Internat. J. Math. & Math. Sci.
VOL. 19 NO. 2 (1996) 263-266

ON NON-PARALLEL s-STRUCTURES

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(Received October 13, 1992)

ABSTRACT. Using algebraic topology, we find out the number of all non-parallel s-structures which an n-dimensional Euclidean space E^n admits The obtaining results are generalized on a manifold M which is CW-complex

KEY WORDS AND PHRASES. Non-parallel s-structure, CW-complex
1991 AMS SUBJECT CLASSIFICATION CODES. 53C15, 53C35, 57N65

0. INTRODUCTION

Let E^n be an n-dimensional Euclidean space and $f : E^n \to E^n$ an automorphism

If $x_0 \in E^n$ then the expression $f(x_0) + Df_{x_0}(x - x_0)$ is the linear approximation of f at x_0

We assume that x_0 is a fixed point of f and the Jacobian matrix Df is an orthogonal matrix Then, if in a closed neighborhood of x_0 (under the usual topology) there is no other fixed point, f is called s-symmetry on x_0 and it is written

$$f_{x_0}(x) = x_0 + A_{x_0}(x - x_0),$$

where the Jacobian A_{x_0} belongs to $0(n) - \{I\}$, (if $A_{x_0} = I$, then every point of E^n will be a fixed point)

A family $\{f_{x_0} : x_0 \in E^n\}$ of s-symmetries is called an s-structure on E^n

An s-structure is called regular if $f_{x_0} \bullet f_{\psi_0} = f_{u_0} \bullet f_{x_0}$, where $u_0 = f_{x_0}(\psi_0)$

An s-structure is called parallel if A_{x_0} is constant i e does not depend on x_0 It is clear that a parallel s-structure is also a regular one If f_0 is an orthogonal transformation at the origin without fixed vectors and t_{x_0} is a translation on E^n such that $t_{x_0}(0) = x_0$ then

$$f_{x_0} = t_{x_0} \circ f_0 \circ t_{x_0}^{-1}$$

are the only parallel s-structures on E^n

Therefore the following question arises Do there exist non-parallel regular s-structures on E^n?

O Kowalski in [1] proved that the Euclidean spaces E^2, E^3 and E^4 admit only parallel regular s-structures and found out a non-parallel regular s-structure on E^5

S Wegrzynowski in [2] obtained the same results using analytical calculations on Lie algebras

In the present paper, we will give a complete classification of the Euclidean spaces of arbitrary dimension admitting non-parallel s-structures and we will give the number of these ones as well Finally we will generalize the meaning of an s-structure on every manifold which is a CW-complex and we will solve the analogous problem on these manifolds We will prove that the number of the non-parallel s-structures is a dimensional-invariant

1. Euclidean Spaces

In the present paragraph we will prove the following

THEOREM 1. In an n-dimensional Euclidean space E^n the number N of the non-parallel s-structures is given by

$$N = \begin{cases} 0 & \text{for} \quad n = 2, 3, 4 \ , \\ 2^{n-1}(2^n - 1) & \text{for} \quad n \geq 5 \ . \end{cases}$$

PROOF. Let f_{x_0} be an s-symmetry on E^n, i e an isometry with an isolated fixed point x_0

If x_0 and x_0' are two fixed points, then we can find positive numbers ϵ and ϵ' such that (under the usual topology in E^n) $x_0 \notin \overline{N}(x_0', \epsilon')$ and $x_0' \notin \overline{N}(x_0, \epsilon)$, where $\overline{N}(x_0', \epsilon)$ $(\overline{N}(x_0', \epsilon'))$ is the closure of the open neighborhood of $x_0(x_0')$ with radius $\epsilon(\epsilon')$

So, we can substitute the fixed point x_0 with the neighborhood $\overline{N}(x_0, \epsilon)$ preserving the geometrical properties of f_{x_0} Then, f_{x_0} becomes

$$f_{x_0} : E^n - \overline{N}(x_0, \epsilon) \longrightarrow E^n - \overline{N}(x_0, \epsilon) \,,$$

where $\overline{N}(x_0, \epsilon)$ is invariant under the action of f_{x_0}

Denoting $\tilde{E}^n_{x_0}(\epsilon) = E^n - \overline{N}(x_0, \epsilon)$, f_{x_0} takes the form

$$f_{x_0} : \tilde{E}^n_{x_0}(\epsilon) \longrightarrow \tilde{E}^n_{x_0}(\epsilon) \,.$$

Using an orthogonal coordinate system in E^n we have

$$\tilde{E}^n_{x_0}(\epsilon) = \left\{ (x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0) \Big/ \sum_{i=1}^{n} (x_i - x_i^0)^2 > \epsilon \right\} .$$

