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UNIT 1-STABLE RANGE FOR IDEALS

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We investigate necessary and sufficient conditions under which a ring satisfies unit 1-stable range for an ideal. As an application, we prove that R satisfies unit 1-stable range for I if and only if $QM_2(R)$ satisfies unit 1-stable range for $QM_2(I)$.

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Let R be a ring with identity 1. We say that R satisfies unit 1-stable range in case $ax+b=1$ with $a,x,b\in R$ implying that $a+bu\in U(R)$. Many authors studied unit 1-stable range such as those of [1, 2, 3, 4, 5, 6]. Following the authors, a ring R satisfies unit 1-stable range for an ideal I provided that $ax+b=1$ with $a\in I$, $x,b\in R$ implying that $x+ub\in U(R)$ for some unit $u\in U(R)$. Let $R=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ and $I=0\oplus\mathbb{Z}/3\mathbb{Z}$. Then R satisfies unit 1-stable range for I, while in fact R does not satisfy unit 1-stable range. Thus the concept of unit 1-stable range for an ideal is a nontrivial generalization of that of rings satisfying such stable range condition. In this note, we investigate necessary and sufficient conditions under which a ring R satisfies unit 1-stable range for an ideal. It is shown that R satisfies unit 1-stable range for I if and only if $QM_2(R)$ satisfies unit 1-stable range for $QM_2(I)$.

Throughout, all rings are associative with identity. $M_n(R)$ denotes the ring of $n\times n$ matrices over R, $GL_n(R)$ denotes the n-dimensional general linear group of R, $TM_n(R)$ denotes the ring of all $n\times n$ lower triangular matrices over R, and $TM_n(I)$ denotes the ideal of all $n\times n$ lower triangular matrices over I. Clearly, $TM_n(I)$ is an ideal of $TM_n(R)$. We begin with the following.

Lemma 1. *Let I be an ideal of a ring R. Then the following are equivalent:*
(1) *R satisfies unit 1-stable range for I;*
(2) *for any $x\in R$, $y\in I$, there exists $a\in R$ such that $1+xa, y+a\in U(R)$.*

Proof. (1)⇒(2). Given any $x\in R$, $y\in I$, we have $yx+(1-yx)=1$; hence, there exists $u\in U(R)$ such that $x+u(1-yx)\in U(R)$, so $1+(u^{-1}-y)x\in U(R)$. Set $a=u^{-1}-y$. Then $1+ax, y+a\in U(R)$. Clearly, $1+ax\in U(R)$ if and only if $1+xa\in U(R)$. Therefore, $1+xa, y+a\in U(R)$.

(2)⇒(1). Suppose that $ax+b=1$ with $a\in I$, $x,b\in R$. Then we have $r\in R$ such that $a+r, 1+(-x)r\in U(R)$. Set $u=a+r$. We get $1-x(a-u)=1+(-x)r\in U(R)$; hence, $1-(a-u)x\in U(R)$. This infers that $b+ux\in U(R)$, and so $x+u^{-1}b\in U(R)$, as asserted. □

In [3, Theorem 2], the authors proved that if R satisfies unit 1-stable range for I, then so does $M_n(R)$ for $M_n(I)$. Now we give a simple proof of this fact.

Theorem 2. *If R satisfies unit 1-stable range for I, then so does $M_n(R)$ for $M_n(I)$.*

