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EXISTENCE THEOREMS FOR A SECOND ORDER m-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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Abstract

Let $f:[0,1]\times R^2\to R$ be a function satisfying Caratheodory's conditions and $e(t)\in L^1[0,1]$. Let $\eta\in(0,1)$, $\xi_i\in(0,1)$, $a_i\geq 0$, $i=1,2,\cdots,m-2$, with $\sum_{i=1}^{m-2}a_i=1$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$ be given. This paper is concerned with the problem of existence of a solution for the following boundary value problems

$x''(t)=f(t,x(t),x'(t))+e(t), 0<t<1,$
$x'(0)=0,\ x(1)=x(\eta),$
$x''(t)=f(t,x(t),x'(t))+e(t), 0<t<1,$
$x'(0)=0,\ x(1)=\sum_{i=1}^{m-2}a_ix(\xi_i).$

Conditions for the existence of a solution for the above boundary value problems are given using Leray Schauder Continuation theorem.

Keywords and Phrases: three-point boundary value problem, m-point boundary value problem, Leray Schauder Continuation theorem, Caratheodory's conditions, Arzela-Ascoli Theorem.

AMS(MOS) Subject Classification: 34B10, 34B15, 34G20.

1 INTRODUCTION.

Let $f:[0,1]\times R^2\to R$ be a function satisfying Caratheodory's conditions, $e:[0,1]\to R$ be a function in $L^1[0,1]$, $a_i\geq 0$, $\xi_i\in(0,1)$, $i=1,2,\cdots,m-2$ with $\sum_{i=1}^{m-2}a_i=1$, $0<\xi_1<\xi_2<\cdots,<\xi_{m-2}<1$ and $\eta\in(0,1)$ be given. We study the problem of existence of solutions for the following boundary value problems

$$\begin{gathered}x''(t)=f(t,x(t),x'(t))+e(t),\ 0<t<1,\\ x'(0)=0,\ x(1)=x(\eta),\end{gathered}\tag{1}$$

$$\begin{gathered}x''(t)=f(t,x(t),x'(t))+e(t),\ 0<t<1,\\ x'(0)=0,\ x(1)=\sum_{i=1}^{m-2}a_ix(\xi_i).\end{gathered}\tag{2}$$

It is well-known, (see, e.g. [1]), that if $x\in C^1[0,1]$ satisfies the boundary conditions in (2), with the a_i's as above, then there exists an $\eta\in[\xi_1,\xi_{m-2}]$, depending on $x\in C^1[0,1]$, such that

$$x(1)=x(\eta).\tag{3}$$

Accordingly, it seems that one can study the problem of existence of a solution for the boundary value problem (2) using the a priori estimates obtained for the three-point boundary value problem (1), as it was done in [2], [3], [4]. But here the m-point boundary value problem (2) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{gathered}x''(t)=0,\ 0<t<1,\\ x'(0)=0,\ x(1)=\sum_{i=1}^{m-2}a_ix(\xi_i),\end{gathered}$$

has $x(t)=A$, $A\in R$, as a non-trivial solution, since $\sum_{i=1}^{m-2}a_i=1$. The result is that $e(t)\in L^1[0,1]$ has to be such that $\sum_{i=1}^{m-2}a_i[\int_0^{\xi_i}(1-\xi_i)e(s)ds+\int_{\xi_i}^1(1-s)e(s)ds]=0$, (in view of the nonlinear Fredholm alternative), so even though there exists an $\eta \in [\xi_1, \xi_{m-2}]$ such that $\int_0^1 (1-\eta)e(s)ds + \int_\eta^1 (1-s)e(s)ds = \sum_{i=1}^{m-2} a_i [\int_0^{\xi_i} (1-\xi_i)e(s)ds + \int_{\xi_i}^1 (1-s)e(s)ds] = 0$, since $\sum_{i=1}^{m-2} a_i = 1$, this η is not necessarily the same η as in (3). We are, accordingly, forced to study the m-point boundary value problem (2) directly and obtain results about the three-point boundary value problem (1) as a corollary to the results for the m-point boundary value probelm. It is interesting to note that while in the nonresonance case we had to study the m-point boundary value problem, using the results for the three -point boundary value problem, it is just the reverse case in the resonance case.

