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RESEARCH NOTES

GENERAL BOUNDEDNESS THEOREMS TO SOME SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION WITH INTEGRABLE FORCING TERM

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ABSTRACT. In this note we present a boundedness theorem to the equation $x'' + c(t,x,x') + a(t)b(x) = e(t)$ where $e(t)$ is a continuous absolutely integrable function over the nonnegative real line. We then extend the result to the equation $x'' + c(t,x,x') + a(t,x) = e(t)$. The first theorem provides the motivation for the second theorem. Also, an example illustrating the theory is then given.

KEY WORDS AND PHRASES. Integrable forcing term, bounded, nonlinear differential equation.
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1. INTRODUCTION.

In this article we shall discuss using standard methods the boundedness properties of a second order nonlinear differential equation with integrable forcing term, i.e. the equation,

$$x'' + c(t,x,x') + a(t)b(x) = e(t) \tag{1.1}$$

Our purpose here is to simplify some of the previous proofs to this well-known equation as well as extending some of the previous results. For example, we are replacing the condition $c(t,x,y)y > 0$ for $y \neq 0$ with $c(t,x,y)y \geq 0$ and letting $a(t)$ be non-increasing (see [1] and [2] for details, especially [2] for its excellent bibliography of previous work). Also, as in [2] we shall not need to make use of any Liapunov function. Finally, the result will be of such a nature that it covers the case when no damping factor appears, i.e. it covers the equation,

$$x'' + a(t)b(x) = e(t) \tag{1.2}$$

Later we shall briefly mention how this result carries over to the more general nonlinear equation,

$$x'' + c(t,x,x') + a(t,x) = e(t) \tag{1.3}$$

However, this case requires a more delicate discussion. We now state and prove the boundedness theorem. Without loss of generality, we shall assume $t \geq 0$.

2. MAIN RESULTS.

THEOREM I. Given the differential equation in (1.1). Suppose $c(t,x,y)$ is continuous on $[0,\infty) \times R \times R$, $c(t,x,y)y \geq 0$ and $e(\bullet)$ is continuous on $[0,\infty)$ with $\int_0^\infty |e(t)|dt < \infty$. Furthermore, if $a(\bullet) \geq a_0 > 0$ for some a_0 and continuous on $[0,\infty)$, $a'(\bullet) \leq 0$, $b(\bullet)$ continuous on R, and $B(x) = \int_0^x b(u)du$ approaches ∞ as $|x| \to \infty$ then all solutions as well as their derivatives are bounded as $t \to \infty$.

PROOF. By standard existence theory, there is a solution to (1) which exists on $[0,T)$ for some $T > 0$ for any initial conditions $x(0)$ and $x'(0)$. Multiply equation (1) by x' and perform an integration by parts on the last term from 0 to $t < T$ in order to obtain,

$$x'(t)^2/2 + \int_0^t c(s,x(s),x'(s))x'(s)ds + a(t)B(x(t)) - \int_0^t a'(s)B(x(s))ds$$
$$= x'(0)^2/2 + \int_0^t e(s)x'(s)ds \le x'(0)^2/2 + \int_0^t |e(s)x'(s)|ds \tag{2.1}$$

Now if $x(t)$ becomes unbounded then we must have that all terms on the LHS of (2.1) become positive from our hypotheses. By the mean value theorem, equation (2.1) may be rewritten as,

$$x'(t)^2/2 + \int_0^t c(s,x(s),x'(s))ds + a(t)B(x(t)) - \int_0^t a'(s)B(x(s))ds$$
$$\le x'(0)^2/2 + |x'(\bar{t})|K \quad \left(K = \int_0^\infty |e(t)|dt, 0 < \bar{t} < t\right) \tag{2.2}$$

Now from (2.2) we see that if $|x|$ approaches ∞ then so must $|x'(t)|$. Otherwise, the LHS of (2.2) becomes unbounded while the RHS stays bounded which is impossible. Also, as $|x'(t)|$ approaches ∞ so must $|x'(\bar{t})|$. Now on any compact subinterval choose t where $x'(t)$ is a maximum. Integrate equation (1.1) as before from 0 to t and divide by $x'(t)$ (assume $x(t) > 0$, a similar argument works for $x'(t) < 0$ only the inequality is reversed) in order to obtain,

$$x'(t)/2 + 1/x'(t)\left(\int_0^t c(s,x(s),x'(s))ds + a(t)B\left(x(t) - \int_0^t a'(s)B(x(s))ds\right)\right.$$
$$\le (x'(0)^2/2 + |x'(\bar{t})|K)/x'(t) \tag{2.3}$$

Now if $x'(t)$ approaches ∞ then the LHS of (2.3) becomes unbounded while the RHS of (2.3) stays bounded which is a contradiction. Thus, $|x|$ and $|x'|$ must stay bounded on $[0,T)$. A standard argument ([3, pp. 17-18]) now permits the solution to be extended on all of $[0,\infty)$.

As for equation (1.3) we may multiply it by x' and integrate as before obtaining the following,

$$x'(t)^2/2 + \int_0^t c(s,x(s),x'(s))x'(s)ds + \int_{x(0)}^{x(t)} a(t,u)du - \int_0^t \int_{x(0)}^{x(s)} \frac{\partial a(s,u(s))}{\partial s} duds$$
$$= x'(0)^2/2 + \int_0^t e(s)x'(s)ds\ . \tag{2.4}$$

We see here that as long as $\int_0^{\pm\infty} a(t,u)du = \infty$ uniformly in t and $x\frac{\partial}{\partial t}a(t,x) \le 0$ then we may use the same argument as in our first theorem. We now state this final result.

THEOREM II. Given equation (1.3). Suppose $c(t,x,y)$ is continuous on $[0,\infty)\times R\times R$, $c(t,x,y)y > 0$, $a(t,x)$ continuous on $[0,\infty)\times R$ with $x\frac{\partial}{\partial t}a(t,x) \le 0$. Furthermore, if $\int_0^{\pm\infty} a(t,u)du = \infty$ uniformly in t and $e(\bullet)$ is continuous on $[0,\infty)$ with $\int_0^\infty |e(t)|dt < \infty$, then all solutions to equation (1.3) as well as their derivatives are bounded as $t \to \infty$.

EXAMPLE. Consider the nonlinear differential equation,

$$x'' + cx^{2m-1}x' + bx^{2n-1} = \exp(-t) \tag{2.5}$$

where $t > 0$, c, b are positive and m, n are positive integers. By Theorem I we see that all solution to equation (2.5) are bounded.

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