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INGHAM TAUBERIAN THEOREM WITH AN ESTIMATE FOR THE ERROR TERM

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We estimate the error term in the Ingham Tauberian theorem. This estimation of the error term is accomplished by considering an elementary proof of a weak form of Wiener's general Tauberian theorem and by using a zero-free region for the Riemann zeta function.

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1. Introduction. As an important application of his general Tauberian theorem (GTT), in 1932, Wiener [6] gave a new proof of the prime number theorem (PNT). In 1945, Ingham [2] applied Wiener's GTT to formulate a new Tauberian theorem (now bearing his name) and deduced the PNT as a special case. In 1964, Levinson [3] rediscovered Ingham's Tauberian theorem with a different proof. On the other hand, in 1973, Levinson [4] did show that a weak formulation of Wiener's GTT is enough for the proof of the PNT. In 1981, Balog [1] formulated Ingham's Tauberian theorem with an estimate for the error term.

In this paper, we use Levinson's approach to Ingham's theorem, as well as Levinson's approach to Wiener's GTT, to prove the following effective version of Ingham's Tauberian theorem.

Theorem 1.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be a nondecreasing right-continuous function, that is, $F(x+)=F(x)$. Suppose that $F(x)=0$ when $x<1$. Let α be a fixed positive number. Assume that*

$$T(x):=\int_{1-}^{x}\left[\frac{x}{y}\right]dF(y)=x\log x+Ax+O\left(\frac{x}{\log^{\alpha}x}\right),\tag{1.1}$$

as $x\to\infty$, for some constant $A\in\mathbb{R}$. Let $\beta<\alpha/3$. Then (as $x\to\infty$)

$$F(x)=x+O\left(\frac{x}{\log^{\beta}x}\right).\tag{1.2}$$

Balog [1] proves that (1.1) implies (1.2) with $\alpha=2$ and $\beta<1/4$. If the error term in (1.1) is assumed to be only $o(x)$, then the proof of Ingham's theorem would require the full strength of Wiener's GTT.

2. Proof of Theorem 1.1. For the proof of the theorem, we follow Levinson [3, 4] and Ingham [2]. In particular, we have adopted the method in Levinson [4] so that we can obtain an estimate for the error term in the above theorem.

If $k \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then we write $k * g(x) = \int_{-\infty}^{+\infty} k(x-y)g(y)\,dy$.

We notice that if $q, k \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then $(q * k) * g = q * (k * g)$. This follows from Fubini's theorem.

Lemma 2.1. *If $F(x)$ is as in (1.1), then $F(x) = O(x)$.*

Proof. Let $T(x)$ be as in (1.1). Since $[2y] - 2[y] = 0$ or 1 and $F(x)$ is nondecreasing, then

$$T(x) - 2T\left(\frac{x}{2}\right) \geq \int_{x/2}^{x} \left[\frac{x}{y}\right] dF(y) = F(x) - F\left(\frac{x}{2}\right). \tag{2.1}$$

On the other hand,

$$T(x) - 2T\left(\frac{x}{2}\right) = x \log x - 2\frac{x}{2} \log \frac{x}{2} + O(x) = O(x). \tag{2.2}$$

Let $M > 0$ be such that $F(x) - F(x/2) \leq xM$. Then we have

$$F(x) = \sum_{j=0}^{\infty} F\left(\frac{x}{2^j}\right) - F\left(\frac{x}{2^{j+1}}\right) \leq \sum_{j=0}^{\infty} x \frac{M}{2^j} = 2Mx. \tag{2.3}$$

□

Lemma 2.2. *Let $F(x)$ be as in the statement of Theorem 1.1. For every $x \in \mathbb{R}$, let*

$$g(x) = \frac{F(e^x)}{e^x} \in L^\infty(\mathbb{R}). \tag{2.4}$$

Then there exists a function $k \in L^1(\mathbb{R})$ satisfying

$$k * g(x) = \int_{-\infty}^{+\infty} k(x-y)g(y)\,dy = 1 + O\left(\frac{1}{x^\alpha}\right) \tag{2.5}$$

and also

$$K(u) := \int_{-\infty}^{+\infty} k(x)e^{-iux}\,dx = \frac{2iu\zeta(1+iu)}{(1+iu)(2+iu)}, \tag{2.6}$$

where $\zeta(1+iu)$ is the Riemann zeta function. Therefore, $K(u) \neq 0$ for all $u \in \mathbb{R}$ and in particular $K(0) = 1$.