Proof. Assume that $\begin{pmatrix} a_1 & N_1 \\ M_1 & B_1 \end{pmatrix} \in M_n(I), \begin{pmatrix} a_2 & N_2 \\ M_2 & B_2 \end{pmatrix} \in M_n(R)$ with $a_1 \in I$, $a_2 \in R$, $B_1 \in M_{n-1}(I)$, $B_2 \in M_{n-1}(R)$, $N_1 \in M_{1\times(n-1)}(I)$, $N_2 \in M_{1\times(n-1)}(R)$, $M_1 \in M_{(n-1)\times 1}(I)$, and $M_2 \in M_{(n-1)\times 1}(R)$. Using Lemma 1, we can choose $a \in R$ such that $a_1 + a = u_1 \in U(R)$, $1 + a_2 a = v_1 \in U(R)$. Assume that $M_{n-1}(R)$ satisfies unit 1-stable range for $M_{n-1}(I)$. Clearly, $B_1 - M_1 u_1^{-1} N_1 \in M_{n-1}(I)$. So we have $B \in M_{n-1}(R)$ such that $(B_1 - M_1 u_1^{-1} N_1) + B = U_2 \in \mathrm{GL}_{n-1}(R)$ and $I_{n-1} + (-M_2 v_1^{-1} N_2 + B_2)B = V_2 \in \mathrm{GL}_{n-1}(R)$. Hence,

$$\begin{gathered}\begin{pmatrix} a_1 & N_1 \\ M_1 & B_1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} u_1 & N_1 \\ M_1 & B_1 + B \end{pmatrix}, \\ I_n + \begin{pmatrix} a_2 & N_2 \\ M_2 & B_2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} v_1 & N_2 B \\ M_2 a & I_{n-1} + B_2 B \end{pmatrix}.\end{gathered} \tag{1}$$

We check that

$$\begin{gathered}\begin{pmatrix} u_1 & N_1 \\ M_1 & B_1 + B \end{pmatrix}^{-1} = \begin{pmatrix} u_1^{-1} + u_1^{-1} N_1 U_2^{-1} M_1 u_1^{-1} & -u_1^{-1} N_1 U_2^{-1} \\ -U_2^{-1} M_1 u_1^{-1} & U_2^{-1} \end{pmatrix}, \\ \begin{pmatrix} v_1 & N_2 B \\ M_2 a & I_{n-1} + B_2 B \end{pmatrix}^{-1} = \begin{pmatrix} v_1^{-1} + v_1^{-1} N_2 B V_2^{-1} M_2 a v_1^{-1} & -v_1^{-1} N_2 B V_2^{-1} \\ -V_2^{-1} M_2 a v_1^{-1} & V_2^{-1} \end{pmatrix}.\end{gathered} \tag{2}$$

By induction and Lemma 1, $M_n(R)$ satisfies unit 1-stable range for $M_n(I)$. □

Corollary 3. *If R satisfies unit 1-stable range, then so does $M_n(R)$ for all $n \in \mathbb{N}$.*

Proof. By choosing $I = R$ in Theorem 2, we complete the proof. □

Theorem 4. *Let I be an ideal of a ring R. Then the following are equivalent:*
(1) *R satisfies unit 1-stable range for I;*
(2) *$\mathrm{TM}_n(R)$ satisfies unit 1-stable range for $\mathrm{TM}_n(I)$.*

Proof. It suffices to show that the result holds for $n = 2$. Suppose that R satisfies unit 1-stable range for I. Assume that $\begin{pmatrix} a_1 & 0 \\ m_1 & b_1 \end{pmatrix} \in \mathrm{TM}_2(I), \begin{pmatrix} a_2 & 0 \\ m_2 & b_2 \end{pmatrix} \in \mathrm{TM}_2(R)$ with $a_1, b_1, m_1 \in I$, $a_2, b_2, m_2 \in R$. Using Lemma 1, we can choose $a \in R$ such that $a_1 + a = u_1 \in U(R)$, $1 + a_2 a = v_1 \in U(R)$ and we have $b \in R$ such that $b_1 + b = u_2 \in U(R)$ and $1 + b_2 b = v_2 \in U(R)$. Hence,

$$\begin{gathered}\begin{pmatrix} a_1 & 0 \\ m_1 & b_1 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ m_1 & b_1 + b \end{pmatrix}, \\ I_2 + \begin{pmatrix} a_2 & 0 \\ m_2 & b_2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} v_1 & 0 \\ m_2 a & I_2 + b_2 b \end{pmatrix}.\end{gathered} \tag{3}$$

Analogously to Theorem 2, we have

$$\begin{pmatrix} u_1 & 0 \\ m_1 & b_1+b \end{pmatrix}, \begin{pmatrix} v_1 & 0 \\ m_2 a & I_2+b_2 b \end{pmatrix} \in \mathrm{GL}_2(\mathrm{TM}_2(R)). \tag{4}$$

By Lemma 1 again, $\mathrm{TM}_2(I)$ is unit 1-stable.

