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COMPLEMENTED SUBSPACES OF p-ADIC SECOND DUAL BANACH SPACES

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ABSTRACT. Let K be a non-archimedean non-trivially valued complete field. In this paper we study Banach spaces over K. Some of main results are as follows:

(1) The Banach space $BC((l^{\infty})_1)$ has an orthocomplemented subspace linearly homeomorphic to c_0.

(2) The Banach space $BC((c_0)_1)$ has an orthocomplemented subspace linearly homeomorphic to l^{∞}.

KEY WORDS AND PHRASES. non-archimedean valued fields, non-archimedean (p-adic) Banach spaces, polar spaces, spherically complete, complemented subspaces.

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1. INTRODUCTION.

Throught this paper K is a non-archimedean non-trivially valued complete field with a valuation $|\ |$, and E, F are Banach spaces over K with a non-archimedean norm denoted by $\|\ \|$. Let L(E,F) be the space consisting of all continuous linear maps of E to F. The dual space of E is $E'=L(E,K)$. The dual operator $T'\in L(F',E')$ of $T\in L(E,F)$ is defined as usual. If there exists a linear isometry from E onto F, then E and F are said to be isomorphic and we denote $E\sim F$. For a Banach space E, if there exists a (ortho)complemented subspace of F which is isomorphic to E, then E is said to be (ortho)complemented in F. Let S be a topological space and let BC(S) be the Banach space consisting of all bounded continuous functions $S\to K$ with a norm

$$\|f\|=\sup\{|f(s)| : s\in S\} \ (f\in BC(S)). \tag{1.1}$$

Let E″ be the second dual Banach space of E and let $J_E : E\to E''$ be the natural map.

DEFINITION. If J_E is linearly homeomorphic from E into E″, then E is said to be polar (see [6]).

DEFINITION. A Banach space E is said to be strongly polar if every continuous seminorm p on E satisfies the following equality (see [7]).

$$p=\sup\{|f| : f\in E',\ |f|\leq p\} \tag{1.2}$$

These spaces were first introduced by Schikhof [5] for locally convex topological spaces over K and were studied by some authors (e.g. [1], [2]).

DEFINITION. Let D be a subspace of E. If every $x'\in D'$ has an extension $\bar{x}'\in E'$, then D has the weak extension property in E. In addition, if $\bar{x}'$ can be chosen such that $\|\bar{x}'\|=\|x'\|$, then we say that D has the extension property in E.

For any r>0 we put $E_r=\{x\in E : \|x\|\leq r\}$. Let π denote an arbitrary fixed element of K with $0<|\pi|<1$. Other terms will be used as in Rooij [4]. In this paper we deal with complemented subspaces of $BC((E')_1)$ and E″. Throught this paper, when we consider a subset $(E')_r$ (r>0) of E′, $(E')_r$ is assumed to have the weak * topology. In section 2 we show that there exists a Banach space E such that $BC((l^{\infty})_1)$ is linearly homeomorphic to $c_0\oplus E$. And in section 3, we show that there exists a Banach space F such that $BC((c_0)_1)$ is linearly homeomorphic to $l^{\infty}\oplus F$.

2. COMPLEMENTED SUBSPACES OF BC(S).

For every $T\in L(E,BC(S))$, for every $s\in S$ and for every $x\in E$, let

$$(\psi_T(s))(x)=(T(x))(s). \tag{2.1}$$

Then the map $\psi_T(s)$ is a linear functional on E. Since $\|\psi_T(s)\|\leq\|T\|$, $\psi_T(s)\in(E')_{\|T\|}$. Hence ψ_T is a weak * continuous map from S to $(E')_{\|T\|}$. Conversely, for every weak * continuous map $\psi : S \to (E')_r$ $(r>0)$, let

$$(T_\psi(x))(s)=(\psi(s))(x) \quad (x\in E,\ s\in S). \tag{2.2}$$

Then $T_\psi(x)$ is a map from S to K. Since for each $x\in E$

$$\sup\{|(T_\psi(x))(s)| : s\in S\}\leq r\|x\|, \tag{2.3}$$

$T_\psi(x)\in BC(S)$. Hence T_ψ is a linear map from E to BC(S). By (2.3), $\|T_\psi\|\leq r$. It follows that $T_\psi\in L(E,BC(S))$.

