Nasef [8] has introduced and studied contra γ-continuous function. In This direction, we will introduce the concept of almost contra ω-continuous functions via the notion of ω-open set and study some properties of contra ω-continuous and almost contra ω-continuous.

All through this paper, (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A is regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$ and A is regular closed if its complement is regular open; equivalently

A is regular closed if $A = \mathrm{Cl}(\mathrm{Int}(A))$, see [9]. Let (X,τ) be a space and let A be a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [10] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X,τ), denoted by τ_ω, forms a topology on X finer than τ. We set $\omega O(X,x) = \{U : x \in U \text{ and } U \in \tau_\omega\}$. The ω-closure and ω-interior, that can be defined in a manner to $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathrm{Cl}_\omega(A)$ and $\mathrm{Int}_\omega(A)$, respectively. Several characterizations and properties of ω-closed subsets were provided in [10–12].

2. Contra ω-continuous

Definition 2.1. A function $f : X \to Y$ is called ω-continuous [12] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq V$.

Definition 2.2. A function $f : X \to Y$ is called contra-ω-continuous (resp., contra-continuous [1]) if $f^{-1}(V)$ is ω-closed (resp., closed) in X for each open set of Y.

Definition 2.3. A function $f : X \to Y$ is said to be almost continuous [13] if $f^{-1}(V)$ is open in X for each regular open set V of Y.

Lemma 2.4 [4]. *The following properties hold for subsets A, B of a space X:*
(1) $x \in \mathrm{Ker}(A)$ *if and only if* $A \cap F \neq \phi$ *for any* $F \in C(X,x)$;
(2) $A \subseteq \mathrm{Ker}(A)$ *and* $A = \mathrm{Ker}(A)$ *if A is open in X*;
(3) *if* $A \subseteq B$, *then* $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(B)$.

Theorem 2.5. *The following are equivalent for a function* $f : X \to Y$:
(1) f *is contra-ω-continuous;*
(2) *for every closed subset F of Y,* $f^{-1}(F) \in \omega O(X)$;
(3) *for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \omega O(X,x)$ such that* $f(U) \subseteq F$;
(4) $f(\mathrm{Cl}_\omega(A)) \subseteq \mathrm{Ker}(f(A))$ *for every subset A of X;*
(5) $\mathrm{Cl}_\omega(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Ker}(B))$ *for every subset B of Y.*

Proof. The implications (1)⇔(2) and (2)⇒(3) are obvious.

(3)⇒(2) Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \omega O(X,x)$ such that $f(U_x) \subseteq F$. Therefore, we obtain $f^{-1}(F) = \cup\{U_x \mid x \in f^{-1}(F)\}$ and $f^{-1}(F)$ is ω-open, since τ_ω is a topological space.

(2)⇒(4) Let A be any subset of X. Suppose that $y \notin \mathrm{Ker}(f(A))$. Then by Lemma 2.4 there exists $F \in C(Y, f(x))$ such that $f(A) \cap F = \phi$. Thus, we have $A \cap f^{-1}(F) = \phi$ and since $f^{-1}(F)$ is ω-open then we have $\mathrm{Cl}_\omega(A) \cap f^{-1}(F) = \phi$. Therefore, we obtain $f(\mathrm{Cl}_\omega(A)) \cap F = \phi$ and $y \notin f(\mathrm{Cl}_\omega(A))$. This implies that $f(\mathrm{Cl}_\omega(A)) \subseteq Ker(f(A))$.

(4)⇒(5) Let B be any subset of Y. By (4) and Lemma 2.4, we have $f(\mathrm{Cl}_\omega(f^{-1}(B))) \subseteq \mathrm{Ker}(f(f^{-1}(B))) \subseteq \mathrm{Ker}(B)$ thus $\mathrm{Cl}_\omega(f^{-1}(B)) \subseteq f^{-1}(\mathrm{Ker}(B))$.

(5)⇒(1) Let V be any open set of Y. Then, by Lemma 2.4 we have $\mathrm{Cl}_\omega(f^{-1}(V)) \subseteq f^{-1}(\mathrm{Ker}(V)) = f^{-1}(V)$ and $\mathrm{Cl}_\omega(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is ω-closed in X. □

The following examples show that contra-ω-continuous and contra-precontinuous functions [4] (resp., contra-semicontinuous [2], contra-α-continuous [5], contra-γ-continuous [8]) are independent notions.

Example 2.6. Let $X = \{a,b\}$ with $\tau = \{X,\phi,\{a\}\}$ and the real number $\mathbb{R}$ with the standard topology, consider the map $f : \mathbb{R} \to X$ defined by $f(x) = b$ if $x \in \mathbb{Q}$ where $\mathbb{Q}$ is the set of all rational numbers and $f(x) = a$ if $x \notin \mathbb{Q}$. Then f is contra-precontinuous but not f contra-ω-continuous since $\{b\}$ is a closed set of (X,τ) and $f^{-1}(\{b\}) = \mathbb{Q}$ is not ω-open. but $\mathbb{Q}$ is preopen set in $\mathbb{R}$.

