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INCLUSIONS IN ELASTO-PLASTIC SOLIDS UNDER THE INFLUENCE OF INTERNAL AND EXTERNAL PRESSURES

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ABSTRACT. An attempt is made to study the problems of spherical and circular inclusions in elasto-plastic solids under the action of internal increasing pressure and external constant pressure, taking into consideration of the work-hardening effect. Particular attention is given to the linear work hardening effect on both problems. It is shown that results of this analysis are in good agreement with those of ideal plastic solids.

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1. INTRODUCTION.

Several authors including Mott and Nabarro [1], Eshelby [2 - 5], Jaswon and Bhargava [6] Willis [7 - 8] have made systematic investigations of the problems of spherical and cylindrical inclusions in an elastic medium. Both Bhargava [9] and Sengupta [10] have used the principle of minimum energy to study problems of symmetric anisotropic inclusions in an elastic medium. Tresca [11] has shown that a metal yields plastically when the maximum shear stress attains a critical value. Bhargava [12] has investigated the problems of inclusions in elasto-plastic solids under the assumptions of the infinitesimal theory of strain and perfect plasticity conditions satisfied by an elastic-plastic solid. From a physical point of view, the work-hardening effect is important in a plastic material even though the consequences of this effect are quite complex. Hopkins [13] has included the work-hardening effect of the material in his problem of dynamical expansion of spherical cavities in metals. Sengupta and his associates [14 - 15] have studied the problems of inclusions in elastic-plastic solids of work-hardening material of finite and infinite extent. On the other hand, Tokuoka [16] has investigated the plastic deformations and instability of spherical shells under internal pressure. In spite of this progress, further investigation of the inclusion problems in an elastic-plastic medium is needed.

This paper is concerned with the problems of spherical and circular inclusions in elasto-plastic solids under the action of internal increasing pressure and external constant pressure, taking into consideration of the work-hardening effect. Special attention is given to the linear work-hardening effect on both problems.

2. SPHERICAL INCLUSION IN ELASTO-PLASTIC SOLIDS.

We consider a spherical region of radius a in a finite elasto-plastic material which tends to undergo a dimensional change to a sphere of radius $a\ (1 + \delta)$ where δ lies in the region of plastic strain. We designate the spherical region as **inclusion,** and the outer material as **matrix**, where its external boundary is a spherical surface of radius b. The sphere is under constant external pressure, because of the constraints of matrix, stresses appear both in the inclusion and the matrix. Hill [17] considered the problem of spherical shell under uniform pressure on its cavity surface under the assumptions of finite plastic strains.

We assume that the internal boundary of the matrix (a sphere of radius a) is under a gradual increasing pressure p. At first at moderate pressure p, the matrix behaves like an elastic material. Introducing spherial polar co-ordinates (r, θ, ϕ) and the corresponding displacement components (u, v, w), for radial symmetry of deformation we suppose

$$u = u(r), \quad v=0 \quad \text{and} \quad w = 0 \tag{2.1}$$

In absence of body force, the only non-vanishing equation of equilibrium satisfied by the displacement component u is

$$\frac{d}{dr}\left(\frac{du}{dr}+\frac{2u}{r}\right)=0. \tag{2.2}$$

The solution of the above differential equation is

$$u = Ar+\frac{B}{r^2}, \tag{2.3}$$

where A and B are constants. The stress components are, therefore,

$$\sigma_r = (3\lambda+2\mu)-4\mu\frac{B}{r^3}, \tag{2.4ab}$$

$$\sigma_\theta = \sigma_\phi = (3\lambda+2\mu)+2\mu\frac{B}{r^3}.$$

According to Hencky and von Mises, the yielding commences when the maximum of $|\sigma_\theta - \sigma_r|$ reaches a critical value ν, where ν is a material constant.

Now $\sigma_\theta - \sigma_r = 6\mu\frac{B}{r^3}$ is maximum when r = a. Therefore, the yielding commences at r = a and the corresponding pressure p_0 is determined by the following conditions

$$[\sigma_r]_{r=a} = -p_0, \quad [\sigma_r]_{r=b} = -p_e, \quad \left[6\mu\frac{B}{r^3}\right]_{r=a} = \gamma \tag{2.5abc}$$

where $p_0 >> p_e$.

Thus, yielding begins when

$$p_0 = \frac{2}{3}\gamma\left(1 - \frac{a^3}{b^3}\right) + p_e \tag{2.6}$$

and the corresponding displacement at the inner boundary is given by

$$U_0(a) = \frac{2}{3}\gamma a\left(\frac{1}{3\lambda + 2\mu}\frac{a^3}{b^3} + \frac{1}{4\mu}\right) - \frac{a}{3\lambda + 2m}p_e \tag{2.7}$$

With increasing pressure a plastic region spreads into the shell. The plastic boundary be a spherical surface, its radius at any moment is denoted by c.

The stresses and displacement in the elastic region $c \le \gamma \le b$ are still of the form

$$\begin{aligned}
\sigma_r &= -A_1\left(\frac{b^3}{r^3} - 1\right) + B_1\left(\frac{a^3}{r^3} - 1\right), \\
\sigma_\theta = \sigma_\phi &= A_1\left(\frac{b^3}{2r^3} + 1\right) - B_1\left(\frac{a^3}{2r^3} + 1\right), \\
u &= A_1\left(\frac{1}{3r + 2\mu}r + \frac{b^3}{4\mu}\frac{1}{r^2}\right) - B_1\left(\frac{1}{3\lambda + 2\mu}r + \frac{a^3}{4\mu}\frac{1}{r^2}\right).
\end{aligned} \tag{2.8abc}$$

in the region $c \le r \le b$.