If we define

$$\tilde{E}^{n-1}_{x_0}(\epsilon, j) = \left\{ (x_1 - x_1^0, \ldots, x_{j-1} - x_{j-1}^0; x_{j+1} - x_{j+1}^0, \ldots, x_n - x_n^0) \Big/ \sum_{\substack{i=1 \\ i \neq j}}^{n} (x_i - x_i^0)^2 > \epsilon \right\}$$

then $\tilde{E}^n_{x_0}(\epsilon)$ can be decomposed to the "direct sum" of the above $(n-1)$-dimensional subspaces as

$$\tilde{E}^n_{x_0}(\epsilon) = \bigoplus_{j=1}^{n} \tilde{E}^{n-1}_{x_0}(\epsilon, j) \ \underline{\underline{\text{def}}}$$

$$= \left\{ \frac{1}{n-1} \left[(0, x_2 - x_2^0, \ldots, x_n - x_n^0)^T + \ldots + (x_1 - x_1^0, x_2 - x_2^0, \ldots, 0)^T \right] \Big/ \sum_{j=1}^{n} (x_j - x_j^0)^2 > \epsilon \right\} .$$

The action of an orthogonal matrix A_{x_0} on $\tilde{E}^n_{x_0}(\epsilon)$ has the form.

$$A_{x_0} \begin{pmatrix} x_1 - x_1^0 \\ \vdots \\ x_n - x_n^0 \end{pmatrix} = \frac{1}{n-1} A_{x_0} \left[\begin{pmatrix} 0 \\ \vdots \\ x_n - x_n^0 \end{pmatrix} + \ldots + \begin{pmatrix} x_1 - x_1^0 \\ \vdots \\ 0 \end{pmatrix} \right] .$$

We observe that this action can be decomposed to a sum of mutually independent parts

Choosing the j-th part where $1 \leq j \leq n$, we shall prove that if x_0, x_0' are two different fixed points in E^n we can pass from $\tilde{E}^{n-1}_{x_0}(\epsilon, j)$ to ${}^*\tilde{E}^{n-1}_{x_0'}(\epsilon', j)$ for every j, where,

$${}^*\tilde{E}^{n-1}_{x_0'}(\epsilon', j) = A_{x_0}(\tilde{E}^{n-1}_{x_0'}(\epsilon', j)) \,.$$

Taking $\tilde{E}^{n-1}_{x_0}(\epsilon^*, j)$ with $\epsilon^* = \min\{\epsilon, \epsilon'\}$ we have the following commutative diagram

$$\begin{array}{ccc}
\tilde{E}_{x_0}^{n-1}(\epsilon^*, j) & \xrightarrow{\quad \phi_j \quad} & {}^*\tilde{E}_{x_0'}^{n-1}(\epsilon^*, j) \\
\downarrow h_1 & & \\
\mathbb{R} \times S_{x_0}^{n-2}(\epsilon^*) & & \\
\downarrow q_1 x I & & \\
\mathbb{R}/\mathbb{Z} \times S_{x_0}^{n-2}(\epsilon^*) & & \\
\downarrow h_2 & \alpha_j & \\
S^1 \times S_{x_0}^{n-2}(\epsilon^*) & & \\
\downarrow q_2 & & \\
S^1 \wedge S_{x_0}^{n-2}(\epsilon^*) & & \\
\downarrow h_3 & & \\
S_{x_0}^{n-1}(\epsilon^*) & \nearrow &
\end{array}$$

where, $\quad a_j : S_{x_0}^{n-1}(\epsilon^*) \overset{st}{\to} E^{n-1} \overset{h_4}{\to} E^{n-2} \times E^1 \overset{q}{\to} E^{n-2} \times S^1 \overset{h_5}{\to} {}^*\tilde{E}_{x_0'}^{n-1}(\epsilon^*, j)$, and h_i are homeomorphisms, $q_1 \times I$ is the natural map, q_2 is the quotient map, $q = (id.q_{S^1})$ is the quotient map and $S_{x_0}^{n-1}(\epsilon^*)$ is the $(n-1)$-sphere with center x_0 and radius ϵ^* under quotient topology ($V \subset S^{n-1}$ is open if and only if $q_2^{-1}(V)$ is open)

Repeating the above diagram for every j, it turns out that the existence of ϕ_j depends on the existence of α_j, which are classified by definition from the $n-1$ homotopy group of $\tilde{E}_{x_0}^{n-1}(\epsilon, j)$ But $\cdot \tilde{E}_{x_0}^{n-1}(\epsilon, j)$ is of the same homotopy type with $S_{x_0}{}^{n-2}(\epsilon)$, hence

$$\pi_{n-1}(\tilde{E}_{x_0}^{n-1}(\epsilon, j)) \cong \pi_{n-1}(S^{n-2}).$$

Finally, we obtain that the mapping.

$$\bigoplus_{j=1}^{n} \tilde{E}_{x_0}^{n-1}(\epsilon, j) \to \bigoplus_{j=1}^{n} {}^*\tilde{E}_{x_0}^{n-1}(\epsilon, j)$$

exists if the corresponding maps α_j belong to the same homotopy equivalence class.

