IJMMS 2004:30, 1599–1611
PII. S0161171204305053
http://ijmms.hindawi.com

MINIMIZING ENERGY AMONG HOMOTOPIC MAPS

PENGZI MIAO

Received 6 May 2003

We study an energy minimizing sequence $\{u_i\}$ in a fixed homotopy class of smooth maps from a 3-manifold. After deriving an approximate monotonicity property for $\{u_i\}$ and a continuous version of the Luckhaus lemma (Simon, 1996) on S^2, we show that, passing to a subsequence, $\{u_i\}$ converges strongly in $W^{1,2}$ topology wherever there is small energy concentration.

2000 Mathematics Subject Classification: 53C43, 58E20.

1. Introduction. Let $\phi : M \to N$ be a continuous map between two compact Riemannian manifolds. In general, there may not exist a harmonic map homotopic to ϕ (see [2]). Hence, a map u that minimizes energy among all smooth maps homotopic to ϕ may not exist. However, it is still a basic question to understand the analytical property of a minimizing sequence. If the domain M is a compact surface, it is known to experts that any minimizing sequence that converges weakly indeed converges strongly in $W^{1,2}$ topology away from a finite number of points, where energy concentrates and bubble forms (see [3, 7]). If the domain is of higher dimension, B. White showed that the infimum of the energy functional over the homotopy class of ϕ is determined only by the restriction of ϕ to a 2-skeleton of M (see [8]). It is our goal in this paper to apply White's result to derive a similar theorem for 3-manifolds.

Theorem 1.1. *Let $\phi : M \to N$ be a smooth map between two compact Riemannian manifolds without boundary. Assume that M has dimension 3. Then there exists a constant $\epsilon_0 = \epsilon_0(M,N) > 0$ such that, for any sequence of maps $\{u_i\}$ which minimizes energy among all smooth maps homotopic to ϕ and converges weakly in $W^{1,2}(M,N)$, if*

$$\liminf_{i\to\infty} \frac{1}{\sigma}\int_{B_\sigma(x)} |du_i|^2 dV \le \epsilon_0, \tag{1.1}$$

then $\{u_i\}$ converges strongly in $W^{1,2}(B_{\sigma/4}(x),N)$.

As an application of Theorem 1.1, we prove a partial regularity result for the weak limit of an energy minimizing sequence.

Corollary 1.2. *Let $\phi : M \to N$ be a smooth map between two compact Riemannian manifolds without boundary, where M has dimension 3. Let $\{u_i\}$ be a sequence of maps which minimizes energy among all smooth maps homotopic to ϕ. Suppose that $\{u_i\}$ converges weakly to some $u \in W^{1,2}(M,N)$; then there exists a closed set $\Sigma \subset M$ with finite*

1-dimensional Hausdorff measure such that u is a smooth harmonic map from $M \setminus \Sigma$ to N. In particular, u is a weakly harmonic map from M to N.

We remark that the dimension restriction of the domain space comes only from the lemma proved in Section 4. If a similar lemma could be established on a general sphere $S^{n-1} \subset \mathbb{R}^n$, the rest of the argument in this paper would imply that Theorem 1.1 holds for arbitrary dimension n with the small energy concentration assumption (1.1) replaced by

$$\liminf_{i\to\infty} \frac{1}{\sigma^{n-2}} \int_{B_\sigma(x)} |du_i|^2 dV \le \epsilon_0. \tag{1.2}$$

2. Preliminaries. Let (M^3, g) and (N^m, h) be compact Riemannian manifolds of dimensions 3 and m. We assume that M and N have no boundary. By the Nash embedding theorem, it is convenient to regard N as isometrically embedded in some Euclidean space $\mathbb{R}^K$. We define

$$W^{1,2}(M,N) = \{u \in W^{1,2}(M,\mathbb{R}^K) \mid u(x) \in N \text{ a.e. } x \in M\}, \tag{2.1}$$

where $W^{1,2}(M,\mathbb{R}^K)$ is the separable Hilbert space of maps $u : M \to \mathbb{R}^K$ whose component functions are $W^{1,2}$ Sobolev functions on M. We note that $W^{1,2}(M,N)$ inherits both strong and weak topologies from $W^{1,2}(M,\mathbb{R}^K)$. Moreover, it is a strongly closed set with the property that, for any $C > 0$,

$$\{u \in W^{1,2}(M,N) \mid \|u\|_{W^{1,2}} \le C\} \tag{2.2}$$

is weakly compact in $W^{1,2}(M,N)$ (see [4]).

For any $u \in W^{1,2}(M,N)$, the energy of u is defined by

$$E(u) = \int_M \operatorname{Tr}_g(u^*h)\, dV = \int_M |du^2|\, dV, \tag{2.3}$$

where u^*h is the pullback of h by u and dV is the volume measure determined by g on M.

Let $C^\infty(M,N) \subset W^{1,2}(M,N)$ be the space of smooth maps. For any $\phi \in C^\infty(M,N)$, we define

$$\begin{aligned} \mathcal{F}_\phi &= \{u \in C^\infty(M,N) \mid u \text{ is homotopic to } \phi\}, \\ E_\phi &= \inf\{E(u) \mid u \in \mathcal{F}_\phi\}. \end{aligned} \tag{2.4}$$

The following result, which is due to White [8], gives a fundamental characterization of E_ϕ.

