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Research Article

Further Properties of β-Pascu Convex Functions of Order α

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We obtain several further properties of β-Pascu convex functions of order α which were recently introduced and studied by Ali et al. in (2006).

1. Introduction

Let $\mathbb{N} := \{1,2,\dots\}$, for $m,p \in \mathbb{N}$, $m \geq p+1$, let $\mathcal{A}(p,m)$ be the class of all p-valent analytic functions $f(z) = z^p + \sum_{n=m}^{\infty} a_n z^n$ defined on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A} := \mathcal{A}(1,2)$.

Let $\mathcal{T}(p,m)$ be the subclass of $\mathcal{A}(p,m)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=m}^{\infty} a_n z^n, \quad a_n \geq 0 \text{ for } n \geq m, \tag{1.1}$$

and let $\mathcal{T} := \mathcal{T}(1,2)$.

A function $f \in A(p,m)$ is β-Pascu convex function of order α if

$$\frac{1}{p}\operatorname{Re}\left\{\frac{(1-\beta)zf'(z)+(\beta/p)z(zf'(z))'}{(1-\beta)f(z)+(\beta/p)zf'(z)}\right\} > \alpha \quad (\beta \geq 0,\ 0 \leq \alpha < 1). \tag{1.2}$$

We denote by $\mathcal{TPC}(p,m,\alpha,\beta)$ the subclass of $\mathcal{T}(p,m)$ consisting of β-Pascu convex function of order α. Clearly, $\mathcal{TS}^*(\alpha) := \mathcal{TPC}(1,2,\alpha,0)$ is the class of starlike functions with negative coefficients of order α and $\mathcal{TC}(\alpha) := \mathcal{TPC}(1,2,\alpha,1)$ is the class of convex functions with negative coefficients of order α (studied by Silverman [1]).

For the class $\mathcal{TPC}(p,m,\alpha,\beta)$, the following characterization was given by Ali et al. [2].

Lemma 1.1. *Let the function f be defined by (1.1). Then f is in the class $\mathcal{TPC}(p,m,\alpha,\beta)$ if and only if*

$$\sum_{n=m}^{\infty}(n-p\alpha)[(1-\beta)p+\beta n]a_n \leq p^2(1-\alpha). \tag{1.3}$$

The result is sharp.

Lemma 1.2. *Let $f(z)$ be given by (1.1). If $f \in \mathcal{TPC}(p,m,\alpha,\beta)$, then*

$$a_n \leq \frac{p^2(1-\alpha)}{(n-p\alpha)[(1-\beta)p+\beta n]} \tag{1.4}$$

with equality only for functions of the form

$$f_n(z) = z^p - \frac{p^2(1-\alpha)}{(n-p\alpha)[(1-\beta)p+\beta n]}z^n. \tag{1.5}$$

Many interesting properties such as coefficient estimate and distortion theorems for the class $\mathcal{TPC}(p,m,\alpha,\beta)$ were given by Ali et al. [2]. In the present sequel to these earlier works, we will derive several interesting properties and characteristic of the δ-neighborhood associated with the class $\mathcal{TPC}(p,m,\alpha,\beta)$.

2. Integral properties of the class $\mathcal{TPC}(p,m,\alpha,\beta)$

We recall the following definition of integral operator before we give integral properties of the class $\mathcal{TPC}(p,m,\alpha,\beta)$.