It is well known that

$$\begin{aligned}
&\text{for} \quad n = 2 \quad \pi_1(S^0) = 0, \\
&\text{for} \quad n = 3 \quad \pi_2(S^1) = 0, \\
&\text{for} \quad n = 4 \quad \pi_3(S^2) = \mathbb{Z}.
\end{aligned}$$

Hence, it is clear that the spaces E^2, E^3 and E^4 admit only parallel s-structures because α_j's exist and belong to the same homotopy equivalence class.

For $n \geq 5$ we have $\pi_{n-1}(S^{n2}) = \mathbb{Z}_2$, so the spaces E^n for $n \geq 5$ admit non-parallel s-structures.

To find out the number of the non-parallel s-structures of a Euclidean space $E^n (n \geq 5)$ we consider the case $n = 5$

Composing again the 4-dimensional spaces we have

$$\tilde{E}_{x_0}^5(\epsilon) = \tilde{E}_{x_0}^4(\epsilon, 1) \oplus \dots \oplus \tilde{E}_{x_0}^4(\epsilon, 5),$$

and

$${}^*\tilde{E}_{x_0'}^5(\epsilon) = {}^*\tilde{E}_{x_0'}^4(\epsilon, 1) \oplus \dots \oplus {}^*\tilde{E}_{x_0'}^4(\epsilon, 5).$$

If 0 and 1 are the classes of $\mathbb{Z}_2$ then every $\tilde{E}_{x_0}^4(\epsilon, j)$ and ${}^*\tilde{E}_{x_0'}^4(\epsilon, j)$ corresponds to 0 or 1 Hence

$$\tilde{E}_{x_0}^5 = \alpha_1 \oplus \alpha_2 \oplus \alpha_3 \oplus \alpha_4 \oplus \alpha_5,$$

and

$$^*\tilde{E}^5_{i'_j} = {}^*\alpha_1 \oplus {}^*\alpha_2 \oplus {}^*\alpha_3 \oplus {}^*\alpha_4 \oplus {}^*\alpha_5 \,,$$

where α_i, $^*\alpha_i$ are 0 or 1

The passing from $^*\tilde{E}^5_{i'_j}$ to $^*\tilde{E}^5_{i'_j}$ can be done by a parallel way, if and only if $^*\alpha_i = \alpha_i$ for every $i = 1, \ldots, 5$

Obviously, there exist $2^5 = 32$ 5-tuples and $\binom{32}{2} = 496$ non-parallel mappings

The above proof we can apply to the n-dimensional Euclidean space, and so the proof of the theorem is completed

2. CW-COMPLEXES

In the present paragraph we generalize the results of the first one

THEOREM 2. Let M be an n-dimensional manifold which is a CW-complex Then, M admits N non-parallel s-structures where N is given by

$$N = \begin{cases} 0 & \text{if} \quad n < 5 \;, \\ 2^{n-1}(2^n - 1) & \text{if} \quad n \geq 5 \;. \end{cases}$$

PROOF. M is a CW-complex, hence it can be decomposed as

$$M = e_{i_1} \sqcup e_{i_2} \sqcup \ldots \sqcup e_{i_n} \,, \quad i_1 \leq i_2 \leq \ldots \leq i_n \,,$$

where e_{i_n} is the maximal-dimension cell and $\dim M = \dim e_{i_n}$

We have to take the fixed point on the cell e_{i_n} Otherwise the fixed point will not be isolated

We consider the diagram

$$\begin{array}{ccccccc} e_{i_1} & \ldots & e_{i_{n-1}} & e_{i_n} & \xrightarrow{h_1} & E^n_{x_0} \\ I & & I & f & & f_{x_0} \\ e_{i_1} & \ldots & e_{i_{n-1}} & e'_{i_n} & \xrightarrow{h_2} & {}^*E^n_{x_0} \end{array}$$

where h_1 and h_2 are homeomorphisms and f_{x_0} is defined as in Theorem 1 Also, we define f to be non-parallel if and only if it does not depend on x_0

Thus, we can study the maps f_{x_0} instead of f The last suggestion completes the proof of Theorem 2.

Examples:

1 $S^6 = e_1 \sqcup e_6$, $N = \binom{2^6}{2} = 2016$,

2 $\mathbb{C}P(10) = e_0 \sqcup e_2 \sqcup e_4 \sqcup e_6 \sqcup e_8 \sqcup e_{10}$, $N = 2^9(2^{10} - 1) = 523.776$,

3 $\mathbb{R}P(6) = e_0 \sqcup e_1 \sqcup e_2 \sqcup e_3 \sqcup e_4 \sqcup e_5 \sqcup e_6$, $N = 2^5(2^6 - 1) = 2016$.

Considering the first and third example of the second paragraph we observe that two manifolds which are CW-complexes $(S^6, \mathbb{R}P(6))$ have the same number of non-parallel s-structures although they have different geometrical and topological structures Thus, the following questions arises: "Do there exist manifolds admitting N non-parallel s-structures where $N \neq \binom{2^n}{n}$, $n \geq 5$?

ACKNOWLEDGMENT. The authors wish to thank Professor O Kowalski for his suggestion to study this problem

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