Example 2.7. Let $X = \{a,b,c\}, \tau = \{\phi,\{a\},\{b\},\{a,b\},X\}$, and $Y = \{1,2\}$ be the Sierpinski space with the topology $\sigma = \{\phi,\{1\},Y\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be defined by $f(a) = 1$ and $f(b) = 2 = f(c)$. Then f is contra ω-continuous but not contra-precontinuous, since $\{2\}$ is a closed set of (Y,σ) and $f^{-1}(\{2\}) = \{c,b\}$ is not preopen (X,τ).

Example 2.8. Let $X = \{a,b,c\}, \tau = \{\phi,\{a\},X\}$, and $\sigma = \{\phi,\{c\},\{b\},\{c,b\},X\}$. Then the identity function $f : (X,\tau) \to (X,\sigma)$ is contra-ω-continuous but not contra-continuous.

Example 2.9. $X = \{a,b\}$ with $\tau = \{X,\phi,\{a\}\}$ and the real number $\mathbb{R}$ with the standard topology, consider the map $f : \mathbb{R} \to X$ defined by $f(x) = b$ if $x \in [0,1)$ and $f(x) = a$ if $x \notin [0,1)$. Then f is contra-semicontinuous but not f contra-ω-continuous since $\{b\}$ is a closed set of (X,τ) and $f^{-1}(\{b\}) = [0,1)$ is not ω-open. but $[0,1)$ is semi-open set in $\mathbb{R}$.

Example 2.10. Let $X = \{a,b\}$ with the indiscrete topology τ and $\sigma = \{\phi,\{a\},X\}$. Then the identity function $f : (X,\tau) \to (X,\sigma)$ is contra ω-continuous but not contra semicontinuous, since $A = \{a\} \in \sigma$ but A is not semiclosed in (X,τ).

Example 2.11. Let $X = \{a,b,c,d\}, \tau = \{\phi,\{b\},\{c\},\{b,c\},\{a,b\},\{a,b,c\},\{b,c,d\},X\}$. Define a function $f : (X,\tau) \to (X,\tau)$ as follows: $f(a) = b, f(b) = a, f(c) = d$, and $f(d) = c$. Then f is contra ω-continuous but not contra α-continuous, since $\{c,d\}$ is a closed set of (x,τ) and $f^{-1}(\{c,d\}) = \{c,d\}$ is not α-open.

$$
\begin{array}{ccccc}
\text{contra-}\omega\text{-continuity} & & & & \\
\Uparrow & & & & \\
\text{contra-continuity} & \Longrightarrow & \text{contra-}\alpha\text{-continuity} & \Rightarrow & \text{contra-percontinuity} \\
 & & \Downarrow & & \Downarrow \\
 & & \text{contra-semicontinuity} & \Rightarrow & \text{contra-}\gamma\text{-continuity} \\
 & & & & \Downarrow \\
 & & & & \text{contra-}\beta\text{-continuity}
\end{array}
\tag{2.1}
$$

Theorem 2.12. *If a function $f : X \to Y$ is contra-ω-continuous and Y is regular, then f is ω-continuous.*

Proof. Let x be an arbitrary point of X and let V be an open set of Y containing $f(x)$; since Y is regular, there exists an open set W in Y containing $f(x)$ such that $\mathrm{Cl}(W) \subseteq V$.

Since f is contra-ω-continuous, so by Theorem 2.5(3) there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq \mathrm{Cl}(W)$. Then $f(U) \subseteq \mathrm{Cl}(W) \subseteq V$. Hence, f is ω-continuous. □

Definition 2.13. A space (X,τ) is said to be ω-space (resp., locally ω-indiscrete) if every ω-open set is open (resp., closed) in X.

For any space (X,τ), we have $\tau \subseteq \tau_\omega$. So the following results follows immediately.

Theorem 2.14. *A function $f:(X,\tau)\to(Y,\sigma)$ is contra-ω-continuous if and only if $f:(X,\tau_\omega)\to(Y,\sigma)$ is contra-continuous.*

Theorem 2.15. *If a function $f:X\to Y$ is contra-ω-continuous and X is ω-space, then f is contra-continuous.*

Theorem 2.16. *Let X be locally ω-indiscrete. If a function $f:X\to Y$ is contra-ω-continuous, then f is continuous.*

Definition 2.17. A function $f:X\to Y$ is called almost-ω-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq \mathrm{Int}_\omega(\mathrm{Cl}(V))$.

Definition 2.18. A function $f:X\to Y$ is said to be pre-ω-open if the image of each ω-open set is ω-open.

Theorem 2.19. *If a function $f:X\to Y$ is a pre-ω-open contra-ω-continuous function, then f is almost ω-continuous.*

Proof. Let x be any arbitrary point of X and V be an open set containing $f(x)$. Since f is contra-ω-continuous, then by Theorem 2.5(3) there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq \mathrm{Cl}(V)$. Since f is pre-ω-open, $f(U)$ is ω-open in Y. Therefore, $f(U) = \mathrm{Int}_w f(U) \subseteq \mathrm{Int}_w(\mathrm{Cl}(f(U))) \subseteq \mathrm{Int}_w(\mathrm{Cl}(V))$. This shows that f is almost ω-continuous. □

Definition 2.20. A function $f:X\to Y$ is said to be almost weakly ω-continuous if for each $x \in X$ and each open V of $f(x)$ there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq \mathrm{Cl}(V)$.