White's theorem. *Let $\mathcal{F}_\phi^{(2)} = \{u \in C^\infty(M,N) \mid u$ is 2-homotopic to $\phi\}$, where two continuous maps v and w are said to be 2-homotopic if their restrictions to the 2-dimensional skeleton of some triangulation of M are homotopic. Then*

$$\inf\{E(u) \mid u \in \mathcal{F}_\phi^{(2)}\} = E_\phi. \tag{2.5}$$

Let $\{u_i\} \subset \mathcal{F}_\phi$ be an arbitrary sequence which minimizes the energy functional, that is,

$$\lim_{i\to\infty} E(u_i) = E_\phi. \tag{2.6}$$

Then the above theorem suggests that $\{u_i\}$ is also a minimizing sequence in $\mathcal{F}_\phi^{(2)}$. This fact is very useful since it allows more competitors to be compared with u_i.

By the weak compactness of bounded sets in $W^{1,2}(M,N)$, we may assume that, passing to a subsequence, $\{u_i\}$ converges weakly in $W^{1,2}(M,N)$, strongly in $L^2(M,N)$, and pointwise almost everywhere to some $u \in W^{1,2}(M,N)$, which has the property that

$$E(u) \le \lim_{i\to\infty} E(u_i) = E_\phi. \tag{2.7}$$

Moreover, by the Riesz representation theorem, we know that there exists a Radon measure μ on M so that

$$|du_i|^2(x)dV \rightharpoonup \mu. \tag{2.8}$$

Throughout the paper, we use $c_1, c_2, c_3, \ldots$ to denote constants depending only on (M,g) and (N,h).

3. Approximate monotonicity of $\{|du_i(x)|^2 dV\}$**.** Given a C^1 vector field X on M, we let $\{F_t\}$ denote the one-parameter group of diffeomorphism on M generated by X. For any $v \in W^{1,2}(M,N)$, we define $E_v(t,X) = E(v \circ F_t)$, where $v \circ F_t(x) = v(F_t(x))$. The first variation formula for the energy functional (see [4]) then gives that

$$\frac{d}{dt}E_v(t,X) = \int_M \left\langle v^*h, -g'(t) + \frac{1}{2}\{\mathrm{Tr}_{g(t)}g'(t)\}g(t)\right\rangle_{g(t)} dV(t), \tag{3.1}$$

where v^*h is the pullback of h by v, $g(t) = F_{-t}^*(g)$, and $dV(t)$ is the volume measure determined by $g(t)$. In particular, at $t = 0$, we have that

$$\frac{d}{dt}E_v(0,X) = \int_M \left\langle v^*h, \mathcal{L}_X g - \frac{1}{2}\{\mathrm{Tr}_g(\mathcal{L}_X g)\}g\right\rangle_g dV. \tag{3.2}$$

The following lemma says that, for large n, u_n is "almost stationary" with respect to a large class of domain variations.

Lemma 3.1. *Given $\Lambda > 0$, let $V_\Lambda = \{C^1$ vector field X with $\|X\|_{C^1} \le \Lambda\}$. Then*

$$\sup_{X\in V_\Lambda}\left\{\frac{d}{dt}E_n(0,X)\right\} \longrightarrow 0 \quad as\ n \longrightarrow \infty, \tag{3.3}$$

where $E_n(t,X) = E_{u_n}(t,X)$.

Proof. Let σ_0 be a sufficiently small positive constant depending only on (M,g) such that for any geodesic ball $B_\sigma(x_0) \subset M$ with $\sigma \le \sigma_0$ and any geodesic normal coordinate chart $\{x^1,x^2,x^3\}$ in $B_\sigma(x_0)$, all the eigenvalues of the matrix $[g_{ij}(x)]_{3\times 3}$ lie in $[1/2,2]$ for each $x \in B_\sigma(x_0)$. With such a choice of σ, we have that

$$\begin{aligned}\int_{B_\sigma(x_0)} |dv|^2 dV &= \sum_{\alpha=1}^{K} \int_{B_\sigma(x_0)} \frac{\partial v^\alpha}{\partial x^i}\frac{\partial v^\alpha}{\partial x^j} g^{ij}(x)\,dV \\ &\geq \frac{1}{2}\sum_{\alpha=1}^{K}\int_{B_\sigma(x_0)}\sum_{i=1}^{3}\left|\frac{\partial v^\alpha}{\partial x^i}\right|^2 (x)\,dx\end{aligned} \tag{3.4}$$

for any $v \in W^{1,2}(M,\mathbb{R}^K)$, where dx denotes the Lebesgue measure in $\mathbb{R}^3$.

To prove the lemma, we first consider $V_{\Lambda,\sigma}$ instead of V_Λ, where $\sigma \leq \sigma_0$ and

$$V_{\Lambda,\sigma} = \{X \in V_\Lambda \mid \text{support}\{X\} \subset B_\sigma(x_0) \text{ for some } x_0 \in M\}. \tag{3.5}$$

For any $X \in V_{\Lambda,\sigma}$, we write

$$\begin{aligned} G(t) &= -g'(t) + \frac{1}{2}\{\mathrm{Tr}_{g(t)}\, g'(t)\} g(t), \\ H^{ij}(t,x) &= G_{kl}(t,x) g^{ik}(t,x) g^{jl}(t,x)\sqrt{\det\left(g_{ij}(t,x)\right)}. \end{aligned} \tag{3.6}$$

It follows from (3.1) that

$$\frac{d}{dt}E_m(t,X) - \frac{d}{dt}E_m(0,X) = \int_{B_\sigma(x_0)} (u_m^* h)_{ij}(x)\{H^{ij}(t,x) - H^{ij}(0,x)\}\,dx, \tag{3.7}$$

where

$$(u_m^* h)_{ij}(x) = \sum_{\alpha=1}^{K} \frac{\partial u_m^\alpha}{\partial x^i}(x)\frac{\partial u_m^\alpha}{\partial x^j}(x). \tag{3.8}$$