For the natural map $J_E : E \to E''$ and for every $x\in E$, let $R_E(x)$ denote the restriction of $J_E(x)$ to $(E')_1$, that is,

$$R_E(x)=J_E(x)|(E')_1. \tag{2.4}$$

Then R_E is a linear map from E into $BC((E')_1)$. Since for every $x\in E$

$$\begin{aligned}\|R_E(x)\|&=\sup\{|(R_E(x))(x')| : x'\in(E')_1\}\\ &\leq\sup\{\|x'\|\|x\| : x'\in(E')_1\}\\ &\leq\|x\|,\end{aligned} \tag{2.5}$$

we have $\|R_E\|\leq 1$ and $R_E\in L(E,BC((E')_1))$.

The next theorem follows from Schikhof [7].

THEOREM 1. Let E be a strongly polar Banach space and let D be a closed subspace of E. Then for each $\varepsilon>0$, each $f\in D'$ can be extended to an $\bar{f}\in E'$ with $|\bar{f}(x)| \leq(1+\varepsilon)\|f\|\|x\|$ $(x\in E)$.

A norm $\|\ \|_p$ on E is said to be polar if

$$\|\ \|_p=\sup\{|f| : f\in E',\ |f|\leq\|\ \|_p\}. \tag{2.6}$$

We recall that if E is polar, then there exists a polar norm $\|\ \|_p$ on E such that it is equivalent to the original norm $\|\ \|$ (see [1, p.75]), and so there exists a real number d $(d\geq 1)$ such that for every $x\in E$ $\|x\|\leq\|x\|_p\leq d\|x\|$.

THEOREM 2. Let E be a polar Banach space. Then there exists a real number c $(c>1)$ satisfying the following (1) and (2).

(1) For each finite-dimensional subspace D of E and for each $f\in D'$ there exists an extension $\bar{f}\in E'$ such that $\|\bar{f}\|\leq c\|f\|$.

(2) For each finite-dimensional subspace D of E there exists a projection $P : E \to D$ with $\|P\|\leq c$.

PROOF. (1) Since $f\in D'$, it is trivial that $f\in(D,\|\ \|_p)'$. Let $\varepsilon>0$ be an arbitrarily given real number and put $c=(1+\varepsilon)d$. By Theorem 2.1 in Garcia [1], there exists an extension $\bar{f}\in(E,\|\ \|_p)'$ such that $\|\bar{f}\|_p\leq(1+\varepsilon)\|f\|_p$. Then we have that $\|\bar{f}\|/d\leq(1+\varepsilon)\|f\|$.
(2) Using again Theorem 2.1 in [1], there exists a projection $P : E \to D$ such that $\|P\|_p\leq 1+\varepsilon$. It follows that $\|P\|\leq d\|P\|_p\leq c$.

THEOREM 3. If E is a polar space, then R_E is a linear homeomorphism. And if the norm on E is polar, then R_E is a linear isometry.

PROOF. In section 1, it is proved that for all $x\varepsilon E$

$$\|R_E(x)\|\leq\|x\|. \tag{2.7}$$

Note that for every $x'\varepsilon E'$, $x'\neq 0$, there exists an integer m with $|\pi|^{m+1}\leq\|x'\|\leq|\pi|^m$, then

$$|\pi|\frac{|x'(x)|}{\|x'\|}\leq|\pi|^{-m}|x'(x)|=|(\pi^{-m}x')(x)|$$
$$=|(R_E(x))(\pi^{-m}x')|\leq\|R_E(x)\|. \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$|\pi|\,\|J_E(x)\|\leq\|R_E(x)\|\leq\|x\|. \tag{2.9}$$

Since E is polar, J_E is a homeomorphism, so is R_E. Next, if the norm $\|\ \|$ of E is polar, then for all $x\varepsilon E$ we have

$$\|x\|=\sup\{|x'(x)|\ :\ x'\varepsilon E',\ \|x'\|\leq 1\}$$
$$=\sup\{|x'(x)|\ :\ x'\varepsilon(E')_1\}=\|R_E(x)\|. \tag{2.10}$$

Therefore R_E is a isometry.