Theorem 2.21. *If a function $f:X\to Y$ is contra-ω-continuous, then f is almost weakly ω-continuous.*

Proof. Let V be any open set of Y. Since $\mathrm{Cl}(V)$ is closed in Y, by Theorem 2.5(3) $f^{-1}(\mathrm{Cl}(V))$ is ω-open in X and set $U = f^{-1}(\mathrm{Cl}(V))$, then we have $f(U) \subseteq \mathrm{Cl}(V)$. This shows that f is almost weakly ω-continuous.

Since the family of all ω-open subsets of a space (X,τ), denoted by τ_ω, forms a topology on X finer than τ, then the ω-frontier of A, where $A \subseteq X$, is defined by $Fr_w(A) = \mathrm{Cl}_w(A) \cap \mathrm{Cl}_w(X-A)$. □

Theorem 2.22. *The set of all points of x of X at which $f:X\to Y$ is not contra-ω-continuous is identical with the union of the ω-frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. Suppose f is not contra-ω-continuous at $x \in X$. There exists $F \in C(Y,f(x))$ such that $f(U) \cap (Y-F) \neq \phi$ for every $U \in \omega O(X,x)$ by Theorem 2.5. This implies that $U \cap f^{-1}(Y-F) \neq \phi$. Therefore, we have $x \in \mathrm{Cl}_w(f^{-1}(Y-F)) = \mathrm{Cl}_w(X - f^{-1}(F))$. However, since $x \in f^{-1}(F) \subseteq \mathrm{Cl}_w(f^{-1}(F))$, thus $x \in \mathrm{Cl}_w(f^{-1}(F)) \cap \mathrm{Cl}_w(f^{-1}(Y-F))$. Therefore, we obtain $x \in Fr_\omega(f^{-1}(F))$. Suppose that $x \in Fr_\omega f(f^{-1}(F))$ for some $F \in C(Y, f(x))$, and f is contra-ω-continuous at x, then there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq F$. Therefore, we have $x \in U \subseteq f^{-1}(F)$ and hence $x \in \mathrm{Int}_\omega(f^{-1}(F)) \subseteq X - Fr_\omega(f^{-1}(F))$. This is a contradiction. This mean that f is not contra-ω-continuous. □

Theorem 2.23. *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra ω-continuous, then f is contra ω-continuous.*

Proof. Let U be an open set in Y, then $X \times U$ is an open set in $X \times Y$. Since g is contra ω-continuous. It follows that $f^{-1}(U) = g^{-1}(X \times U)$ is an ω-closed in X. Thus, f is contra ω-continuous. □

Theorem 2.24. *If $f : X \to Y$ and $g : X \to Y$ are contra ω-continuous and Y is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is ω-closed in X.*

Proof. Let $x \in X - E$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V, g(x) \in W$, and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \phi$. Since f and g is contra ω-continuous, then $f^{-1}(\mathrm{Cl}(V))$ and $g^{-1}(\mathrm{Cl}(W))$ are ω-open sets in X. Let $U = f^{-1}(\mathrm{Cl}(V))$ and $G = g^{-1}(\mathrm{Cl}(W))$. Then U and V are ω-open sets containing x. Set $A = U \cap G$, thus A is ω-open in X. Hence, $f(A) \cap g(A) = f(U \cap G) \cap g(U \cap G) \subseteq f(U) \cap g(G) = \mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \phi$; therefore, $A \cap E = \phi$ and $x \notin \mathrm{Cl}_\omega(E)$. Hence, E is ω-closed in X. □

A subset A of a topological space X is said to be ω-dense in X if $\mathrm{Cl}_w(A) = X$.

Theorem 2.25. *Let $f : X \to Y$ and $g : X \to Y$ be functions. If Y is Urysohn, f and g are contra ω-continuous and $f = g$ on ω-dense set $A \subseteq X$, then $f = g$ on X.*

Proof. Since f and g are contra ω-continuous and Y is Urysohn, by the previous theorem, $E = \{x \in X : f(x) = g(x)\}$ is ω-closed in X. By assumption, we have $f = g$ on ω-dense set $A \subseteq X$. Since $A \subseteq E$ and A is ω-dense set in X, then $X = \mathrm{Cl}_\omega(A) \subseteq \mathrm{Cl}_\omega(E) = E$. Hence, $f = g$ on X. □

Definition 2.26. A space X is called ω-connected provided that X is not the union of two disjoint nonempty ω-open sets.