Hence,

$$\begin{aligned} &\left|\frac{d}{dt}E_m(t,X) - \frac{d}{dt}E_m(0,X)\right| \\ &\quad \leq 6\sum_{i,j=1}^{3}\left(\sup_{x\in B_\sigma(x_0)} |H^{ij}(t,x) - H^{ij}(0,x)|\right)\cdot\left(\int_{B_\sigma(x_0)} |du_m|^2 dV\right) \end{aligned} \tag{3.9}$$

by the Cauchy-Schwartz inequality and (3.4). We note that $H^{ij}(t,x)$ is a known function of $\{g_{ij}(t,x)\}$ and $\{(d/dt)g_{ij}(t,x)\}$, while $g(t,x) = F_{-t}^* g(x)$ and $(d/dt)g(t,x) = F_{-t}^*(\mathcal{L}_X g)(x)$. Since $\|X\|_{C^1} \leq \Lambda$, it follows from the standard ODE theory that, for any $\epsilon > 0$, there exists t_0 depending only on ϵ, Λ, and g so that, for any $t \in [-t_0, t_0]$, we have that $\|g(t) - g\|_{C^1} \leq \epsilon$, hence $|H^{ij}(t,x) - H^{ij}(0,x)| \leq C\epsilon$ for some constant C depending only on the algebraic expression of H^{ij}.

Now assume that the lemma is not true for $V_{\Lambda,\sigma}$; then there exist $\delta_0 > 0$, a sequence of $\{X_k\} \subset V_{\Lambda,\sigma}$, and a subsequence $\{u_{i_k}\}$ of $\{u_i\}$ such that

$$\left|\frac{d}{dt}E_{i_k}(0,X_k)\right| > \delta_0. \tag{3.10}$$

Our above analysis then shows that there exists $t_0 = t_0(\delta_0, g, \Lambda, E_\phi)$ such that

$$\left| \frac{d}{dt} E_{i_k}(t, X_k) \right| > \frac{1}{2}\delta_0 \quad \forall t \in [-t_0, t_0]. \tag{3.11}$$

Since $\lim_{k\to\infty} E(u_{i_k}) = E_\phi$, we conclude that, for some k large enough and some $t \in [-t_0, t_0]$, $E(u_{i_k} \circ F_t) < E_\phi - (1/4)\delta_0 t_0$, which is a contradiction to the fact $u_{i_k} \circ F_t \in \mathcal{F}_\phi$ and the definition of E_ϕ.

To replace $V_{\Lambda,\sigma}$ by V_Λ, we can simply apply a partition of unity argument considering that $(d/dt)E_n(0, X)$ is linear in X. Hence, the lemma is proved. □

Now we are ready to derive an approximate monotonicity property for $\{u_i\}$. Let $\xi(t)$ be any C^1 decreasing function on $[0, +\infty)$ whose support lies in $[0, 1]$. We fix $x_0 \in M$ and let $\{x^1, x^2, x^3\}$ be a geodesic normal coordinate chart in $B_\sigma(x_0)$. For $0 < \rho < \sigma \le \sigma_0$ and $x \in B_\sigma(x_0)$, we define $X_\rho(x) = \xi(|x|/\rho)x^i(\partial/\partial x^i)$ and view X_ρ as a vector field defined globally on M. It is easily checked that $\|X_\rho\|_{C^1} \le \Lambda$ for some constant $\Lambda = \Lambda(\xi) > 0$. Thus Lemma 3.1 implies that there exists a sequence $\{\kappa_i\}$ depending on $\Lambda(\xi)$ but not on ρ such that

$$\left| \frac{d}{dt} E_i(0, X_\rho) \right| \le \kappa_i, \quad \lim_{i\to\infty} \kappa_i = 0. \tag{3.12}$$

A direct calculation shows that

$$\begin{aligned} \frac{d}{dt} E_i(0, X_\rho) = \operatorname{error}(\rho) &+ (-1) \int_{B_\sigma(x_0)} \xi\left(\frac{|x|}{\rho}\right) |du_i|^2 dV \\ &+ (-1) \int_{B_\sigma(x_0)} \xi'\left(\frac{|x|}{\rho}\right)\left(\frac{|x|}{\rho}\right) |du_i|^2 dV \\ &+ 2 \int_{B_\sigma(x_0)} \xi'\left(\frac{|x|}{\rho}\right)\left(\frac{|x|}{\rho}\right) \left|\frac{\partial u_i}{\partial \nu}\right|^2 dV, \end{aligned} \tag{3.13}$$

where $\nu = (x^i/|x|)(\partial/\partial x^i)$, $|\operatorname{error}(\rho)| \le \bar{c}\rho^2(\int_{B_\sigma(x_0)} |du_i|^2 dV)$, and $\bar{c} = \bar{c}(\xi, g)$. We then define

$$E_i(\rho) = \frac{1}{\rho} \int_{B_\sigma(x_0)} \xi\left(\frac{|x|}{\rho}\right) |du_i|^2 dV. \tag{3.14}$$

It follows from (3.12) and (3.13) that

$$E_i'(\rho) + \bar{c} \int_{B_\sigma(x_0)} |du_i|^2 dV \ge -\kappa_i \frac{1}{\rho^2}, \tag{3.15}$$

which gives that

$$E_i(\tau) \le E_i(\rho) + \bar{c}(\rho - \tau) \int_{B_\sigma(x_0)} |du_i|^2 dV + \kappa_i \left(\frac{1}{\tau} - \frac{1}{\rho}\right) \tag{3.16}$$

for any $0 < \tau < \rho < \sigma$. Hence, we have proved the following proposition.