COROLLARY 4. (1) For any strongly polar space E, R_E is a linear isometry. (2) For any topological space S, $R_{BC(S)}$ is a linear isometry.

THEOREM 5. For every $T\varepsilon L(E,BC(S))$, there exists a $\overline{T}\varepsilon L(BC((E')_1),BC(S))$ such that $\overline{T}\circ R_E=T$. In particular, if $\|T\|=1$, then $\overline{T}$ satisfies $\|\overline{T}\|=1$.

PROOF. At first, we notice that $(E')_1$ is supposed to carry the weak * topology. To show theorem, we may assume that $\|T\|\leq 1$. Then ψ_T is a weak * continuous map from S into $(E')_1$. Define

$$\overline{T}\ :\ BC((E')_1)\rightarrow BC(S), \tag{2.11}$$

by

$$\overline{T}(f)=f\circ\psi_T\quad (f\varepsilon BC(E')_1)). \tag{2.12}$$

For every $x\varepsilon E$ and for every $s\varepsilon S$, we have

$$(T(R_E(x))(s)=(R_E(x))(\psi_T(s))=(\psi_T(s))(x)=(T(x))(s). \tag{2.13}$$

Then $\overline{T}\circ R_E=T$. Further,

$$\|\overline{T}\|=\sup\{\frac{\sup\{|f(\psi_T(s))|\ :\ s\varepsilon S\}}{\|f\|}\ :\ f\varepsilon BC((E')_1)\}$$
$$\leq 1. \tag{2.14}$$

Hence if $\|T\|=1$, then

$$1=\|T\|\leq\|\overline{T}\circ R_E\|\leq\|\overline{T}\|\,\|R_E\|\leq\|\overline{T}\|\leq 1. \tag{2.15}$$

The proof is complete.

LEMMA 6. Let E, F and X be Banach spaces. Let $A : E\rightarrow X$ be a linear homeomorphism onto X and $H : E\rightarrow F$ be a linear homeomorphism into F. If there exists an $\overline{A}\varepsilon L(F,X)$ such that $\overline{A}\circ H=A$, then the closed subspace H(E) of F is complemented. In particular, if A and H are linear isometries and $\|\overline{A}\|=1$, then E is orthocomplemented in F.

PROOF. Put $P=H\circ A^{-1}\circ\overline{A}\ :\ F\rightarrow H(E)\subset F$. Then P is a projection onto H(E). If A and H are linear isometries and $\|\overline{A}\|=1$, then $\|P\|\leq 1$. Hence P is an orthoprojection.

THEOREM 7. Let E be of countable type. Then $R_E(E)$ is complemented in $BC((E')_1)$ Especially, c_0 is orthocomplemented in $BC((l^{\infty})_1)$.