Let $\mathcal{I}_c : \mathcal{T}(p,m) \to \mathcal{T}(p,m)$ be *integral operator* defined by $g = \mathcal{I}_c(f)$, where $c \in (-p,\infty)$, $f \in \mathcal{T}(p,m)$ and

$$g(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{2.1}$$

We note that if $f \in \mathcal{T}(p,m)$ is a function of the form (1.1), then

$$g(z) = \mathcal{I}_c(f)(z) = z^p - \sum_{n=m}^{\infty}\frac{c+p}{c+n}a_n z^n. \tag{2.2}$$

Theorem 2.1. *Let $p,m \in \mathbb{N}$, $m \geq p+1$, $\alpha \in [0,1)$, $\beta \in [0,\infty)$, and $c \in (-p,\infty)$. If $f \in \mathcal{TPC}(p,m,\alpha,\beta)$ and $g = \mathcal{I}_c(f)$, then $g \in \mathcal{TPC}(p,m,\lambda,\beta)$, where*

$$\lambda = \lambda(p,m,\alpha,c) = 1 - \frac{(1-\alpha)(c+p)(m-p)}{(m-p\alpha)(c+m)-(1-\alpha)(c+p)p} \tag{2.3}$$

and $\alpha < \lambda$. The result is sharp.

Proof. From Lemma 1.1 and (2.2), we have $g \in \mathcal{TPC}(p,m,\lambda,\beta)$ if and only if

$$\sum_{n=m}^{\infty}\frac{(n-p\lambda)[(1-\beta)p+\beta n](c+p)}{p^2(1-\lambda)(c+n)}a_n \leq 1. \tag{2.4}$$

We find the largest λ such that (2.4) holds. We note that the inequalities

$$\frac{(n-p\lambda)[(1-\beta)p+\beta n](c+p)}{p^2(1-\lambda)(c+n)} \le \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} \tag{2.5}$$

imply (2.4), because $f \in \mathcal{TPC}(p,m,\alpha,\beta)$ and satisfy (1.3). But inequalities (2.5) are equivalent to

$$\frac{(n-p\lambda)(c+p)}{(1-\lambda)(c+n)} \le \frac{(n-p\alpha)}{(1-\alpha)}. \tag{2.6}$$

Since $(n-p\alpha) > p(1-\alpha)$ and $c+n > c+p$, we obtain $\lambda \le \lambda(p,n,\alpha,c)$, where

$$\lambda(p,n,\alpha,c) = \frac{(n-p\alpha)(c+n)-(1-\alpha)(c+p)n}{(n-p\alpha)(c+n)-(1-\alpha)(c+p)p}. \tag{2.7}$$

Now we show that $\lambda(p,n,\alpha,c)$ is an increasing function of n, $n \ge m$. Indeed,

$$\lambda(p,n,\alpha,c) = 1-(1-\alpha)(c+p)E(p,n,\alpha,c), \tag{2.8}$$

where

$$E(p,n,\alpha,c) = \frac{(n-p)}{(n-p\alpha)(c+n)-(1-\alpha)(c+p)p}, \tag{2.9}$$

and $\lambda(p,n,\alpha,c)$ increases when n increases if and only if $E(p,n,\alpha,c)$ is a strictly decreasing function of n.

Let $h(x) = E(p,x,\alpha,c)$, $x \in [m,\infty) \subset [p+1,\infty)$, we have

$$h'(x) = -\frac{(x-p)^2}{\left[(x-p\alpha)(c+x)-(1-\alpha)(c+p)p\right]^2} < 0. \tag{2.10}$$

We obtained

$$\lambda = \lambda(p,m,\alpha,c) \le \lambda(p,n,\alpha,c), \quad n \ge m. \tag{2.11}$$

The result is sharp because

$$\mathcal{I}_c(f_\alpha) = f_\lambda, \tag{2.12}$$

where

$$\begin{aligned} f_\alpha(z) &= z^p - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]}z^m, \\ f_\lambda(z) &= z^p - \frac{p^2(1-\lambda)}{(m-p\lambda)[(1-\beta)p+\beta m]}z^m \end{aligned} \tag{2.13}$$

are extremal functions of $\mathcal{TPC}(p,m,\alpha,\beta)$ and $\mathcal{TPC}(p,m,\lambda,\beta)$, respectively, and $\lambda = \lambda(p,m,\alpha,c)$.