We now establish the converse. Given any $x \in R$, $y \in I$, we have $\mathrm{diag}(x,1) \in \mathrm{TM}_2(R)$ and $\mathrm{diag}(y,0) \in \mathrm{TM}_2(I)$. So there exists $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in \mathrm{TM}_2(R)$ such that $\mathrm{diag}(1,1)+\mathrm{diag}(x,1)\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$, $\mathrm{diag}(y,0)+\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathrm{TM}_2(R))$. Therefore, $1+xa, y+a \in U(R)$, as required. □

Let I be an ideal of a unital complex C^*-algebra R. By Theorem 4 and [3, Corollary 6], we prove that if every element of I is a sum of a unitary and a unit, then every square lower-triangular matrix over I is a sum of two invertible matrices. Let I be an ideal of a ring R. Define $\mathrm{QM}_2(R) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a+c = b+d,\ a,b,c,d \in R\}$ and $\mathrm{QM}_2(I) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a+c = b+d,\ a,b,c,d \in I\}$. It is easy to verify that $\mathrm{QM}_2(I)$ is an ideal of $\mathrm{QM}_2(R)$.

Corollary 5. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit 1-stable range for I;*

(2) *$\mathrm{QM}_2(R)$ satisfies unit 1-stable for $\mathrm{QM}_2(I)$.*

Proof. (1)⇒(2). We construct a map $\psi : \mathrm{QM}_2(R) \to \mathrm{TM}_2(R)$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \to \left(\begin{smallmatrix} a+c & 0 \\ c & d-c \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{QM}_2(R)$. For any $\left(\begin{smallmatrix} x & 0 \\ z & y \end{smallmatrix}\right) \in \mathrm{TM}_2(R)$, we have $\psi(\left(\begin{smallmatrix} x-z & x-y-z \\ z & y+z \end{smallmatrix}\right)) = \left(\begin{smallmatrix} x & 0 \\ z & y \end{smallmatrix}\right)$. Thus we prove that ψ is a ring isomorphism. Also $\psi\,|_{\mathrm{QM}_2(I)}: \mathrm{QM}_2(I) \cong \mathrm{TM}_2(I)$. Thus we complete the proof by Theorem 4. □

As an immediate consequence of Corollary 5, we prove that R satisfies unit 1-stable range if and only if so does $\mathrm{QM}_2(R)$. This result gives a new kind of rings satisfying unit 1-stable range.

Theorem 6. *Let I be an ideal of a ring R. Then the following are equivalent:*

(1) *R satisfies unit 1-stable range for I;*

(2) *there exists a complete set of idempotents $\{e_1,\dots,e_n\}$ such that all e_iRe_i satisfy unit 1-stable range for e_iIe_i.*

Proof. (1)⇒(2) is clear.

(2)⇒(1). One easily checks that

$$I \cong \begin{pmatrix} e_1Ie_1 & \cdots & e_1Ie_n \\ \vdots & \ddots & \vdots \\ e_nIe_1 & \cdots & e_nIe_n \end{pmatrix}, \qquad R \cong \begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}. \tag{5}$$