Proof. Let $k_0(x) = [x] - x + 1/2$ when $x \geq 1$ and let $k_0(x) = 0$ when $x < 1$. Then

$$\begin{aligned} x \log x + Ax + O\left(\frac{x}{\log^\alpha x}\right) &= \int_{1^-}^{x} k_0\left(\frac{x}{y}\right) dF(y) + \int_{1^-}^{x} \frac{x}{y}\, dF(y) - \frac{1}{2}F(x) \\ &= \int_{1^-}^{x} k_0\left(\frac{x}{y}\right) dF(x) + h(x), \end{aligned} \tag{2.7}$$

where, integrating by parts, $h(x) := x\int_1^x (F(y)/y^2)\,dy + (1/2)F(x)$. Hence we have

$$h(x) = x\log x + Ax - \int_{1^-}^{x} k_0\left(\frac{x}{y}\right)dF(y) + O\left(\frac{x}{\log^{\alpha} x}\right). \tag{2.8}$$

From the definition of $h(x)$, one can show by taking derivatives that

$$\frac{1}{2}x^2\int_1^x \frac{F(y)}{y^2}dy = \int_1^x h(y)\,dy. \tag{2.9}$$

Thus we can write

$$F(x) = 2h(x) - 2x\int_1^x \frac{F(y)}{y^2}dy = 2h(x) - \frac{4}{x}\int_1^x h(y)\,dy. \tag{2.10}$$

From (2.8) and (2.10), we obtain

$$\begin{aligned} F(x) &= 2x\log x + 2Ax - 2\int_{1^-}^{x} k_0\left(\frac{x}{y}\right)dF(y) \\ &\quad - \frac{4}{x}\left\{\frac{1}{2}x^2\log x - \frac{1-2A}{4}x^2 - \int_1^x\int_{1^-}^{y} k_0\left(\frac{y}{t}\right)dF(t)\,dy\right\} + O\left(\frac{x}{\log^{\alpha} x}\right). \end{aligned} \tag{2.11}$$

Therefore,

$$\begin{aligned} F(x) &= x - 2\int_{1^-}^{x} k_0\left(\frac{x}{y}\right)dF(y) + \frac{4}{x}\int_{1^-}^{x}\int_t^x k_0\left(\frac{y}{t}\right)dy\,dF(t) + O\left(\frac{x}{\log^{\alpha} x}\right) \\ &= x - 2\int_1^x k_0\left(\frac{x}{y}\right)dF(y) + \frac{4}{x}\int_{1^-}^{x} t\int_1^{x/t} k_0(y)\,dy\,dF(t) + O\left(\frac{x}{\log^{\alpha} x}\right). \end{aligned} \tag{2.12}$$

If we let $k_1(x) = 1 + 2k_0(x) - (4/x)\int_1^x k_0(y)\,dy$ when $x \ge 1$ and $k_1(x) = 0$ otherwise, then we can write

$$\int_{1^-}^{x} k_1\left(\frac{x}{y}\right)dF(y) = x + O\left(\frac{x}{\log^{\alpha} x}\right). \tag{2.13}$$

In this equation, we make the substitution $x \mapsto t$, then we multiply by $1/t$, and finally we integrate from $t = 1$ to $t = x$. We get

$$\int_{1^-}^{x}\frac{1}{t}\int_1^t k_1\left(\frac{t}{y}\right)dF(y)\,dt = \int_{1^-}^{x}\int_y^x k_1\left(\frac{t}{y}\right)\frac{dt}{t}dF(y) = \int_{1^-}^{x}\int_1^{x/y}\frac{k_1(t)}{t}dt\,dF(y). \tag{2.14}$$

If we integrate the last expression by parts, then formula (2.13) becomes

$$\int_1^x \frac{F(y)}{y}k_1\left(\frac{x}{y}\right)dy = x + O\left(\frac{x}{\log^{\alpha} x}\right). \tag{2.15}$$

Since $F(y) = 0$ for $y < 1$ and $k_1(x) = 0$ for $x < 1$, then we can make the substitutions $x \mapsto e^x$ and $y \mapsto e^y$ to obtain

$$\int_{-\infty}^{+\infty} F(e^y)k_1(e^{x-y})\,dy = e^x + O\Big(\frac{e^x}{x^\alpha}\Big). \tag{2.16}$$

Now we write

$$k(x) = \frac{k_1(e^x)}{e^x}. \tag{2.17}$$

Since $k_0(x)$ and $k_1(x)$ are bounded, then $k \in L^1(\mathbb{R})$. The first assertion in Lemma 2.2 follows from (2.16). Now we prove the second assertion. Since $k_1(x) = 0$ when $x < 1$, then

$$K(u) = \int_0^\infty \frac{k_1(e^x)}{e^x} e^{-iux}\,dx = \int_1^\infty \frac{k_1(x)}{x^{s+1}}\,dx, \tag{2.18}$$

where $s = 1 + iu$. Recall that $k_1(x) = 1 + 2k_0(x) - (4/x)\int_1^x k_0(y)\,dy$ and that $k_0(x) = [x] - x + 1/2$. It is a well-known fact that

$$\int_1^\infty \frac{k_0(x)}{x^{s+1}}\,dx = \frac{\zeta(s)}{s} - \frac{1}{s(s-1)} - \frac{1}{2s}. \tag{2.19}$$