We obtain conditions for the existence of a solution for the boundary value problem (2), using Mawhin's version of the Leray Schauder Continuation theorem [5] or [6] or [7]. Recently, Gupta, Ntouyas and Tsamatos studied the m-point boundary value problem

$$\begin{aligned} x''(t) &= f(t, x(t), x'(t)) + e(t),\ 0 < t < 1, \\ x'(0) &= 0,\ x(1) = \textstyle\sum_{i=1}^{m-2} a_i x(\xi_i), \end{aligned} \tag{4}$$

with $\xi_i \in (0,1)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in R$, all a_i having the same sign, given, and $\sum_{i=1}^{m-2} a_i \neq 1$, in [3]. The boundary value problem (2) differs from the boundary value problem (4) in that the associated linear boundary value problem with (2), namely,

$$\begin{aligned} x''(t) &= 0,\ 0 < t < 1, \\ x'(0) &= 0,\ x(1) = \textstyle\sum_{i=1}^{m-2} a_i x(\xi_i), \end{aligned} \tag{5}$$

has $x(t) = A$, for $A \in R$, as non-trivial solutions, since $\sum_{i=1}^{m-2} a_i = 1$, while the corresponding linear boundary value problem associated with (4), namely,

$$\begin{aligned} x''(t) &= 0,\ 0 < t < 1, \\ x'(0) &= 0,\ x(1) = \textstyle\sum_{i=1}^{m-2} a_i x(\xi_i), \end{aligned} \tag{6}$$

with $\sum_{i=1}^{m-2} a_i \neq 1$, has $x(t) \equiv 0$, as its only solution. It is for this reason we call the boundary value problem (2) to be at resonance. For some recent results on m-point and three-point boundary value problems we refer the reader to [2], [3], [4], [8], [9], [10], (and [11]).

We use the classical spaces $C[0,1]$, $C^k[0,1]$, $L^k[0,1]$, and $L^\infty[0,1]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose k-th power of the absolute value is Lebesgue integrable on $[0,1]$, or measurable functions that are essentially bounded on $[0,1]$. We also use the Sobolev space $W^{2,k}(0,1)$, $k = 1,2$ defined by

$$W^{2,k}(0,1) = \{x : [0,1] \to R \,|\, x, x' \text{ abs. cont. on } [0,1] \text{ with } x'' \in L^k[0,1]\}$$

with its usual norm. We denote the norm in $L^k[0,1]$ by $\|\cdot\|_k$, and the norm in $L^\infty[0,1]$ by $\|\cdot\|_\infty$.

2 EXISTENCE THEOREMS.

Let X, Y denote Banach spaces $X = C^1[0,1]$ and $Y = L^1[0,1]$ with their usual norms. Let Y_2 be the subspace of Y spanned by the function 1, i.e.

$$Y_2 = \{x(t) \in Y \mid x(t) = A, \text{ a.e. on } [0,1], A \in R\} \tag{7}$$

and let Y_1 be the subspace of Y such that $Y = Y_1 \oplus Y_2$. Let $a_i \geq 0$, $\xi_i \in (0,1)$, $i = 1,2,\cdots,m-2$ with $\sum_{i=1}^{m-2} a_i = 1$, $0 < \xi_1 < \xi_2 < \cdots, < \xi_{m-2} < 1$, be given. We note that for $x(t) \in Y$ we can write

$$x(t) = (x(t) - A) + A, \tag{8}$$

with $A = \frac{2}{\sum_{i=1}^{m-2} a_i(1-\xi_i^2)} \sum_{i=1}^{m-2} a_i [\int_0^{\xi_i} (1-\xi_i)x(s)ds + \int_{\xi_i}^1 (1-s)x(s)ds]$, for $t \in [0,1]$. We define the canonical projection operators $P : Y \to Y_1$, $Q : Y \to Y_2$ by

$$\begin{aligned} P(x(t)) &= x(t) - \frac{2}{\sum_{i=1}^{m-2} a_i(1-\xi_i^2)} [\textstyle\sum_{i=1}^{m-2} a_i [\int_0^{\xi_i} (1-\xi_i)x(s)ds + \int_{\xi_i}^1 (1-s)x(s)ds]], \\ Q(x(t)) &= \frac{2}{\sum_{i=1}^{m-2} a_i(1-\xi_i^2)} [\textstyle\sum_{i=1}^{m-2} a_i [\int_0^{\xi_i} (1-\xi_i)x(s)ds + \int_{\xi_i}^1 (1-s)x(s)ds]], \end{aligned} \tag{9}$$

for $x(t) \in Y$. We note that if $Q(x(t)) = 0$, there exists a $\zeta \in (0,1)$ such that $x(\zeta) = 0$. Clearly, $Q = I - P$, where I denotes the identity mapping on Y, and the projections P and Q are continuous. Now let $X_2 = X \cap Y_2$. Clearly X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projections $P \mid X : X \to X_1$ and $Q \mid X : X \to X_2$ are continuous. In the following, X, Y, P, Q will refer to the Banach spaces and the projections as defined and we shall not distinguish between P, $P \mid X$ (resp. Q, $Q \mid X$) and depend on the context for the proper meaning.