Proposition 3.2. *For any C^1 decreasing function $\xi(t)$ with support in $[0,1]$, there exists a sequence $\{\kappa_i\}$ such that $\lim_{i\to 0}\kappa_i = 0$ and*

$$\frac{1}{\tau}\int_{B_\sigma(x_0)} \xi\left(\frac{|x|}{\tau}\right)|du_i|^2 dV \le \frac{1}{\rho}\int_{B_\sigma(x_0)} \xi\left(\frac{|x|}{\rho}\right)|du_i|^2 dV + \bar{c}(\rho-\tau)\int_{B_\sigma(x_0)}|du_i|^2 dV + \kappa_i\left(\frac{1}{\tau}-\frac{1}{\rho}\right) \tag{3.17}$$

for any $x_0 \in M$ and any $0<\tau<\rho<\sigma\le\sigma_0$. Here $\{\kappa_i\}$ is independent of ρ and τ, and $\bar{c}$ is a constant depending only on ξ and g.

Letting i go to ∞, we have the following "monotonicity" formula for the limiting measure μ.

Corollary 3.3. *For any C^1 decreasing function $\xi(t)$ with its support in $[0,1]$,*

$$\frac{1}{\tau}\int_{B_\sigma(x_0)} \xi\left(\frac{|x|}{\tau}\right)d\mu \le \frac{1}{\rho}\int_{B_\sigma(x_0)} \xi\left(\frac{|x|}{\rho}\right)d\mu + \bar{c}\mu(B_\sigma(x_0)) \tag{3.18}$$

for any $x_0\in M$, $0<\tau<\rho<\sigma\le\sigma_0$, and some constant $\bar{c}=\bar{c}(\xi,g)$. Choosing ξ to be 1 on $[0,1/2]$,

$$\frac{1}{\tau}\mu(B_\tau(x_0)) \le \frac{2}{\rho}\mu(B_\rho(x_0)) + \bar{c}\mu(B_\sigma(x_0)) \tag{3.19}$$

for any $0<2\tau<\rho<\sigma\le\sigma_0$, where $\bar{c}=\bar{c}(g)$.

As an application of this "monotonicity" property of μ, we show that u can be well approximated by smooth maps into N from the region where $\{u_i\}$ has small energy concentration.

Proposition 3.4. *There exists a number ϵ_1 depending only on M and N such that if $\mu(B_\sigma(x_0))/\sigma < \epsilon_1$, then there exists a sequence of smooth maps $\{u_\tau\}_{0<\tau<\tau_0}$ from $B_{\sigma/2}(x_0)$ to N such that $\lim_{\tau\to 0}\|u_\tau - u\|_{W^{1,2}(B_{\sigma/2}(x_0))} = 0$.*

Proof. We use the idea in [5] to mollify u. Let $\varphi:\mathbb{R}^3\to\mathbb{R}^+$ be a smooth radial mollifying function so that support $(\varphi)\subset B_1$ and $\int_{\mathbb{R}^3}\varphi\,dx = 1$. Assume that $\mu(B_\sigma(x_0))/\sigma < \epsilon_1$ for some ϵ_1 to be determined later; by Corollary 3.3, we have that

$$\frac{\mu(B_\tau(y))}{\tau} \le 4\frac{\mu(B_{\sigma/2}(y))}{\sigma} + \bar{c}\sigma\epsilon_1 \le 4\frac{\mu(B_\sigma(x_0))}{\sigma} + \bar{c}\sigma\epsilon_1 \le 5\epsilon_1 \tag{3.20}$$

for any $y\in B_{\sigma/2}(x_0)$ and $0<2\tau<\sigma/2$ provided $\bar{c}\sigma\le 1$. Now define $u^\tau(y) = (1/\tau^3)\int_{B_\tau(y)}\varphi(|y-z|/\tau)u(z)dz$ inside a normal coordinate chart around x_0; we can apply a version of the Poincare inequality to assert that

$$\frac{1}{\tau^3}\int_{B_\tau(y)}|u(x)-u^\tau(y)|^2 dx \le c_2\frac{1}{\tau}\int_{B_\tau(y)}|du|^2 dx \le c_2\frac{\mu(B_\tau(y))}{\tau}, \tag{3.21}$$

where the last inequality holds because of the lower semicontinuity of energy with respect to weak convergence. It follows from (3.20) and (3.21) that $u^\tau(y)$ lies near many values of $u(z)$ for $z \in B_\tau(y)$. In particular, we see that

$$\operatorname{dist}(u^\tau(y), N) \le c_3 \epsilon_1^{1/2}. \tag{3.22}$$

Let $\mathbb{O}_\varepsilon$ be a ε-tubular neighborhood of N in $\mathbb{R}^K$, and let $\Phi : \mathbb{O}_\varepsilon \to N$ denote the smooth nearest point projection map. We see that if $c_3\epsilon_1^{1/2} < \varepsilon$, then $u^{(\tau)}(y) \in \mathbb{O}_\varepsilon$ for all $y \in B_{\sigma/2}(x_0)$. Hence, we can define a smooth map $u_\tau : B_{\sigma/2}(x_0) \to N$ by $u_\tau(y) = \Phi \circ u^\tau(y)$. Since $u^\tau(y)$ is the standard mollification of u by φ with a scaling factor τ, we see immediately that $\lim_{\tau\to 0} \|u_\tau - u\|_{W^{1,2}(B_{\sigma/2}x_0)} = 0$. □