Thus, changing the order of integrals, one shows that

$$\int_1^\infty \frac{1}{x^{s+1}} \cdot \frac{1}{x}\int_1^x k_0(y)\,dy\,dx = \frac{\zeta(s)}{s(s+1)} - \frac{1}{2s(s-1)}. \tag{2.20}$$

Adding $\int_1^\infty (1/x^{s+1})\,dx = 1/s$ to two times (2.19) minus four times (2.20), we obtain

$$K(u) = \frac{2(s-1)\zeta(s)}{s(s+1)}, \tag{2.21}$$

which is as claimed in the lemma. □

Lemma 2.3. *For $\ell > 0$, let*

$$\delta_\ell(x) = \frac{1}{2}\sqrt{\frac{\ell}{\pi}}\,e^{-(\ell/4)x^2}, \qquad \Delta_\ell(t) = \int_{-\infty}^{+\infty} \delta_\ell(x)e^{-ixt}\,dx. \tag{2.22}$$

Then $\Delta_\ell(t) = e^{-(1/\ell)t^2}$ and in particular $\Delta_\ell(0) = 1$.

Lemma 2.4. *Let $k(x)$ be as in Lemma 2.2 and δ_ℓ as in Lemma 2.3. Then there exists a function $q_\ell \in L^1(\mathbb{R})$ such that $\delta_\ell = q_\ell * k$. Let $n \in \mathbb{N}$. If ϵ_1 is a fixed positive number, however small, then (as $x \to \infty$)*

$$q_\ell(x) \ll \frac{n!}{x^n}\ell^{1+\epsilon_1 n}\Gamma(2\epsilon_1 n + 1). \tag{2.23}$$

Proof. Let $k(x)$ be as in Lemma 2.2. Let

$$q_\ell(x) = \int_{-\infty}^{+\infty} \frac{\Delta_\ell(u)}{K(u)} e^{ixu} du, \quad K(u) = \int_{-\infty}^{+\infty} k(t) e^{-iut} dt. \tag{2.24}$$

That the integral defining q_ℓ does exist follows from (2.28) and (2.31). Then we have

$$\begin{aligned} q_\ell * k(x) &= \int_{-\infty}^{+\infty} k(t) \int_{-\infty}^{+\infty} \frac{\Delta_\ell(u)}{K(u)} e^{i(x-t)u} du\, dt \\ &= \int_{-\infty}^{+\infty} \frac{\Delta_\ell(u)}{K(u)} e^{ixu} \int_{-\infty}^{+\infty} k(t) e^{-itu} dt\, du = \delta_\ell(x). \end{aligned} \tag{2.25}$$

We can integrate n times by parts to obtain

$$q_\ell(x) = \left(\frac{i}{x}\right)^n \int_{-\infty}^{+\infty} \left(\frac{\Delta_\ell(u)}{K(u)}\right)^{(n)} e^{ixu} du. \tag{2.26}$$

Now we must show that

$$I := \int_{-\infty}^{+\infty} \left| \left(\frac{\Delta_\ell(u)}{K(u)}\right)^{(n)} \right| du \ll n!\ell^{1+\epsilon_1 n} \Gamma(2\epsilon_1 n + 1). \tag{2.27}$$

Let $s = 1 + iu$. Then

$$Z(s) := \frac{s(s+1)}{2(s-1)\zeta(s)} e^{(1/\ell)(s-1)^2} = \frac{\Delta_\ell(u)}{K(u)} \tag{2.28}$$

is an analytic function of s in the region (see [5, Theorem 15, page 157])

$$s = \sigma + it \quad \text{such that } \sigma \geq 1 - \frac{c}{\log|t|}, \tag{2.29}$$

where c is a suitable positive real number. Therefore,

$$\frac{d^n}{ds^n} Z(s) = \frac{n!}{2\pi i} \int Z(s+\xi) \frac{d\xi}{\xi^{n+1}}, \tag{2.30}$$

where the integral is over the small circle $|\xi| = c/2\log|u|$. Since (see [5, Theorem 16, page 158])

$$Z(s) \ll u e^{-(1/\ell)u^2} \log u \quad (\text{as } u \longrightarrow \infty), \tag{2.31}$$

then we also have

$$\left(\frac{\Delta_\ell(u)}{K(u)}\right)^{(n)} \ll n! u e^{-(1/\ell)u^2} \log^{n+1} u. \tag{2.32}$$