Define a linear operator L: $D(L) \subset X \to Y$ by setting

$$D(L) = \{x \in W^{2,1}(0,1) \mid x'(0) = 0,\ x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)\}, \tag{10}$$

and for $x \in D(L)$,

$$Lx = x''. \tag{11}$$

Let, now, for $e \in Y_1$, i.e. $e \in L^1[0,1]$ with $\sum_{i=1}^{m-2} a_i[\int_0^{\xi_i}(1-\xi_i)e(s)ds + \int_{\xi_i}^1(1-s)e(s)ds] = 0$, Ke denote the unique solution of the boundary value problem

$$x''(t) = e(t),\ 0 < t < 1,$$
$$x'(0) = 0,\ x(1) = \textstyle\sum_{i=1}^{m-2} a_i x(\xi_i),$$

such that $\sum_{i=1}^{m-2} a_i[\int_0^{\xi_i}(1-\xi_i)x(s)ds + \int_{\xi_i}^1(1-s)x(s)ds] = 0$. Indeed, for $t \in [0,1]$,

$$(Ke)(t) = \int_0^t (t-s)e(s)ds + A, \tag{12}$$

where $A = -\frac{2}{\sum_{i=1}^{m-2} a_i(1-\xi_i^2)}[\sum_{i=1}^{m-2} a_i[\int_0^{\xi_i}\int_0^t(1-\xi_i)(t-s)e(s)dsdt + \int_{\xi_i}^1\int_0^t(1-t)(t-s)e(s)dsdt]]$. Accordingly the linear mapping $K : Y_1 \to X_1$ defined by the equation (12) is a bounded linear mapping and is such that for

$$x \in Y,\ KPx \in D(L), \text{ and } LKP(x) = P(x).$$

DEFINITION 1 :- *A function $f : [0,1] \times R^2 \to R$ satisfies Caratheodory's conditions if (i) for each $(x,y) \in R^2$, the function $t \in [0,1] \to f(t,x,y) \in R$ is measurable on $[0,1]$, (ii) for a.e. $t \in [0,1]$, the function $(x,y) \in R^2 \to f(t,x,y) \in R$ is continuous on R^2, and (iii) for each $r > 0$, there exists $\alpha_r(t) \in L^1[0,1]$ such that $|f(t,x,y)| \le \alpha_r(t)$ for a.e. $t \in [0,1]$ and all $(x,y) \in R^2$ with $\sqrt{x^2+y^2} \le r$.*

Let $f : [0,1] \times R^2 \to R$ be a function satisfying Caratheodory's conditions. Let $N: X \to Y$ be the non-linear mapping defined by

$$(Nx)(t) = f(t, x(t), x'(t)),\ t \in [0,1],$$

for $x(t) \in X$.

For $e(t) \in Y_1$, i.e. $e(t) \in L^1[0,1]$ with $\sum_{i=1}^{m-2} a_i[\int_0^{\xi_i}(1-\xi_i)e(s)ds + \int_{\xi_i}^1(1-s)e(s)ds] = 0$, the boundary value problem (2) reduces to the functional equation

$$Lx = Nx + e, \tag{13}$$

in X, with $e(t) \in Y_1$, given.