Indeed, we have

$$\mathcal{I}_c(f_\alpha(z)) = z^p - \frac{p^2(1-\alpha)(c+p)}{(m-p\alpha)[(1-\beta)p+\beta m](c+m)} z^m. \tag{2.14}$$

We deduce

$$\frac{p^2(1-\lambda)}{(m-p\lambda)} = \frac{p^2(1-\alpha)(c+p)}{(m-p\alpha)(c+m)}, \tag{2.15}$$

and this implies (2.14).

From $\lambda = 1-(1-\alpha)(c+p)(m-p)/((m-p\alpha)(c+m)-(1-\alpha)(c+p)p)$ we obtain $\lambda < 1$ and also $\lambda > \alpha$. Indeed,

$$\begin{aligned}\lambda - \alpha &= (1-\alpha)\left\{1 - \frac{(c+p)(m-p)}{(m-p\alpha)(c+m)-(1-\alpha)(c+p)p}\right\} \\ &= (1-\alpha)\frac{(m-p\alpha)(m-p)}{(m-p\alpha)(c+m)-(1-\alpha)(c+p)p} > 0.\end{aligned} \tag{2.16}$$

□

3. Integral means inequalities for the class $\mathcal{TPC}(p,m,\alpha,\beta)$

An analytic function g is said to be subordinate to an analytic function f (written $g \prec f$) if $g(z) = f(w(z))$, $z \in \mathbb{U}$, for some analytic function w with $|w(z)| \le |z|$. In 1925, Littlewood [3] proved the following subordination result which will be required in our present investigation.

Lemma 3.1. *If f and g are analytic in $\mathbb{U}$ with $g \prec f$, then*

$$\int_0^{2\pi} |g(re^{i\theta})|^\delta d\theta \le \int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta, \tag{3.1}$$

where $\delta > 0$, $z = re^{i\theta}$, and $0 < r < 1$.

Applying Lemmas 1.1 and 3.1, we prove the following.

Theorem 3.2. *Let $\delta > 0$. If $f \in \mathcal{TPC}(p,m,\alpha,\beta)$ and $f_m(z) = z^p - (p^2(1-\alpha)/(m-p\alpha)[(1-\beta)p+\beta m])z^m$, then for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^\delta d\theta \le \int_0^{2\pi} |f_m(re^{i\theta})|^\delta d\theta. \tag{3.2}$$

Proof. Let

$$\begin{aligned} f(z) &= z^p - \sum_{n=m}^{\infty} a_n z^n, \quad a_n \ge 0,\ n \ge m, \\ f_m(z) &= z^p - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^m, \end{aligned} \tag{3.3}$$

then we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=m}^{\infty} a_n z^{n-p} \right|^{\delta} d\theta \leq \int_0^{2\pi} \left| 1 - \frac{p^2(1-\alpha)}{(m-p\alpha)\left[(1-\beta)p+\beta m\right]} z^{m-p} \right|^{\delta} d\theta. \tag{3.4}$$

By Lemma 3.1, it suffices to show that

$$1 - \sum_{n=m}^{\infty} a_n z^{n-p} \prec 1 - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^{m-p}. \tag{3.5}$$

Set

$$1 - \sum_{n=m}^{\infty} a_n z^{n-p} = 1 - \frac{p^2(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} w(z)^{m-p}. \tag{3.6}$$

From (3.6) and (1.3), we obtain

$$\begin{aligned} |w(z)|^{m-p} &= \left| \frac{(m-p\alpha)[(1-\beta)p+\beta m]}{p^2(1-\alpha)} \right| \left| \sum_{n=m}^{\infty} a_n z^{n-p} \right| \\ &\leq |z^{m-p}| \sum_{n=m}^{\infty} \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} a_n \leq |z^{m-p}| \leq |z|. \end{aligned} \tag{3.7}$$

This completes the proof of the theorem. □

The proof for the first derivative is similar.