4. A continuous version of Luckhaus lemma. In this section, we use $\nabla(\cdot)$ to denote the gradient operator on $S^2 \subset \mathbb{R}^3$ and $d\omega$ to denote the Euclidean surface measure on S^2. For a map u defined on a cylinder $[a,b]\times S^2$, we use $\nabla_x u$, $\nabla_t u$ to denote the partial x, t gradient of u, where $(t,x) \in [a,b]\times S^2$. The following technical lemma, which may be viewed as a continuous version of the 2-dimensional Luckhaus lemma (see [6]) in the study of energy minimizing maps, will help us construct comparison maps in the proof of the main theorem.

Lemma 4.1. *Assume that $N \subset \mathbb{R}^K$ is an isometrically embedded compact manifold. Then there exists $\epsilon_2 = \epsilon_2(N) > 0$ such that if $v, w \in W^{1,2}(S^2,N)\cap C^0(S^2,N)$ and*

$$\int_{S^2} |\nabla v|^2 d\omega \le \epsilon_2, \qquad \int_{S^2} |\nabla w|^2 d\omega \le \epsilon_2, \tag{4.1}$$

then for all $\beta > 0$, there exists $\eta = \eta(\epsilon_2,\beta) > 0$, where η does not depend on the choice of v and w, such that if

$$\int_{S^2} |v - w|^2 d\omega < \eta, \tag{4.2}$$

then there exist $\beta' \in [0,\beta)$ and $v' \in W^{1,2}([0,\beta']\times S^2,N)\cap C^0([0,\beta']\times S^2,N)$ with properties that

$$\begin{gathered} v'(0,x) = v(x), \qquad v'(\beta',x) = w(x), \\ \int_{[0,\beta']\times S^2} \left|\nabla_{(t,x)} v'\right|^2 d\omega\, dt \le \beta. \end{gathered} \tag{4.3}$$

Proof. Let $v, w \in W^{1,2}(S^2,N)\cap C^0(S^2,N)$ such that (4.1) holds for some ϵ_2 to be determined later. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ be a smooth radial mollifying function so that $\operatorname{support}(\varphi) \subset B_1$ and $\int_{\mathbb{R}^2} \varphi\, dx = 1$. For any $0 < h \ll \pi/2$ and any $(t,x) \in (0,h]\times S^2$, we define

$$v(t,x) = \int_{S^2} v(y)\varphi^t(\operatorname{dist}(x,y))\, d\omega(y), \tag{4.4}$$

where $\operatorname{dist}(x,y)$ represents the sphere distance between x and y on S^2 and $\varphi^t(r) = (1/t^2)\varphi(r/t)$. Let $\mathbb{O}_{2\varepsilon}$ be a 2ε-tubular neighborhood of N in $\mathbb{R}^K$; by the argument used in the proof of Proposition 3.4, we know that, if we choose $\epsilon_2 = \epsilon_2(N)$ to be sufficiently small, then $v(t,x) \in \mathbb{O}_\varepsilon$ for all $(t,x) \in (0,h]\times S^2$. (We note that the monotonicity of the energy of v, which is crucial in the proof of Proposition 3.4, is automatically satisfied in this case because the domain of v is of 2-dimensional.) Since v is continuous on S^2, we have that

$$\lim_{(t,z)\to(0,x)} v(t,z) = v(x). \tag{4.5}$$

Thus $v(t,x)$ is a continuous map on the closed cylinder $[0,h]\times S^2$ with $v(0,x) = v(x)$. On the other hand, if $B_\sigma(z)$ is a geodesic ball with a normal coordinate chart such that $x \in B_{\sigma/2}(z)$, then for $0 < h = h(\epsilon_2) \ll 1$, we have that

$$v(t,x) = \int_{S^2} v(y)\varphi^t(\operatorname{dist}(x,y))d\omega(y) \approx \int_{B_\sigma(0)\subset\mathbb{R}^2} v(x-ty)\varphi(|y|)dy, \tag{4.6}$$

which implies that

$$\begin{aligned} |\nabla_x v(t,x)|^2 &\le c_4 \int_{S^2} |\nabla v(y)|^2 \varphi^t(\operatorname{dist}(x,y))d\omega(y) + \epsilon_2, \\ |\nabla_t v(t,x)|^2 &\le c_4 \int_{S^2} |\nabla v(y)|^2 \varphi^t(\operatorname{dist}(x,y))d\omega(y) + \epsilon_2 \end{aligned} \tag{4.7}$$

by the Cauchy-Schwartz inequality. Then it follows from (4.7) that

$$\int_{[0,h]\times S^2} |\nabla_{(t,x)} v(t,x)|^2 d\omega(x)dt \le c_5 h\left(\int_{S^2} |\nabla v(y)|^2 d\omega(y) + \epsilon_2\right) \tag{4.8}$$

by the Fubini theorem. Similarly, we define $w(t,x) : [l+h, l+2h]\times S^2 \to \mathbb{O}_\varepsilon$ by

$$w(t,x) = \int_{S^2} w(y)\varphi^{(l+2h-t)}(\operatorname{dist}(x,y))d\omega(y) \tag{4.9}$$

for some l determined later.