Theorem 2.27. *If $f : X \to Y$ is a contra ω-continuous function from an ω-connected space X onto any space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper nonempty open and closed subset of Y. Then $f^{-1}(A)$ is a proper nonempty ω-clopen subset of X, which is a contradiction to the fact that X is ω-connected. □

Theorem 2.28. *If $f : X \to Y$ is contra-ω-continuous surjection and X is ω-connected, then Y is connected.*

Proof. Suppose that Y is not connected space. Then there exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is contra-ω-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are ω-open in X. Moreover, $f^{-1}(V_1)$ and

$f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not ω-connected. This is a contradiction. This means that Y is connected. □

Theorem 2.29. *A space X is ω-connected, if every contra-ω-continuous from a space X into any T_0-space Y is constant.*

Proof. Suppose that X is not ω-connected and every contra-ω-continuous function from X into Y is constant. Since X is not ω-connected, there exists a proper nonempty ω-clopen subset A of X. Let $Y = \{a,b\}$ and $\tau = \{Y,\phi,\{a\},\{b\}\}$ be a topology for Y. Let $f : X \to Y$ be a function such that $f(A) = \{a\}$ and $f(X - A) = \{b\}$. Then f is nonconstant and contra-ω-continuous such that Y is T_0 which is a contradiction. Hence, X must be ω-connected. □

Definition 2.30. A space X is said to be ω-T_2 if for each pair of distinct points x and y in X, there exist $U \in \omega O(X,x)$ and $V \in \omega O(X,y)$ such that $U \cap V = \phi$.

Theorem 2.31. *Let X and Y be topological spaces. If*

(1) *for each pair of distinct points x and y in X there exists a function f of X into Y such that $f(x) \neq f(y)$,*
(2) *Y is an Urysohn space,*
(3) *f is contra-ω-continuous at x and y, then X is ω-T_2.*

Proof. let x and y be any distinct points in X. Then, there exists a Urysohn space Y and a function $f : X \to Y$ such that $f(x) \neq f(y)$ and f is contra-ω-continuous at x and y. Let $a = f(x)$ and $b = f(y)$. Then $a \neq b$. Since Y is Urysohn space, there exist open sets V and W containing a and b, respectively, such that $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \phi$. Since f is contra-ω-continuous at x and y, then there exist ω-open sets A and B containing a and b, respectively, such that $f(A) \subseteq \mathrm{Cl}(V)$ and $f(B) \subseteq \mathrm{Cl}(W)$. Then $f(A) \cap f(B) = \phi$, so $A \cap B = \phi$. Hence, X is ω-T_2. □

Corollary 2.32. *Let $f : X \to Y$ be contra-ω-continuous injection. If Y is an Urysohn space, then X is ω-T_2.*

3. Almost contra ω-continuous

In this section, we introduce a new type of continuity called almost contra ω-continuous which is weaker than contra ω-continuous.

Definition 3.1. A function $f : X \to Y$ is said to be almost contra-ω-continuous (resp., almost contra-precontinuous [6]) $f^{-1}(V) \in \omega C(X)$ (resp., $f^{-1}(V) \in PC(X)$) for every $V \in RO(X)$.

Theorem 3.2. *The following are equivalents for a function $f : X \to Y$:*

(1) *f is almost contra-ω-continuous;*
(2) *$f^{-1}(F) \in \omega O(X,x)$ for every $F \in RC(Y)$;*
(3) *for each $x \in X$ and each regular closed set F in Y containing $f(x)$, there exists an ω-open set U in X containing x such that $f(U) \subseteq F$;*
(4) *for each $x \in X$ and each regular open set V in Y noncontaining $f(x)$, there exists an ω-closed set K in X noncontaining x such that $f^{-1}(V) \subseteq K$.*

Proof. (1)$\Leftrightarrow$(2). Let F be any regular closed set of Y. Then $Y-F$ is regular open. By (1), $f^{-1}(Y-F)=X-f^{-1}(F)\in\omega C(X)$. We have $f^{-1}(F)\in\omega O(X)$. The converse is obvious.

(2)$\Rightarrow$(3). Let F be any regular closed set in Y containing $f(x)$. Then by (2) $f^{-1}(F)\in\omega O(X)$ and $x\in f^{-1}(F)$. Take $U=f^{-1}(F)$. Then $f(U)\subseteq F$.

(3)$\Rightarrow$(2). Let F be any regular closed set in Y and $x\in f^{-1}(F)$. From (3) there exists an ω-open U_x in X containing x such that $f(U_x)\subseteq F$, thus $U_x\subseteq f^{-1}(F)$. We have $f^{-1}(F)\subseteq\cup_{x\in f-1(F)}U_x$. This implies that $f^{-1}(F)$is ω-open.

(3)$\Leftrightarrow$(4). Let V be any regular open set in Y noncontaining $f(x)$. Then $Y-V$ is a regular closed set containing $f(x)$. By (3), there exists an ω-open set U in X containing x such that $f(U)\subseteq Y-V$. Hence, $U\subseteq f^{-1}(Y-V)\subseteq X-f^{-1}(V)$ and then $f^{-1}(V)\subseteq X-U$. Take $H=X-U$. We obtain that H is an ω-closed set in X noncontaining x. The converse is obvious. □

The following examples show that almost contra-ω-continuous and almost contra-precontinuous functions are independent notions.

Example 3.3. Let $X=\{a,b,c\}, \tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$. Then $RC(X,\tau)=\{X,\phi,\{b,c\},\{a,c\}\}$ and $\omega O(X,\tau)=\mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X, $PO(X,\tau)=\{X,\phi,\{a\},\{b\},\{a,b\}\}$. Let $f:(X,\tau)\to(X,\tau)$ be the identity map. Then f is almost contra-ω-continuous function which is not almost contra-precontinuous, since $\{a,c\}$ is a regular closed set of (X,τ) and $f^{-1}(\{a,c\})=\{a,c\}\notin PO(X,\tau)$.