PROOF. If E is finite-dimensional, then the assertion of this theorem is clear. Hence we may assume E is infinite-dimensional. Since E is of countable type, E is a polar space. Then by Theorem 3 the map $R_E : E \to BC((E')_1)$ is a linear homeomorphism into $BC((E')_1)$. Further, since E is infinite-dimensional, for an infinite compact ultrametrizable space S, E is linearly homeomorphic to BC(S) (see [4, p.190]). Let $H_0 : E \to BC(S)$ be a linear homeomorphism onto BC(S). By Theorem 5, there exists an $\bar{H}_0 \in L(BC((E')_1), BC(S))$ such that $\bar{H}_0 \circ R_E = H_0$. Hence by Lemma 6, $R_E(E)$ is complemented in $BC((E')_1)$. If $E=c_0$, then the above H_0 can be taken as a linear isometric from c_0 onto BC(S). Since c_0 is strongly polar, by Corollary 4, the map R_{c_0} is linearly isometric. Hence by Theorem 5, there exists an $\bar{H}_0 \in L(BC(((c_0)')_1), BC(S))$ with $\|\bar{H}_0\|=1$. Thus, by Lemma 6, $R_{c_0}(c_0)$ is orthocomplemented in $BC(((c_0)')_1)$. Since $(c_0)' \sim l^\infty$, $BC(((c_0)')_1) \sim BC((l^\infty)_1)$. Hence c_0 is orthocomplemented in $BC((l^\infty)_1)$.

The following corollary follows immediately from Theorem 7.

COROLLARY 8. Let E be of countable type. Then there exists a Banach space X such that $BC((l^\infty)_1)$ and $E \oplus X$ are linearly homeomorphic.

Since c_0 is linearly isometric to some BC(S), the second part of Theorem 7 is a special case of the following corollary.

COROLLARY 9. For any topological space S, let E=BC(S). Then E is orthocomplemented in $BC((E')_1)$.

PROOF. Let $I : E \to BC(S)$ be the identity. Then there exists an $\bar{I} \in L(BC((E')_1), BC(S))$ such that $\bar{I} \circ R_E = 1$ and $\|\bar{I}\|=\|I\|=1$. By Corollary 4, $R_E : E \to BC((E')_1)$ is linearly isometric. Put $P=R_E \circ I^{-1} \circ \bar{I}$. Then P is an orthoprojection of $BC((E')_1)$ onto $R_E(E)$. Hence E is orthocomplemented in $BC((E')_1)$.

COROLLARY 10. The Banach space $BC((c_0)_1)$ contains an orthocomplemented subspace linearly homeomorphic to l^∞. In particular if K is spherically complete, then the Banach space $BC((c_0)_1)$ contains an orthocomplemented subspace linearly isometric to l^∞.

PROOF. Suppose that K is not spherically complete. Applying the extended version of Corollary 9 to S=N (N denotes the set of all natural numbers) and observing that $E=l^\infty$ and $E' \sim c_0$, we can obtain this corollary. Furthermore, if K is spherically complete, then so is l^∞; it follows easily that the second part holds.

3. COMPLEMENTED SUBSPACES IN SECOND DUAL SPACES.

Let $T \in L(E,F')$. Then T determins a map

$$\phi_T : F \to E' \tag{3.1}$$

defined by $(\phi_T(y))(x)=(T(x))(y)$ $(x \in E,\ y \in F)$. Clearly, ϕ_T is linear and $\|\phi_T\| \le \|T\|$. Hence $\phi_T \in L(F,E')$. Let D be a closed subspace and let $D^\perp$ be the annihilator of D in F', i.e. $D^\perp=\{x' \in F' : x'(d)=0,\ d \in D\}$. A subset A of E is said to be compactoid if for every $\varepsilon>0$, there exists a finite subset X of E such that $A \subset B_\varepsilon + Co(X)$, where $B_\varepsilon=\{x \in E : \|x\| \le \varepsilon\}$ and Co(X) is the absolutely convex hull of X. Let $T \in L(E,F)$. If $T(E_1)$ is compactoid in F, then T is said to be compact. A Banach space E is said to be (0)-space if every $T \in L(E,c_0)$ is compact.

PROPOSITION 11. Let E, F be Banach spaces and let D be a closed subspace of F. Then for every $T \in L(E,D^\perp)$, there exists a $\bar{T} \in L(E'',D^\perp)$ such that $\bar{T} \circ J_E=T$ and $\|\bar{T}\|=\|T\|$.