THEOREM 2 :- *Let $f : [0,1] \times R^2 \to R$ be a function satisfying Caratheodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,x_1,x_2)| \le p(t)|x_1| + q(t)|x_2| + r(t) \tag{14}$$

for a.e. $t \in [0,1]$ and all $(x_1,x_2) \in R^2$. Also let $a_i \ge 0$, $\xi_i \in (0,1)$, $i = 1,2,\dots,m-2$ with $\sum_{i=1}^{m-2} a_i = 1$, $0 < \xi_1 < \xi_2 < \dots, < \xi_{m-2} < 1$ be given, and assume that for every $x(t) \in X$,

$$(Qx)(t).(QNx)(t) \ge 0, \text{ for } t \in [0,1]. \tag{15}$$

Then for $e(t) \in Y_1$, i.e. $e(t) \in L^1[0,1]$ with $\sum_{i=1}^{m-2} a_i[\int_0^{\xi_i}(1-\xi_i)e(s)ds + \int_{\xi_i}^1(1-s)e(s)ds] = 0$, given, the boundary value problem (2) has at least one solution in $C^1[0,1]$ provided

$$\|p\|_1 + \|q\|_1 < 1. \tag{16}$$

PROOF:- We first note that the bounded linear mapping $K : Y_1 \to X_1$ defined by the equation (12) is such that the mapping $KPN : X \to X$ maps bounded subsets of X into relatively compact subsets of X, in view of Arzela-Ascoli Theorem. Hence $KPN : X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the boundary value problem (2) if and only if x is a solution to the operator equation

$$Lx = Nx + e.$$

Now, to solve the operator equation $Lx = Nx + e$, it suffices to solve the system of equations

$$\begin{aligned} Px &= KPNx + e_1, \\ QNx &= 0, \end{aligned} \tag{17}$$

$x \in X$, $e_1 = Ke$ (note that since $e \in Y_1$, $Pe = e$, $Qe = 0$). Indeed, if $x \in X$ is a solution of (17) then $x \in D(L)$ and

$$\begin{aligned} LPx = Lx = LKPNx + Le_1 &= PNx + e, \\ QNx &= 0, \end{aligned}$$

which gives on adding that $Lx = Nx + e$.

Now, (17) is clearly equivalent to the single equation

$$Px + QNx - KPNx = e_1, \tag{18}$$

which has the form of a compact perturbation of the Fredholm operator P of index zero. We can, therefore, apply the version given in ([5], Theorem 1, Corollary 1) or ([6], Theorem IV.4) or ([7]) of the Leray-Schauder Continuation theorem which ensures the existence of a solution for (18) if the set of all possible solutions of the family of equations

$$Px + (1-\lambda)Qx + \lambda QNx - \lambda KPNx = \lambda e_1, \tag{19}$$

$\lambda \in (0,1)$, is a priori bounded, independently of λ. Notice that (19) is then equivalent to the system of equations

$$\begin{aligned} Px &= \lambda KPNx + \lambda e_1, \\ (1-\lambda)Qx + \lambda QNx &= 0. \end{aligned} \tag{20}$$

Let, now, $x(t)$ be a solution of (20) for some $\lambda \in (0,1)$. We see on multiplying the second equation in (20) and using (15) that $(1-\lambda)((Qx)(t))^2 \leq 0$ for every $t \in [0,1]$. Hence $(Qx)(t) = 0$ for every $t \in [0,1]$ and accordingly there exists a $\zeta \in (0,1)$ such that $x(\zeta) = 0$. Since, now, $x'(0) = 0$ it follows that $\| x \|_\infty \leq \| x' \|_\infty \leq \| x'' \|_1$. Also since $Qx = 0$, we have $QNx = 0$. It follows that $x \in D(L)$, i.e., $x \in W^{2,1}(0,1)$ with $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ and $x''(t) = \lambda f(t, x(t), x'(t)) + \lambda e(t)$. Accordingly, we get that

$$\begin{aligned} \|x''\|_1 &= \lambda \|f(t, x(t), x'(t)) + e(t)\|_1 \\ &\leq \|p\|_1 \|x\|_\infty + \|q\|_1 \|x'\|_\infty + \|r\|_1 + \|e\|_1 \\ &\leq (\|p\|_1 + \|q\|_1)\|x''\|_1 + \|r\|_1 + \|e\|_1 \end{aligned}$$

It follows from the assumption (16) that there is a constant c, independent of $\lambda \in (0,1)$ and $x(t)$, such that

$$\|x''\|_1 \leq c.$$

It is now immediate from $\| x \|_\infty \leq \| x' \|_\infty \leq \| x'' \|_1$ that the set of solutions of the family of equations (20) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in (0,1)$.

This completes the proof of the theorem.//

REMARK 1:- We remark that the Theorem 2 remains valid if we replace (15) by the condition

$$(Qx)(t).(QNx)(t) \leq 0, \text{ for } t \in [0,1]. \tag{21}$$

for every $x \in X$.