Example 3.4. Let $\mathbb{R}$ be the real number with usual topology and $X=\{a,b,c\}$ with $\tau=\{X,\phi,\{a\},\{b\},\{a,b\}\}$, then $RO(X)=\{\phi,X,\{a\},\{b\}\}$. Let $f:\mathbb{R}\to X$ be defined as $f(x)=a$ if $x\in\mathbb{Q}$ and $f(x)=c$ if $x\notin\mathbb{Q}$. Then f is almost contra-precontinuous function which is not almost contra ω-continuous, since $\{a\}$ is a regular closed set in (X,τ) and $f^{-1}(\{a\})=\mathbb{Q}$ which is not ω-open but preopen in $\mathbb{R}$.

$$
\begin{array}{ccccc}
\text{contra-}\omega\text{-continuity} & \Rightarrow & \text{almost contra-}\omega\text{-continuity} & \Rightarrow & \text{almost week-}\omega\text{-continuity}\\
\Uparrow & & & & \Uparrow\\
\text{contra-continuity} & \Longrightarrow & (\theta,s)\text{-continuity} & \Longrightarrow & \text{week-continuity}\\
\Downarrow & & \Downarrow & & \Downarrow\\
\text{contra-percontinuity} & \Rightarrow & \text{almost contra-precontinuity} & \Rightarrow & \text{almost week-continuity}\\
\Downarrow & & \Downarrow & & \Downarrow\\
\text{contra-}\gamma\text{-continuity} & \Rightarrow & \text{almost contra-}\gamma\text{-continuity} & \Rightarrow & \text{almost week-}\gamma\text{-continuity}
\end{array}
\tag{3.1}
$$

A space (X,τ) is anti-locally countable [11] if all nonempty open subsets are uncountable. Note that $\mathbb{R}$ with usual topology is anti-locally countable space.

Lemma 3.5 [11]. *If (X,τ) is an anti-locally countable space, then* $\mathrm{Cl}_\omega(A)=\mathrm{Cl}(A)$ *for every ω-open subset of X and* $\mathrm{Int}(A)=\mathrm{Int}_\omega(A)$ *for every ω-closed subset of X.*

Definition 3.6 [11]. A space (X,τ) is called locally countable, if each point $x \in X$ has a countable open neighborhood.

Lemma 3.7 [11]. *If (X,τ) is a locally countable space, then τ_ω is the discrete topology on X.*

Definition 3.8. A function $f : X \to Y$ is said to be regular set-connected if $f^{-1}(V)$ is clopen in X for each regular open set V of Y.

Theorem 3.9. *Let (X,τ) be an anti-locally countable space, if a function $f : X \to Y$ is almost contra-ω-continuous and almost continuous, then f is regular set-connected.*

Proof. Let V be any regular open set in Y. Since f is almost contra-ω-continuous and contra continuous $f^{-1}(V)$ is ω-closed and open. Thus $\mathrm{Cl}_\omega(f^{-1}(V)) = (f^{-1}(V))$, since (X,τ) be an anti-locally countable space then by Lemma 3.5, we have $\mathrm{Cl}_\omega(f^{-1}(V)) = \mathrm{Cl}(f^{-1}(V))$. Hence $f^{-1}(V)$ is clopen. We obtain that f is regular set-connected. □

Definition 3.10 [14]. A space X is said to be weakly Hausdorff if each element of X is an intersection of regular closed sets.

Definition 3.11. A space X is said to be ω-T_1if for each pair of distinct points x and y of X, there exists ω-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

Theorem 3.12. *If $f : X \to Y$ is an almost contra-ω-continuous injection and Y is weakly Hausdorff, then X is ω-T_1.*

Proof. Suppose that Y is weakly Hausdorff. For any distinct points x and y in X, there exists V, W which are regular closed in Y such that $f(x) \in V, f(y) \notin V, f(x) \notin W$, and $f(y) \in W$. Since f is almost contra-ω-continuous, then $f^{-1}(V)$ and $f^{-1}(W)$ are ω-open subsets of X such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$, and $y \in f^{-1}(W)$. This show that X is ω-T_1. □

Corollary 3.13. *If $f : X \to Y$is an contra-ω-continuous injection and Y is weakly Hausdorff, then X is ω-T_1.*

Theorem 3.14. *If $f : X \to Y$ is almost contra-ω-continuous surjection and X is ω-connected, then Y is connected.*

Proof. Suppose that Y is not connected space. There exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen sets. Thus they are regular open in Y. Since f is almost contra-ω-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are ω-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not ω-connected. This is a contradiction. This means that Y is connected. □

Definition 3.15. A space X is said to be

(1) ω-compact if every ω-open cover of X has a finite subcover;
(2) countably ω- compact if every countable cover of X by ω-open sets has a finite subcover;
(3) ω-Lindelof if every ω-open cover of X has a countable subcover;
(4) S-Lindelof [6] if every cover of X by regular closed sets has a countable subcover;

(5) countably S-closed [15] if every countable cover of X by regular closed sets has a finite subcover;
(6) S-closed [16] if every regular closed cover of X has a finite subcover.