If ϵ_1 is a positive number, then we have

$$\begin{aligned}\frac{I}{n!} &\ll \int_0^\infty u e^{-(1/\ell)u^2} \log^{n+1}(3+u)\,du \\ &\ll \ell \int_0^\infty u e^{-u^2} \log^{n+1}(3+\sqrt{\ell}u)\,du \\ &\ll \ell \int_0^\infty u e^{-u^2} (\sqrt{\ell}u)^{2\epsilon_1 n}\,du \\ &= \ell^{1+\epsilon_1 n} \int_0^\infty u^{2\epsilon_1 n} e^{-u}\,du \\ &= \ell^{1+\epsilon_1 n}\Gamma(2\epsilon_1 n+1).\end{aligned} \tag{2.33}$$

□

Proof of Theorem 1.1. We apply Lemma 2.4 with $n+1 \in \mathbb{N}$ and ϵ_1 small. Let $\phi = g-1$. Then we have

$$\delta_\ell * \phi(x) = q_\ell * k * \phi(x) = q_\ell * h(x), \tag{2.34}$$

where $h(x) = k * \phi(x) = O(1/x^\alpha)$, as it follows from (2.5). Since

$$\begin{aligned}q_\ell * h(x) &= \left\{\int_{-\infty}^{x/2} + \int_{x/2}^{\infty}\right\} q_\ell(x-t)h(t)\,dt \\ &\ll \sup_{t\in\mathbb{R}} |h(t)| \int_{x/2}^{\infty} |q_\ell(t)|\,dt + \sup_{t>x/2} |h(t)| \int_{-\infty}^{+\infty} |q_\ell(t)|\,dt,\end{aligned} \tag{2.35}$$

then we see that $\delta_\ell * \phi(x) \ll \ell^\tau(1/x^\alpha + 1/x^n)$ as $x \to \infty$ holds with any constant $\tau > 1$. Letting $n = [\alpha]+1$, we obtain

$$\delta_\ell * g(x) = 1 + O\left(\frac{\ell^\tau}{x^\alpha}\right). \tag{2.36}$$

Let ϵ be real and positive. Since $e^t g(t) = F(e^t)$ is nondecreasing and nonnegative, then $x-\epsilon \le t \le x+\epsilon$ implies

$$e^{x-\epsilon} g(x-\epsilon) \le e^t g(t) \le e^{x+\epsilon} g(t) \tag{2.37}$$

so that $x-\epsilon \le t \le x+\epsilon$ implies

$$e^{-2\epsilon} g(x-\epsilon) \le g(t). \tag{2.38}$$

Therefore, we have

$$e^{-2\epsilon} g(x-\epsilon) \int_{-\epsilon}^{+\epsilon} \delta_\ell(t)\,dt \le \int_{x-\epsilon}^{x+\epsilon} g(t)\delta_\ell(x-t)\,dt. \tag{2.39}$$

Now we would like to extend these integrals from the finite range $|x-t| \le \epsilon$ to the whole real line. From Lemma 2.1, we know that $g(t)$ is a bounded function. Thus

$$\int_{|t|\ge\epsilon} g(x-t)\delta_\ell(t)\,dt \ll \sqrt{\ell}\int_\epsilon^\infty e^{-(\ell/4)x^2}\,dx \ll e^{-(\ell/4)\epsilon^2}, \tag{2.40}$$

where the implied constant is independent of x. Hence,

$$e^{-2\epsilon}g(x-\epsilon) \le \delta_\ell * g(x) + O\left(e^{-(\ell/4)\epsilon^2}\right). \tag{2.41}$$

This inequality, together with (2.36), implies

$$g(x) \le 1 + O\left(\epsilon + \frac{\ell\tau}{x^\alpha} + e^{-(\ell/4)\epsilon^2}\right). \tag{2.42}$$

Let $\beta < \alpha/3$. Letting $\epsilon = x^{-\beta}$ and $\ell\tau = x^{\alpha-\beta}$, we obtain the right-hand-side inequality of

$$1 + O\left(\frac{1}{x^\beta}\right) \le g(x) \le 1 + O\left(\frac{1}{x^\beta}\right). \tag{2.43}$$

One can prove the left-hand-side inequality in a similar fashion. Recalling that $g(x) = F(e^x)/e^x$, we now have that

$$F(x) = x + O\left(\frac{x}{\log^\beta x}\right) \tag{2.44}$$

holds with $\beta < \alpha/3$. □

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