By induction, it suffices to prove that the result holds for $n=2$. Assume that $\left(\begin{smallmatrix} a_1 & n_1\\ m_1 & b_1\end{smallmatrix}\right) \in \left(\begin{smallmatrix} e_1Ie_1 & e_1Ie_2\\ e_2Ie_1 & e_2Ie_2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a_2 & n_2\\ m_2 & b_2\end{smallmatrix}\right) \in \left(\begin{smallmatrix} e_1Re_1 & e_1Re_2\\ e_2Re_1 & e_2Re_2\end{smallmatrix}\right)$. According to Lemma 1, we can choose $a\in e_1Re_1$ such that $a_1+a=u_1\in U(e_1Re_1)$, $e_1+a_2a=v_1\in U(e_1Re_1)$. Also we have $b\in e_2Re_2$ such that $(b_1-m_1u_1^{-1}n_1)+b=u_2\in U(e_2Re_2)$ and $e_2+(-m_2v_1^{-1}n_2+b_2)b=v_2\in U(e_2Re_2)$. Hence,

$$\begin{gathered}\begin{pmatrix} a_1 & n_1\\ m_1 & b_1\end{pmatrix}+\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix} u_1 & n_1\\ m_1 & b_1+b\end{pmatrix},\\ \operatorname{diag}(e_1,e_2)+\begin{pmatrix} a_2 & n_2\\ m_2 & b_2\end{pmatrix}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix} v_1 & n_2b\\ m_2a & e_2+b_2b\end{pmatrix}.\end{gathered}\tag{6}$$

Similarly to Theorem 2, we show that

$$\begin{pmatrix} u_1 & n_1\\ m_1 & b_1+b\end{pmatrix},\begin{pmatrix} v_1 & n_2b\\ m_2a & e_2+b_2b\end{pmatrix}\in U\begin{pmatrix} e_1Re_1 & e_1Re_2\\ e_2Re_1 & e_2Re_2\end{pmatrix}.\tag{7}$$

It follows by Lemma 1 that $\left(\begin{smallmatrix} e_1Ie_1 & e_1Ie_2\\ e_2Ie_1 & e_2Ie_2\end{smallmatrix}\right)$ satisfies unit 1-stable range for $\left(\begin{smallmatrix} e_1Ie_1 & e_1Ie_2\\ e_2Ie_1 & e_2Ie_2\end{smallmatrix}\right)$, as required. □

Corollary 7. *Let R be a ring. Then the following are equivalent:*
(1) *R satisfies unit 1-stable range;*
(2) *there exists a complete set of idempotents $\{e_1,\dots,e_n\}$ such that all e_iRe_i satisfy unit 1-stable range.*

Proof. It is an immediate consequence of Theorem 8. □

Recall that I is a regular ideal of R. In case for any $x\in I$, there exists $y\in R$ such that $x=xyx$. We prove that unit 1-stable range condition for an ideal is right and left symmetric for regular ideals.

Theorem 8. *Let I be a regular ideal of a ring R. Then the following are equivalent:*
(1) *R satisfies unit 1-stable for I;*
(2) *for any $x,y\in I$, there exists some $a\in U(R)$ such that $1+xa, y+a\in U(R)$;*
(3) *$ax+b=1$ with $a\in I$, $b\in R$ implying that there exists $u\in U(R)$ such that $a+bu\in U(R)$.*

Proof. (1)⇒(2) is trivial by Lemma 1.

(2)⇒(3). Suppose that $ax+b=1$ with $a\in I$, $b\in R$. By the regularity of I, we have a $c\in R$ such that $a=aca$. Hence, $a(cax)+b=1$. Since $cax\in I$, it follows by Lemma 1 that there exists a $d\in R$ such that $1+ad, cax+d\in U(R)$. Set $u=cax+d$. Then we have $1+ad=1+a(u-cax)=au+b\in U(R)$. Therefore, $a+bu^{-1}\in U(R)$.