Theorem 3.16. *Let $f : X \to Y$ be an almost contra-ω-continuous surjection. The following statements hold:*
(1) *if X is ω-compact, then Y is S-closed;*
(2) *if X is ω-Lindelof, then Y is S-Lindelof;*
(3) *if X is countably ω-compact, then Y is countably S-closed.*

Proof. We prove only (1). let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra-ω-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an ω-open cover of X and hence there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ therefore we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is S-closed. □

Definition 3.17. A space X is said to be
(1) ω-closed compact if every ω-closed cover of X has a finite subcover;
(2) countably ω-closed conmpact if every countable cover of X by ω-closed sets has a finite subcover;
(3) ω-closed-Lindelof if every cover of X by ω-closed sets has a countable subcover;
(4) nearly compact [17] if every regular open cover of X has a finite subcover;
(5) nearly countably compact [17] if every countable cover of X by regular open sets has a finite subcover;
(6) nearly Lindelof [17] if every cover of X by regular open sets has a countably subcover.

Theorem 3.18. *Let $f : X \to Y$ be an almost contra-ω-continuous surjection. The following statements hold:*
(1) *if X is ω-closed compact, then Y is nearly compact;*
(2) *if X is ω-closed-Lindelof, then Y nearly Lindelof;*
(3) *if X is countably ω-closed compact, then Y is nearly countably compact.*

Proof. We prove only (1). Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra-ω-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an ω-closed cover of X. Since X is ω-closed compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is nearly compact. □

Definition 3.19 [14]. A space X is said to be mildly compact (mildly countably compact, mildly Lindelof) if every clopen cover (resp., clopen countably cover, clopen cover) of X has a finite (resp., a finite, a countable) subcover.

Theorem 3.20. *Let (X,τ) be an anti-locally countable space, if $f : X \to Y$ be an almost contra-ω-continuous and almost continuous surjection and X is mildly compact (resp., mildly countably compact, mildly Lindelof), then Y is nearly compact (resp., nearly countably compact, nearly Lindelof) and S-closed (resp., countably S-closed, S-Lindelof).*

Proof. Let V be any regular closed set on Y. Then since f is almost contra-ω-continuous and almost continuous, then $f^{-1}(V)$ is ω-open and closed in X. By Lemma 3.5, we have $\mathrm{Int}(f^{-1}(V)) = \mathrm{Int}_\omega(f^{-1}(V)) = f^{-1}(V)$. Hence, $f^{-1}(V)$ is clopen. Let $\{V_\alpha : \alpha \in I\}$ be any regular closed (resp., regular open) cover of Y. Then $\{F^{-1}(V_\alpha : \alpha \in I)\}$ is a clopen cover of X and since X is mildly compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. since f is surjection, we obtain $Y = \cup\{V_\alpha : \alpha \in I_0\}$. This shows that Y is S-closed (resp., nearly compact). The other proofs are similar. □

Theorem 3.21. *If $f : X \to Y$ is contra-ω-continuous and A is ω-compact relative to X, then $f(A)$ is strongly S-closed in Y.*

Proof. Let $\{V_i : i \in I\}$ be any cover of $f(A)$, by closed sets of the subspace $f(A)$. For $i \in I$, there exists a closed set A_i of Y such that $V_i = A_i \cap f(A)$. For each $x \in A$, there exists $i(x) \in I$ such that $f(x) \in A_{i(x)}$ and by Theorem 2.5, there exists $U_x \in \omega O(X,x)$ such that $f(U_x) \subseteq A_{i(x)}$. Since the family $\{U_x : x \in A\}$ is a cover of A by ω-open sets of X, there exists a finite subset A_0 of A such that $A \subseteq \cup\{U_x : x \in A_0\}$. Therefore, we obtain $f(A) \subseteq \cup\{f(U_x) : x \in A_0\}$. which is a subset of $\cup\{A_{i(x)} : x \in A_0\}$. Thus $f(A) = \cup\{V_{i(x)} : x \in A_0\}$ and hence $f(A)$is strongly S-closed. □

Corollary 3.22. *If $f : X \to Y$is contra-ω-continuous surjection and X is ω-compacts, then Y is strongly S-closed.*

4. Contra-closed graphs

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be contra-ω-closed if for each $(x,y) \in (X,Y) - G(f)$, there exist $U \in \omega O(X,x)$ and $V \in C(Y,y)$ such that $(U \times V) \cap G(f) = \phi$.

The following results can be easily verified.