PROOF. Let $J_{E'} : E' \to E''$ be the canonical map. Define an operator

$$\bar{T} : E'' \to D^{\perp} \tag{3.2}$$

by $(\bar{T}(x''))(y)=(J_{E'}(\phi_T(y))(x'')$ $(y\in F,\ x''\in E'')$. For every $x''\in E''$, $\bar{T}(x'')$ is a linear functional on F and $\|\bar{T}(x'')\|\le\|T\|\|x''\|$, so $\bar{T}(x'')\in F'$. For every $y\in D$ and for every $x\in E$,

$$(\phi_T(y))(x)=(T(x))(y)=0. \tag{3.3}$$

Hence $(\bar{T}(x''))(y)=0$. This means that $\bar{T}(x'')\in D^{\perp}$. It follows that $\bar{T}\in L(E'',D^{\perp})$ and $\|\bar{T}\|\le\|T\|$. Further, for every $x\in E$ and for every $y\in F$,

$$\begin{aligned}((\bar{T}\circ J_E)(x))(y)&=(J_{E'}(\phi_T(y))(J_E(x))\\ &=(J_E(x))(\phi_T(y))\\ &=(\phi_T(y))(x)\\ &=(T(x))(y).\end{aligned} \tag{3.4}$$

Hence $\bar{T}\circ J_E=T$. Therefore we have

$$\|T\|\le\|\bar{T}\|\|J_E\|\le\|\bar{T}\|. \tag{3.5}$$

Thus we complete the proof.

The following corollary is immediate from Proposition 11.

COROLLARY 12. Let E and F be Banach spaces. For every $T\in L(E,F')$, there exists a $\bar{T}\in L(E'',F')$ such that $\bar{T}\circ J_E=T$ and $\|\bar{T}\|=\|T\|$.

PROOF. In Proposition 11, put $D=\{0\}$. Then $D^{\perp}=F'$.

PROPOSITION 13. Let E be a Banach space and let D be a closed subspace of E. If D is linearly homeomorphic (resp. isometric) to some dual space and is complemented (resp. orthocomplemented) in E, then $J_E(D)$ is complemented (resp. orthocomplemented) in E''. In particular, if K is not spherically complete and D is of countable type and complemented in E, then $J_E(D)$ is complemented in E''.

PROOF. Let D be a complemented closed subspace of E, linearly homeomorphic to a dual Banach space F'. By Lemma 4.23, (ii) and (iii), in Rooij [4], J_D is a homeomorphism and there exists a projection of D'' onto $J_D(D)$, so there is a $Q\in L(D'',D)$ with $Q\circ J_D=I_D$ (= the identity map of D). As D is complemented in E, there is a projection $P : E \to D$. Then $J_E\circ Q\circ P''\in L(E'',J_E(D))$. As

$$\begin{aligned}(Q\circ P'')\circ J_E&=Q\circ(P''\circ J_E)=Q\circ(J_D\circ P)\\ &=(Q\circ J_D)\circ P=I_D\circ P=P,\end{aligned} \tag{3.6}$$

for $x\in D$ we have

$$(J_E\circ Q\circ P'')(J_E(x))=J_E(P(x))=J_E(x), \tag{3.7}$$

so $J_E\circ Q\circ P''$ is the identity on $J_E(D)$. Thus $J_E\circ Q\circ P''$ is a projection of E'' onto $J_E(D)$. If D is orthocomplemented in E'' and linearly isometric to F', we obtain $\|Q\|\le 1$ and $\|P\|\le 1$, whence $\|J_E\circ Q\circ P''\|\le 1$. In particular, if K is not spherically complete and D is of countable type, then D is linearly homeomorphic to $(l^{\infty})'$ or K^n, where n is some positive integer. Hence by the first assertion of this proposition, we can complete the proof.

COROLLARY 14. Suppose K is not spherically complete. Let E be an infinite-dimensional polar space which is not a (0)-space and let F be an infinite-dimensional Banach space of countable type. Then there exists a Banach space X such that E'' is linearly homeomorphic to $F\oplus X$.