Now we want to connect $v(h,x)$ and $w(l+h,x)$ on $[h,l+h]\times S^2$. We first estimate $|v(h,x) - w(l+h,x)|$ pointwise. It follows from the definition and the Cauchy-Schwartz inequality that

$$\begin{aligned} |v(h,x) - w(l+h,x)| &= \int_{S^2} |v(y) - w(y)|\varphi^h(\operatorname{dist}(x,y))d\omega(y) \\ &\le c_6 \frac{1}{h}\left(\int_{S^2} |v(y) - w(y)|^2 d\omega(y)\right)^{1/2}. \end{aligned} \tag{4.10}$$

We define $z(t,x)$ on $[h,l+h]\times S^2$ to be

$$z(t,x) = \left(\frac{t-h}{l}\right)w(l+h,x) + \left(\frac{l+h-t}{l}\right)v(h,x). \tag{4.11}$$

Then (4.7) and (4.10) imply that

$$\begin{aligned} \int_{[h,l+h]\times S^2} |\nabla_{(t,x)} z(t,x)|^2 d\omega(x)dt \le{}& c_7 l\left\{\int_{S^2} [|\nabla v|^2 + |\nabla w|^2]d\omega + 2\epsilon_2\right\} \\ &+ c_7 \frac{1}{l}\frac{1}{h^2}\int_{S^2} |v(y) - w(y)|^2 d\omega(y). \end{aligned} \tag{4.12}$$

Now we consider

$$\tilde{v} = \begin{cases} v(t,x), & 0 \le t \le h, \\ z(t,x), & h \le t \le l+h, \\ w(t,x), & l+h \le t \le l+2h. \end{cases} \tag{4.13}$$

Clearly, $\tilde{v} \in C^0 \cap W^{1,2}([0,l+2h]\times S^2,\mathbb{R}^K)$. Furthermore,

$$\int_{[0,l+2h]\times S^2} |\nabla_{(t,x)}\tilde{v}(t,x)|^2 d\omega\, dt \le c_8 \frac{1}{l}\frac{1}{h^2}\int_{S^2}|v(y)-w(y)|^2 d\omega(y) + c_8(h+l)\epsilon_2. \tag{4.14}$$

For any $\beta > 0$, we first choose $h = h(\beta,\epsilon_2)$ and $l = l(\beta,\epsilon_2)$ such that

$$l+2h < \beta, \qquad c_8(h+l)\epsilon_2 < \frac{\beta}{2}, \tag{4.15}$$

then we let $\|v-w\|_{L^2(S^2)} < \eta$, where $\eta = \eta(l,h,\beta,\epsilon_2)$ is so small that

$$c_8 \frac{1}{l}\frac{1}{h^2}\eta < \frac{\beta}{2}, \qquad c_6 \frac{1}{h}\eta^{1/2} < \varepsilon. \tag{4.16}$$

It follows from (4.10) and (4.14) that $\tilde{v}(t,x) \in \mathbb{O}_{2\varepsilon}$ for all $(t,x) \in [0,l+2h]\times S^2$ and the total energy of $\tilde{v}$ is bounded by β. To get v' finally, we compose $\tilde{v}$ with the nearest point projection map $\Phi : \mathbb{O}_{2\varepsilon} \to N$. Hence, the lemma is proved. □

5. Proof of Theorem 1.1. Throughout this section, we fix a geodesic ball $B_\sigma(x_0)$, where

$$\frac{1}{\sigma}\mu(B_\sigma) = \lim_{i\to\infty}\frac{1}{\sigma}\int_{B_\sigma}|du_i|^2 dV < \epsilon_0 \tag{5.1}$$

for some ϵ_0 to be determined. For each τ, we let B_τ denote $B_\tau(x_0)$.

Assume that $\epsilon_0 < \epsilon_1$; Proposition 3.4 implies that there exists a sequence $\{v_i\} \subset C^\infty(B_{\sigma/2},N)$ such that

$$\lim_{i\to\infty}\|v_i - u\|_{W^{1,2}(B_{\sigma/2})} = 0. \tag{5.2}$$

We then choose $\rho \in (\sigma/4,\sigma/2)$ such that $u|_{\partial B_\rho}, u_i|_{\partial B_\rho} \in W^{1,2}(\partial B_\rho,N)$,

$$\begin{gathered} \lim_{i\to\infty}\|v_i-u\|_{W^{1,2}(\partial B_\rho)} = 0, \\ \lim_{i\to\infty}\|u_i-u\|_{L^2(\partial B_\rho)} = 0, \\ \int_{\partial B_\rho}|du|^2 d\Sigma \le c_9\frac{1}{\sigma}\int_{B_\sigma}|du|^2 dV, \\ \liminf_{i\to\infty}\int_{\partial B_\rho}|du_i|^2 d\Sigma \le c_9 \frac{1}{\sigma}\lim_{i\to\infty}\int_{B_\sigma}|du_i|^2 dV, \end{gathered} \tag{5.3}$$

where $d\Sigma$ is the induced surface measure on $\partial B_\rho \subset M$. After fixing a geodesic normal coordinate chart centered at x_0, we may also view $u|_{\partial B_\rho}, u_i|_{\partial B_\rho}, v_i|_{\partial B_\rho}$ as defined on the Euclidean sphere S_ρ with radius ρ; then we have that

$$\begin{aligned} \int_{S_\rho} |\nabla u|^2 d\omega_\rho &\le c_{10} \int_{\partial B_\rho} |du|^2 d\Sigma, \\ \int_{S_\rho} |\nabla u_i|^2 d\omega_\rho &\le c_{10} \int_{\partial B_\rho} |du_i|^2 d\Sigma, \\ \int_{S_\rho} |\nabla v_i|^2 d\omega_\rho &\le c_{10} \int_{\partial B_\rho} |dv_i|^2 d\Sigma, \end{aligned} \tag{5.4}$$

where $d\omega_\rho$ represents the Euclidean surface measure on S_ρ. Now we choose $\epsilon_0 < \min\{\epsilon_1, (c_9 c_{10})^{-1}\epsilon_2\}$ and we claim that

$$\lim_{i\to\infty} \int_{B_\rho} |du_i|^2 dV = \int_{B_\rho} |du|^2 dV. \tag{5.5}$$