Theorem 3.3. *Let $\delta > 0$. If $f \in \mathcal{TPC}(p,m,\alpha,\beta)$ and $f_m(z) = z^p - (p^2(1-\alpha)/(m-p\alpha)[(1-\beta)p+\beta m])z^m$, then for $z = re^{i\theta}$ and $0 < r < 1$,*

$$\int_0^{2\pi} |f'(re^{i\theta})|^{\delta} d\theta \leq \int_0^{2\pi} |f_m'(re^{i\theta})|^{\delta} d\theta. \tag{3.8}$$

Proof. It suffices to show that

$$1 - \sum_{n=m}^{\infty} \frac{n}{p} a_n z^{n-p} \prec 1 - \frac{mp(1-\alpha)}{(m-p\alpha)[(1-\beta)p+\beta m]} z^{m-p}. \tag{3.9}$$

This follows because

$$\begin{aligned} |w(z)|^{m-p} &= \left| \sum_{n=m}^{\infty} \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} a_n z^{n-p} \right| \\ &\leq |z|^{n-p} \sum_{n=m}^{\infty} \frac{(n-p\alpha)[(1-\beta)p+\beta n]}{p^2(1-\alpha)} a_n \leq |z|^{n-p} \leq |z|. \end{aligned} \tag{3.10}$$

□

4. Neighborhoods of the class $\mathcal{TPC}(p,m,\alpha,\beta)$

For $f \in \mathcal{T}(p,m)$ and $\gamma \geq 0$, Frasin [4] defined

$$M_\gamma^q(f) = \left\{g \in \mathcal{T}(p,m) : g(z) = z^p - \sum_{n=m}^{\infty} b_n z^n,\ \sum_{n=m}^{\infty} n^{q+1} \left| a_n - b_n \right| \leq \gamma \right\}, \tag{4.1}$$

which was called q-γ-neighborhood of f. So, for $e(z) = z$, we see that

$$M_\gamma^q(e) = \left\{g \in \mathcal{T}(p,m) : g(z) = z^p - \sum_{n=m}^{\infty} b_n z^n,\ \sum_{n=m}^{\infty} n^{q+1} \left| b_n \right| \leq \gamma \right\}, \tag{4.2}$$

where q is a fixed positive integer. Note that $M_\gamma^0(f) \equiv N_\gamma(f)$ and $M_\gamma^1(f) \equiv M_\gamma(f)$. $N_\gamma(f)$ is called a γ-neighborhood of f by Ruscheweyh [5] and $M_\gamma(f)$ was defined by Silverman [6].

Now, we consider q-γ-neighborhood for function in the class $\mathcal{TPC}(p,m,\alpha,\beta)$.

Theorem 4.1. *Let*

$$\gamma = \frac{m^{q+1} p^2 (1-\alpha)}{(m - p\alpha)[(1-\beta)p + \beta m]}, \tag{4.3}$$

then $\mathcal{TPC}(p,m,\alpha,\beta) \subset M_\gamma^q(e)$.

Proof. If $f \in \mathcal{TPC}(p,m,\alpha,\beta)$, then

$$\sum_{n=m}^{\infty} n^{q+1} a_n \leq \frac{m^{q+1} p^2 (1-\alpha)}{(m - p\alpha)[(1-\beta)p + \beta m]} = \gamma. \tag{4.4}$$

This gives that $\mathcal{TPC}(p,m,\alpha,\beta) \subset M_\gamma^q(e)$. □

Putting $p = 1$, $m = 2$ and $\beta = 0$ in Theorem 4.1, we have the following.

Corollary 4.2. $\mathcal{TS}^*(\alpha) \subset M_\gamma^q(e)$, *where* $\gamma = 2^{q+1}(1-\alpha)/(2-\alpha)$.

Putting $p = 1, m = 2$, and $\beta = 1$ in Theorem 4.1, we have the following.

Corollary 4.3. $\mathcal{TC}(\alpha) \subset M_\gamma^q(e)$, *where* $\gamma = 2^q(1-\alpha)/(2-\alpha)$.

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