Lemma 4.2 [6]. *Let $G(f)$ be the graph of f, for any subset $A \subseteq X$ and $B \subseteq Y$, we have $f(A) \cap B = \phi$ if and only if $(A \times B) \cap G(f) = \phi$.*

Lemma 4.3. *The graph $G(f)$ of $f : X \to Y$ is contra-ω-closed in $X \times Y$ if and only if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in \omega O(X,x)$ and $V \in C(Y,y)$ such that $f(U) \cap V = \phi$.*

Theorem 4.4. *If $f : X \to Y$ is contra-ω-continuous and Y is Urysohn, then $G(f)$ is contra-ω-closed in $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exists open sets V, W such that $f(x) \in V, y \in W$, and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \phi$. Since f is contra-ω-continuous, there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq \mathrm{Cl}(V)$. Therefore, we obtain $f(U) \cap \mathrm{Cl}(W) = \phi$. This shows that $G(f)$ is contra-ω-closed. □

Theorem 4.5. *If $f : X \to Y$ is ω-continuous and Y is T_1, then $G(f)$ is contra-ω-closed in $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exists open set V of Y, such that $f(x) \in V, y \notin V$. Since f is ω-continuous, there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq V$. Therefore, $f(U) \cap (Y - V) = \phi$ and $Y - V \in C(Y,y)$. This shows that $G(f)$ is contra-ω-closed in $X \times Y$. □

Theorem 4.6. *If $f : X \to Y$ has a contra ω-closed graph, then the inverse image of a strongly S-closed set A of Y is ω-closed in X.*

Proof. Assume that A is a strongly S-closed set of Y and $x \notin f^{-1}(A)$. For each $a \in A, (x,a) \notin G(f)$. By Lemma 4.3 there exist $U_a \in \omega(X,x)$ and $V_a \in C(Y,a)$ such that $f(U_a) \cap V_a = \phi$. Since $\{A \cap V_a \mid a \in A\}$ is a closed cover of the subspace A, there exists a finite subset $A_0 \subseteq A$ such that $A \subseteq \cup \{V_a \mid a \in A_0\}$. Set $U = \cap \{U_a \mid a \in A_0\}$, and U is ω-open since τ_ω is topology and $f(U) \cap A = \phi$. Therefore, $U \cap f^{-1}(A) = \phi$; and hence, $x \notin \mathrm{Cl}_\omega(f^{-1}(A))$. This shows that $f^{-1}(A)$ is ω-closed. □

Theorem 4.7. *Let Y be a strongly S-closed space. If a function $f : X \to Y$ has a contra-ω-closed graph, then f is contra ω-continuous.*

Proof. Suppose that Y is strongly S-closed space and $G(f)$ is contra ω-closed. First we show that an open set of Y is strongly S-closed. Let U be an open set of Y and $\{V_i \mid i \in I\}$ be a cover of U by closed sets V_i of U. For each $i \in I$, there exists a closed set K_i of X such that $V_i = K_i \cap U$. Then the family $\{K_i \mid i \in I\} \cup (Y - U)$ is a closed cover of Y. Since Y is strongly S-closed, there exists a finite subset $I_0 \subseteq I$ such that $Y = \cup \{K_i \mid i \in I_0\} \cup (Y - U)$. Therefore, we obtain $U = \cup \{V_i \mid i \in I_0\}$. This shows that U is strongly S-closed. Now for any open set U by Theorem 4.6 $f^{-1}(U)$ is ω-closed in X; therefore, f is contra ω-continuous. □

Definition 4.8. The graph $G(f)$ of a function $f : X \to Y$ is said to be strongly contra-ω-closed if for each $(x,y) \in (X,Y) - G(f)$, there exist $U \in \omega O(X,x)$ and $V \in RC(Y,y)$ such that $(U \times V) \cap G(f) = \phi$.

Lemma 4.9. *The graph $G(f)$ of $f : X \to Y$ is strongly contra-ω-closed graph in $X \times Y$ if and only if for each $(x,y) \in (X \times Y) - G(f)$, there exist $U \in \omega O(X,x)$ and $V \in RC(Y,y)$ such that $f(U) \cap V = \phi$.*

Theorem 4.10. *If $f : X \to Y$ is almost weakly-ω-continuous and Y is Urysohn, then $G(f)$ is strongly contra-ω-closed in $X \times Y$.*

Proof. Suppose that $(x,y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \phi$. Since f is almost weakly-ω-continuous, by Definition 2.20 there exists $U \in \omega(X,x)$ such that $f(U) \subseteq \mathrm{Cl}(W)$. This shows that $f(U) \cap \mathrm{Cl}(V) = f(U) \cap \mathrm{Cl}(\mathrm{Int}(V)) = \phi$, where $\mathrm{Cl}(\mathrm{Int}(V)) \in RC(Y)$ and hence by Lemma 4.9 we have $G(f)$ is strongly contra-ω-closed. □

Theorem 4.11. *If $f : X \to Y$ is almost contra-ω-continuous, then f is almost weakly-ω-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $\mathrm{Cl}(V)$ is a regular closed set of Y containing $f(x)$. Since f is almost contra-ω-continuous, by Theorem 3.2 there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq \mathrm{Cl}(V)$. By Definition 2.20 f is almost weakly-ω-continuous. □

Corollary 4.12. *If $f : X \to Y$ is almost contra-ω-continuous and Y is Urysohn, then $G(f)$ is strongly contra-ω-closed.*

The following result can be easily verified.