We remark that, once (5.5) is established, it will readily imply that

$$\lim_{i\to\infty} ||u_i - u||_{W^{1,2}(B_\rho)} = 0 \tag{5.6}$$

by the fact that $\{u_i\}$ converges to u weakly and the standard Hilbert space theories.

Assume that (5.5) does not hold; we have that

$$\lim_{i\to\infty} \int_{B_\rho(x_0)} |du_i|^2 dV > \int_{B_\rho(x_0)} |du|^2 dV + 2\delta \tag{5.7}$$

for some $\delta > 0$. The idea for the rest of the proof is to construct a sequence of comparison maps which are almost v_i inside B_ρ and u_i outside B_ρ. For that purpose, we need to connect u_i and v_i on the boundary of B_ρ using Lemma 4.1. We first note that (5.3) imply that there exists a subsequence $\{u_{i_k}\}$, $\{v_{i_k}\}$ such that

$$\begin{gathered} \int_{S_\rho} |\nabla u_{i_k}|^2 d\omega_\rho < \epsilon_2, \quad \int_{S_\rho} |\nabla v_{i_k}|^2 d\omega_\rho < \epsilon_2 \quad \forall k, \\ \lim_{k\to\infty} ||v_{i_k} - u_{i_k}||_{L^2(S_\rho)} = 0. \end{gathered} \tag{5.8}$$

We then consider $\bar{u}_{i_k}(\omega) = u_{i_k}(\rho\omega)$ and $\bar{v}_{i_k}(\omega) = v_{i_k}(\rho\omega)$, where ω denotes the point on S^2. It follows from (5.8) and Lemma 4.1 that for all $\beta > 0$, there exist $k_0 = k_0(\beta, \epsilon_2)$ and $\beta' = \beta'(\beta, \epsilon_2) < \beta$ such that for all $k > k_0$, there exists $\bar{w}_k \in W^{1,2} \cap C^0([0,\beta'] \times S^2, N)$ such that

$$\begin{aligned} \bar{w}_k(0,x) &= \bar{u}_{i_k}(x), \\ \bar{w}_k(\beta',x) &= \bar{v}_{i_k}(x), \\ \int_{[0,\beta']\times S^2} |\nabla_{(t,x)} \bar{w}_k|^2 d\omega\, dt &\le \beta. \end{aligned} \tag{5.9}$$

Next, we use polar coordinates to transplant $\bar{w}_k$ to the shell region between S_ρ and $S_{(1-\beta')\rho}$ by defining $w_k((1-t)\rho\omega) = \bar{w}_k(t,\omega)$. Rescaling $v_{i_k}(x)$ on B_ρ to $v'_{i_k}(x) = v'_{i_k}(r\omega) = v_{i_k}(r\omega/(1-\beta'))$ on $B_{(1-\beta')\rho}$, we then have that

$$\begin{aligned} w_k(x) &= u_{i_k}(x), \quad x \in \partial B_\rho, \\ w_k(x) &= v'_{i_k}(x), \quad x \in \partial B_{(1-\beta')\rho}, \\ \int_{B_\rho \setminus B_{(1-\beta')\rho}} & |dw_k|^2 dV \le c_{10}\rho\beta. \end{aligned} \tag{5.10}$$

Now we consider a new sequence $\{\hat{u}_k\} \subset W^{1,2} \cap C^0(M,N)$ given by

$$\hat{u}_k = \begin{cases} u_{i_k}, & x \notin B_\rho, \\ w_k, & x \in B_\rho \setminus B_{(1-\beta')\rho}, \\ v'_{i_k}, & x \in B_{(1-\beta')\rho}. \end{cases} \tag{5.11}$$

First, we note that the fact that $\hat{u}_k = u_{i_k}$ outside B_ρ and $\pi_2(S^3) = 0$ implies that $\hat{u}_k$ is 2-homotopic to u_{i_k}. Second, we have the following energy estimate:

$$\begin{aligned} E(\hat{u}_k) &= \int_{M\setminus B_\rho} |du_{i_k}|^2 dV + \int_{B_\rho \setminus B_{(1-\beta')\rho}} |dw_k|^2 dV + \int_{B_{(1-\beta')\rho}} |dv'_k|^2 dV \\ &\le E(u_{i_k}) - \int_{B_\rho} |du_{i_k}|^2 dV + c_{10}\rho\beta + c(\beta') \int_{B_\rho} |dv_{i_k}|^2 dV, \end{aligned} \tag{5.12}$$

where $c(\beta')$ is the supremum of the Jacobian of the scaling diffeomorphism from $B_{(1-\beta')\rho}$ to B_ρ, which satisfies $\lim_{\beta' \to 0} c(\beta') = 1$ with the convergence only depending on (M,g). We now fix β such that

$$\beta < (c_{10}\rho)^{-1}\delta, \qquad |c(\beta') - 1| \int_{B_\rho(x_0)} |du|^2 dV < \frac{\delta}{2}. \tag{5.13}$$