REMARK 2:- We remark that the condition (15) can be replaced by the condition

$$f(t,x_1,x_2)x_1 \geq 0, \tag{22}$$

for almost all $t \in (0,1)$ and all $(x_1,x_2) \in R^2$. Indeed, condition (15) was used to show, in the proof of Theorem 2, that if $x(t)$ is a solution of (20) for some $\lambda \in (0,1)$ then there exists a $\zeta \in (0,1)$ such that $x(\zeta) = 0$. We, now, show that (22), implies that if $x(t)$ is a solution of (20) for some $\lambda \in (0,1)$ then there exists a $\zeta \in (0,1)$ such that $x(\zeta) = 0$. Indeed, suppose that $x(t) \neq 0$, for all $t \in (0,1)$. We may, infact, assume without any loss of generality that $x(t) > 0$, for every $t \in (0,1)$. It then follows from (22) that $f(t,x(t),x'(t)) \geq 0$, for a.e. $t \in (0,1)$. Hence $Qx > 0$ and $QNx \geq 0$. Now the second equation in (20) gives that $(1-\lambda)(Qx)^2 + \lambda(QNx)(Qx) = 0$, so that we get $(Qx)^2 \leq 0$, a contradiction. Accordingly, there must exist a $\zeta \in (0,1)$ such that $x(\zeta) = 0$.

THEOREM 3 :- *Let $f : [0,1] \times R^2 \to R$ be a function as in Theorem 2. Assume that the functions $p(t)$, $q(t)$, $r(t)$ in (14) are in $L^2(0,1)$. Let $a_i \geq 0$, $\xi_i \in (0,1)$, $i = 1,2,\cdots,m-2$ with $\sum_{i=1}^{m-2} a_i = 1$, $0 < \xi_1 < \xi_2 < \cdots, < \xi_{m-2} < 1$ be given.*

Then for $e(t) \in L^2[0,1]$ with $\sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 e(s)ds = 0$, given, the boundary value problem (2) has at least one solution in $C^1[0,1]$ provided

$$\frac{2}{\pi}\left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right) < 1. \tag{23}$$

PROOF:- The proof is similar to the proof of Theorem 2, except now one uses the inequalities $\| x \|_2 \leq \frac{2}{\pi} \| x' \|_2 \leq \frac{4}{\pi^2} \| x'' \|_2$ for an $x \in W^{2,2}(0,1)$ with $x(\zeta) = 0$, for some $\zeta \in (0,1)$ and $x'(0) = 0$ (see, Theorem 256 of [12]) to show that the set of solutions of the family of equations (19) is a priori bounded in $C^1[0,1]$ by a constant independent of $\lambda \in (0,1)$.//

THEOREM 4 :- *Let $f : [0,1] \times R^2 \to R$ be a function as in Theorem 2 (respectively, Theorem 3). Let $\eta \in (0,1)$ be given. Then for $e(t) \in L^1[0,1]$ (resectively, $e(t) \in L^2[0,1]$) with $\int_0^\eta (1-\eta)e(s)ds + \int_\eta^1 (1-s)e(s)ds = 0$, given, the three-point boundary value problem (1) has at least one solution in $C^1[0,1]$ provided*

$$\|p\|_1 + \|q\|_1 < 1, \tag{24}$$

(respectively, $\frac{2}{\pi}(\frac{2}{\pi}\|p\|_2 + \|q\|_2) < 1$).

PROOF:- The theorem follows immediately from Theorem 2 (respectively, Theorem 3) with $m = 3$ and $a_1 = 1$, $\xi_1 = \eta$.//

THEOREM 5 :- *Let $f : [0,1] \times R^2 \to R$ be a function as in Theorem 2 (respectively, Theorem 3). Then for $e(t) \in L^1[0,1]$ (resectively, $e(t) \in L^2[0,1]$) with $\int_0^1 (1-s)e(s)ds = 0$, given, the boundary value problem*

$$x''(t) = f(t, x(t), x'(t)) + e(t),\ 0 < t < 1,$$
$$x'(0) = 0,\ x(0) = x(1),$$

has at least one solution in $C^1[0,1]$ provided

$$\|p\|_1 + \|q\|_1 < 1, \tag{25}$$

(respectively, $\frac{2}{\pi}(\frac{2}{\pi}\|p\|_2 + \|q\|_2) < 1$).

PROOF:- The theorem follows immediately from Theorem 2 (respectively, Theorem 3) with $m = 2$ and $a_1 = 1$, $\xi_1 = 0$.//

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