Lemma 4.13. *a function $f : X \to Y$ is almost ω-continuous, if and only if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq V$.*

Theorem 4.14. *If $f : X \to Y$ is almost ω-continuous, and Y is Hausdorff, then $G(f)$ is strongly contra-ω-closed.*

Proof. Suppose that $(x,y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $V \cap W = \phi$; hence, $\mathrm{Cl}(V) \cap \mathrm{Int}(\mathrm{Cl}(W)) = \phi$. Since f is almost ω-continuous, and W is regular open by Lemma 4.13 there exists $U \in \omega O(X,x)$ such that $f(U) = W \subseteq \mathrm{Int}(\mathrm{Cl}(W))$. This shows that $f(U) \cap \mathrm{Cl}(V) = \phi$ and hence by Lemma 4.9 we have $G(f)$ is strongly contra-ω-closed. □

We recall that a topological space (X,τ) is said to be extremely disconnected (E.D) if the closure of every open set of X is open in X.

Theorem 4.15. *Let Y be E.D. Then a function $f : X \to Y$is almost contra-ω-continuous if and only if it is almost ω-continuous.*

Proof. Let $x \in X$ and V be any regular open set of Y containing $f(x)$. Since Y is E.D then V is clopen and hence V is regular closed. By Theorem 3.2, there exists $U \in \omega O(X,x)$ such that $f(U) \subseteq V$. Then Lemma 4.13 implies that f is almost ω-continuous. Conversely, let F be any regular closed set of Y. Since Y is E.D, F is also regular open and $f^{-1}(F)$ is ω-open in X. This shows that f is almost contra-ω-continuous. □

Theorem 4.16. *If $f : X \to Y$ is an injective almost contra-ω-continuous function with the strongly contra-ω-closed graph, then (X,τ) is ω-T_2.*

Proof. Let x and y be distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly contra-ω-closed, by Lemma 4.9 there exists $U \in \omega O(X,x)$ and a regular closed set V containing $f(y)$ such that $f(U) \cap V = \phi$. Since f is almost contra-ω-continuous, by Theorem 3.2 there exists $G \in \omega O(X,y)$such that $f(G) \subseteq V$. Therefore, we have $f(U) \cap f(G) = \phi$; hence, $U \cap G = \phi$. This shows that (X,τ) is ω-T_2. □

Acknowledgment

This work is financially supported by the Malaysian Ministry of Science, Technology and Environment, Science Fund Grant no. 04-01-02-SF0177.

References

[1] J. Dontchev, "Contra-continuous functions and strongly *S*-closed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 303–310, 1996.

[2] J. Dontchev and T. Noiri, "Contra-semicontinuous functions," *Mathematica Pannonica*, vol. 10, no. 2, pp. 159–168, 1999.

[3] M. Caldas and S. Jafari, "Some properties of contra-β-continuous functions," *Memoirs of the Faculty of Science Kochi University. Series A. Mathematics*, vol. 22, pp. 19–28, 2001.

[4] S. Jafari and T. Noiri, "On contra-precontinuous functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 25, no. 2, pp. 115–128, 2002.

[5] S. Jafari and T. Noiri, "Contra-α-continuous functions between topological spaces," *Iranian International Journal of Science*, vol. 2, no. 2, pp. 153–167, 2001.

[6] E. Ekici, "Almost contra-precontinuous functions," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 27, no. 1, pp. 53–65, 2004.

[7] T. Noiri and V. Popa, "Some properties of almost contra-precontinuous functions," *Bulletin of the Malaysian Mathematical Sciences Society.*, vol. 28, no. 2, pp. 107–116, 2005.

[8] A. A. Nasef, "Some properties of contra-γ-continuous functions," *Chaos, Solitons & Fractals*, vol. 24, no. 2, pp. 471–477, 2005.

[9] S. Willard, *General Topology*, Addison-Wesley, Reading, Mass, USA, 1970.

[10] H. Z. Hdeib, "ω-closed mappings," *Revista Colombiana de Matemáticas*, vol. 16, no. 1-2, pp. 65–78, 1982.

[11] K. Al-Zoubi and B. Al-Nashef, "The topology of ω-open subsets," *Al-Manarah Journal*, vol. 9, no. 2, pp. 169–179, 2003.

[12] H. Hdeib, "ω-continuous functions," *Dirasat Journal*, vol. 16, no. 2, pp. 136–153, 1989.

[13] T. Noiri, "On almost continuous functions," *Indian Journal of Pure and Applied Mathematics*, vol. 20, no. 6, pp. 571–576, 1989.

[14] T. Soundararajan, "Weakly Hausdorff spaces and the cardinality of topological spaces," in *General Topology and Its Relations to Modern Analysis and Algebra, III (Proc. Conf., Kanpur, 1968)*, pp. 301–306, Academia, Prague, 1971.

[15] K. Dlaska, N. Ergun, and M. Ganster, "Countably *S*-closed spaces," *Mathematica Slovaca*, vol. 44, no. 3, pp. 337–348, 1994.

[16] J. E. Joseph and M. H. Kwack, "On *S*-closed spaces," *Proceedings of the American Mathematical Society*, vol. 80, no. 2, pp. 341–348, 1980.

[17] M. K. Singal and A. Mathur, "On nearly-compact spaces," *Bollettino della Unione Matematica Italiana*, vol. 2, pp. 702–710, 1969.

Ahmad Al-Omari: School of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
Email address: omarimutah1@yahoo.com

Mohd Salmi Md Noorani: School of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, 43600 Bangi, Selangor, Malaysia
Email address: msn@pkrisc.cc.ukm.my