Letting $k \to \infty$, we then have that

$$\limsup_{k\to\infty} E(\hat{u}_k) \le E_\phi - \frac{\delta}{2}. \tag{5.14}$$

Finally, we note that the fact that $\hat{u}_k \in W^{1,2} \cap C^0(M,N)$ implies that $\hat{u}_k$ can be well approximated in $W^{1,2}$ norm by smooth maps from M to N which are homotopic to $\hat{u}_k$. One way to see this is to consider the standard mollification of $\hat{u}_k$ into $\mathbb{R}^K$, where the uniform continuity of $\hat{u}_k$ on M will guarantee that the image of the mollification will be inside a tubular neighborhood of N. Composing it with the nearest point projection map, we then have the desired approximation. Hence, we know that there exists another sequence $\{\tilde{u}_k\} \subset C^\infty(M,N)$ such that $\tilde{u}_k$ is homotopic to $\hat{u}_k$ and

$$\limsup_{k\to\infty} E(\tilde{u}_k) = \limsup_{k\to\infty} E(\hat{u}_k) < E_\phi - \frac{\delta}{2}. \tag{5.15}$$

Since $\hat{u}_k$ and u_{i_k} are 2-homotopic, we know that $\{\tilde{u}_k\} \subset \mathcal{F}_\phi^{(2)}$. Thus (5.15) gives a contradiction to the fact that $E_\phi = \inf\{E(u) \mid u \in \mathcal{F}_\phi^{(2)}\}$ by White's theorem. Therefore, (5.5) holds and Theorem 1.1 is proved.

Next, we prove Corollary 1.2 on the partial regularity of the weak limit of $\{u_i\}$. We recall the following ϵ-regularity theorem for stationary harmonic maps obtained by Bethuel [1].

Bethuel's theorem. *There exists a number $\epsilon_3 = \epsilon_3(M,N) > 0$ such that if $u : B_\sigma(x_0) \subset M \to N$ is a stationary harmonic map and $(1/\sigma)\int_{B_\sigma(x_0)} |du|^2 dV \le \epsilon_3$, then u is smooth inside $B_{\sigma/2}(x_0)$.*

Proof of Corollary 1.2. Let $\bar{\epsilon}$ be a number to be determined and let $B_\sigma(x_0)$ be a geodesic ball, where $(1/\sigma)\mu(B_\sigma(x_0)) < \bar{\epsilon}$. Assume that $\bar{\epsilon} < \epsilon_0$; our main theorem implies that $\{u_k\}$ converges strongly to u in $W^{1,2}(B_{\sigma/4}(x_0), N)$. Then it follows from this $W^{1,2}$ strong convergence and the fact that $\{u_i\}$ is a minimizing sequence that u is stationary with respect to both the first and the second variation (see [4]) inside $B_{\sigma/4}(x_0)$, hence $u : B_{\sigma/4}(x_0) \to N$ is a stationary harmonic map. We note that

$$\frac{4}{\sigma}\int_{B_{\sigma/4}(x_0)} |du|^2 dV \le \frac{4}{\sigma}\int_{B_\sigma(x_0)} |du|^2 dV \le \frac{4}{\sigma}\mu(B_\sigma(x_0)). \tag{5.16}$$

Hence, assuming that $\bar{\epsilon} < (1/4)\epsilon_3$ and applying Bethuel's theorem, we know that u is smooth inside $B_{\sigma/8}(x_0)$. With such a choice of $\bar{\epsilon}$, we define

$$\Sigma = \left\{x \in M \mid \lim_{\sigma\to 0}\frac{1}{\sigma}\mu(B_\sigma(x)) \ge \bar{\epsilon}\right\}. \tag{5.17}$$

A standard covering argument (see [4]) then shows that Σ is a closed set with finite 1-dimensional Hausdorff measure. Hence, we conclude that u is a smooth harmonic map from $M\setminus\Sigma$ to N, where Σ is a close set with $\mathcal{H}^1(\Sigma) < \infty$. □

Acknowledgments. The author wants to thank Professor Richard Schoen for suggesting this problem. The author also wants to thank the referees for careful reading of the original manuscript.

References

[1] F. Bethuel, *On the singular set of stationary harmonic maps*, Manuscripta Math. **78** (1993), no. 4, 417–443.

[2] J. Eells and J. C. Wood, *Restrictions on harmonic maps of surfaces*, Topology **15** (1976), no. 3, 263–266.

[3] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of* 2*-spheres*, Ann. of Math. (2) **113** (1981), no. 1, 1–24.

[4] R. M. Schoen, *Analytic aspects of the harmonic map problem*, Seminar on Nonlinear Partial Differential Equations (California, 1983), Math. Sci. Res. Inst. Publ., vol. 2, Springer, New York, 1984, pp. 321–358.

[5] R. M. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differential Geom. **17** (1982), no. 2, 307–335.

[6] L. Simon, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1996, based on lecture notes by Norbert Hungerbühler.

[7] C. Wang, *Bubble phenomena of certain Palais-Smale sequences from surfaces to general targets*, Houston J. Math. **22** (1996), no. 3, 559–590.

[8] B. White, *Infima of energy functionals in homotopy classes of mappings*, J. Differential Geom. **23** (1986), no. 2, 127–142.

Pengzi Miao: Department of Mathematics, Stanford University, Palo Alto, CA 94305, USA
E-mail address: mpengzi@math.stanford.edu