(3)⇒(2). For any $x,y\in I$, we have $xy+(1-xy)=1$. By hypothesis, we can find $u\in U(R)$ such that $x+(1-xy)u\in U(R)$. Hence, $xu^{-1}+1-xy\in U(R)$. Set $a=u^{-1}-y$. Then $1+xa, y+a\in U(R)$, as required. □

Corollary 9. *Let I be a regular ideal of a ring R. Then the following are equivalent:*
(1) *R satisfies unit 1-stable range for I;*
(2) *R^{op} satisfies unit 1-stable range for I^{op}.*

Proof. (1)⇒(2). For any $x^{\mathrm{op}}, y^{\mathrm{op}} \in I^{\mathrm{op}}$, we have $x, y \in R$. Hence, there exists some $a \in U(R)$ such that $1+xa, y+a \in U(R)$ by Theorem 8. Clearly, $1+xa \in U(R)$ if and only if $1+ax \in U(R)$. So $1^{\mathrm{op}}+x^{\mathrm{op}}a^{\mathrm{op}}, x^{\mathrm{op}}+a^{\mathrm{op}} \in U(R^{\mathrm{op}})$. Therefore, I^{op} is unit 1-stable by Theorem 8 again.

(2)⇒(1) is proved in the same manner. □

Recall that an ideal I of a ring R has stable rank one in case $aR+bR = R$ with $a \in 1+I$, $b \in R$ implying that there exists $y \in R$ such that $a+by \in U(R)$. It is well known that an ideal I of a regular ring R has stable range one if and only if eRe is unit-regular for all idempotents $e \in I$.

Theorem 10. *Let I be an ideal of a regular ring R. If I has stable range one, then the following are equivalent:*
(1) *R satisfies unit 1-stable range for I;*
(2) *if $e \in I$, $f \in 1+I$ are idempotents such that $eR+fR = R$, then there exist $u, v \in U(R)$ such that $eu+fv = 1$.*

Proof. (1)⇒(2). Suppose that $eR+fR = R$ with idempotents $e \in I$, $f \in 1+I$. Since I has stable rank one, we can find a $y \in R$ such that $ey+f = u \in U(R)$. Hence, $eyu^{-1}+fu^{-1} = 1$. As R satisfies unit 1-stable range for I, there exists $v \in U(R)$ such that $e+fu^{-1}v = w \in U(R)$ by Theorem 8. Hence, $ew^{-1}+fu^{-1}vw^{-1} = 1$, as required.

(2)⇒(1). Given $ax+b = 1$ with $a \in I$, $x, b \in R$, then $b = 1-ax \in 1+I$. Since R is regular, we have $c \in R$ such that $b = bcb$. Clearly, $c \in 1+I$. As I has stable range one, it follows from $bc+(1-bc) = 1$ that $b+(1-bc)y \in U(R)$ for a $y \in R$. Hence, $b+(1-bc)y = u$, and so $b = bcb = bcu$. Similarly, we have $d \in R$ such that $a = ada$. Since $(a+(1-ad))d+(1-ad)(1-d) = 1$ with $a+(1-ad) \in 1+I$, we can find $z \in R$ such that $a+(1-ad)+z(1-ad)(1-d) = v \in U(R)$. This shows that $a = ada = adv$. Let $e = ad$ and $f = bc$. Then $e \in I$ and $f \in 1+I$. Clearly, $eR+fR = R$. By hypothesis, there are $s, t \in U(R)$ such that $es+ft = 1$. Therefore, $av^{-1}s+bu^{-1}t = 1$, and then $a+bu^{-1}ts^{-1}v \in U(R)$. According to Theorem 8, we obtain the result. □

Let I be a bounded ideal of a regular ring R. As a result, we deduce that R satisfies unit 1-stable range for I if and only if $e \in I$, $f \in 1+I$ are idempotents such that $eR+fR = R$; then there exist $u, v \in U(R)$ such that $eu+fv = 1$.

Corollary 11. *Let R be unit-regular. Then the following are equivalent:*
(1) *R satisfies unit 1-stable range;*
(2) *if $e, f \in R$ are idempotents such that $eR+fR = R$, then there exist $u, v \in U(R)$ such that $eu+fv = 1$.*

Proof. It is clear by Theorem 10. □

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