PROOF. By hypothesis, there exists an infinite-dimensional complemented subspace D of E which is of countable type (see [6, p.23]). It follows from Proposition 13 that there exists a subspace X of E'' such that $E''=J_E(D)\oplus X$. Since E is a polar space, J_E is a linear homeomorphism. Therefore, $J_E(D)$ is of countable type. Hence $J_E(D)$ and F are linearly homeomophic, so E'' is linearly homeomorphic to $F\oplus X$.

COROLLARY 15. The subspace $J_E(E)$ of E'' has the extension property in E''.

PROOF. For every continuous linear $x' : J_E(E)\to K$ the function $\bar{x}'=J_{E'}(x'\circ J_E)$ is a continuous linear function $E''\to K$ extending x' and with $\|\bar{x}'\|\le\|x'\|$, hence $\|\bar{x}'\|=\|x'\|$.

The following comment was given by the referee: From the proof of Corollary 15 we obtain a sort of "simultaneous extension", a linear isometry $x'\mapsto\bar{x}'$ of $(J_E(E))'$ onto E''' that assigns to every continuous linear function $J_E(E)\to K$ an extension $E''\to K$. Further, the following question was asked by him: Under what circumstances is there an orthoprojection of E'' onto (the closure of) $J_E(E)$?

COROLLARY 16. Let D be a closed subspace of E. If J_D has an extension T from E into D''. Then D has the weak extension property in E. In particular, if $\|T\|=\|J_D\|$, then D has the extension property in E.

PROOF. By Corollary 12, for every $f\in D'$, there exists an $\bar{f}\in D''$ such that $\bar{f}\circ J_D=f$ and $\|\bar{f}\|=\|f\|$. Put $g=\bar{f}\circ T$. Then $g\in E'$ and $g|D=f$. Hence D has the weak extension property in E. If $\|T\|=\|J_D\|$, then by Corollary 12, for every $x\in E$

$$\begin{aligned}|g(x)|&=|(\bar{f}\circ T)(x)|\le\|\bar{f}\|\|T\|\|x\|\\&=\|\bar{f}\|\|J_D\|\|x\|\le\|\bar{f}\|\|x\|.\end{aligned}\tag{3.8}$$

Hence it holds that $\|g\|\le\|\bar{f}\|=\|f\|\le\|g\|$.

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REFERENCES

1. C. P. GARCIA, Semi-Fredholm operators and the calkin algebra in p-adic analysis I-II, Bull. Soc. Math. Belg., Ser. B. 42 (1990), 69-101.

2. N. DE GRANDE DE KIMPE AND C.P. GARCIA, Weakly closed subspaces and the Hahn-Banach extension property in p-adic analysis, Proc. Kon. Ned. Akad. v. Wet. A91 (1988), 253-261.

3. T. KIYOSAWA, On spaces of compact operators in non-archimedean Banach spaces, Canada. Math. Bull. Vol. 32 (1989), 450-458.

4. A. C. M. VAN ROOIJ, Non-Archimedean Functional Analysis, Marcel Dekker, New York, 1978.

5. W. H. SCHIKHOF, Locally convex spaces over non-spherically complete valued fields I-II, Bull. Soc. Math. Belg. Ser. B. 38 (1986), 187-224.

6. W. H. SCHIKHOF, On p-adic compact operators, Report 8911, Department of Mathematics, Catholic University, Nijmengen, The Netherlands (1989), 1-28.

7. W. H. SCHIKHOF, The complementation property of l^∞ in p-adic Banach spaces, In p-adic Analysis. Proceeding, Trento 1989, endited by F. Baldassarri, S. Bosch and B. Dwork, Springer-Verlag (Lect. Notes 1454), Berlin-Heidelberg-New York (1990), 64-79.

8. M. ZIPPIN, The embedding of Banach spaces into spaces with structure, Ilinois J. Math. 34 (1990), 586-606.