We must consider the plastic solid in the region $a \le r \le c$

$$\frac{\partial\sigma_r}{\partial r} + \frac{2}{r}(\sigma_r - \sigma_\theta) = 0$$

subject to the yields condition $\sigma_\theta - \sigma_r = \gamma$.

Solving the differential equation and using the condition of continuity of the normal stress at the elasto-plastic interface r = c, we have

$$\begin{aligned}
\sigma_r &= -2\gamma\log\frac{c}{r} - \frac{2}{3}\gamma\left(1 - \frac{c^3}{b^3}\right) - p_e \\
\sigma_\theta = \sigma_\phi &= \gamma - 2r\log\frac{c}{r} - \frac{2}{3}\gamma\left(1 - \frac{c^3}{b^3}\right) - p_e
\end{aligned} \tag{2.10ab}$$

If $\sigma_r = -p_1$, at r = a, then

$$p_1 = 2\gamma\log\frac{c}{a} + \frac{2}{3}\gamma\left(1 - \frac{c^3}{b^3}\right) + p_e \tag{2.11}$$

From the condition in the plastic region

$$e_{rr} + 2e_{\theta\theta} = \frac{1 - 2v}{E}(\sigma_r + 2\sigma_\theta)$$

where v and E are the Poisson's ratio and Young's modulus respectively, and the condition of continuity of displacement at the interface, the radial displacement u in the plastic region $a \le r \le c$ is

$$u = \frac{2\gamma}{3K}c^3\left[\left(\frac{4\mu c^3 + 3Kb^3}{12\mu}\right)\frac{1}{b^3} + \frac{1}{3}\frac{c^3}{b^3} + \frac{1}{3}\right]\frac{1}{r^2} - \frac{2\gamma(1 - 2v)}{E}\left[\log\frac{c}{r} - \frac{1}{3}\frac{c^3}{b^3} + \frac{1}{3}\right]r - p_e \tag{2.12}$$

The displacement at inner boundary is

$$u(a) = \frac{2\gamma}{3K}c^3\left[\left(\frac{4\mu c^3 + 3Kb^3}{12\mu}\right)\frac{1}{b^3} - \frac{1}{3}\frac{c^3}{b^3} + \frac{1}{3}\right]\frac{1}{a^2} - \frac{2\gamma(1 - 2v)}{E}\left[\log\frac{c}{a} - \frac{1}{3}\frac{c^3}{b^3} + \frac{1}{3}\right]a - p_e\frac{a}{3K} \tag{2.13}$$

We know that the radial displacement of the inclusion is $a(\delta - \varepsilon)$, and

$$u'(a) = p_1\frac{a}{3K'} \quad \text{and therefore} \quad p_1 = 3K'(\delta - \varepsilon).$$

Now by using the condition of continuity of normal stress at the inclusion and matrix, we get

$$3K'(\delta - \varepsilon) = 2\gamma\log\frac{c}{a} + \frac{2}{3}\gamma\left(1 - \frac{c^3}{b^3}\right) + p_e \tag{2.14}$$

The displacement of the inner boundary of the matrix is $a\varepsilon$ which by equation (2.13) is

$$a\varepsilon = \frac{2\gamma}{3K}c^3\left[\left(\frac{4\mu c^3+3Kb^3}{12\mu}\right)\frac{1}{b^3}-\frac{1}{3}\frac{c^3}{b^3}+\frac{1}{3}\right]\frac{1}{a^2}-\frac{2\gamma(1-2\nu)}{E}\left[\log\frac{c}{a}+\frac{1}{3}-\frac{1}{3}\frac{c^3}{b^3}\right]-p_e\frac{a}{3K} \tag{2.15}$$

Therefore

$$\delta = \frac{Y}{E}\frac{c^3}{a^3}(1-\nu) \tag{2.16}$$

The relation between ε and δ is given by

$$\varepsilon = \delta - \frac{2\gamma}{3K'}\left[\log\frac{c}{a}+\frac{1}{3}-\frac{1}{3}\frac{c^3}{b^3}\right]-\frac{p_e}{3K'} \tag{2.17}$$

The material in the region $a \le r \le c$ is under elasto-plastic strain and satisifes the work-hardening conditions. If we define ε_r^e, ε_θ^e as the radial and tangential elastic strains, ε_r^p, ε_θ^p as the radial and tangential plastic strains and $\varepsilon_r, \varepsilon_\theta$ as the corresponding quantities of total strain and if u be the radial displacement, then we have the following relations

$$\varepsilon_r = \varepsilon_r^p + \varepsilon_r^e,\ \varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p \tag{2.18}$$

$$\varepsilon_r = \frac{\partial u}{\partial r},\ \varepsilon_\theta = \frac{u}{r} \tag{2.19}$$

$$E\varepsilon_r^e = \sigma_r - 2\nu\sigma_\theta,\ E\varepsilon_\theta^e = (1-\nu)\sigma_\theta - \nu\sigma_r \tag{2.20}$$

It is customary to assume that the plastic strain satisfies the incompressibility condition

$$\varepsilon_r^p + 2\varepsilon_\theta^p = 0. \tag{2.21}$$

Then the following compressibility can be derived by using the equations (2.18) - (2.21)

$$\sigma_r + 2\sigma_\theta = 3K\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right), \tag{2.22}$$

where K is the bulk modulus.

The elasto-plastic material satifies the following equilibrium equation

$$\frac{\partial\sigma_r}{\partial_r} = \frac{2}{r}(\sigma_\theta - \sigma_r) \tag{2.23}$$

The stress-strain curve for a work-hardening material in uni-axial compression is of the form (see Hill [7])

$$\sigma = \gamma + H(\varepsilon) \tag{2.24}$$

where σ, ε are compressive stress and strain (both taken as positive), γ i the initial yield stress, H is the degree of hardening expressed as a function of total strain. Evidently, in radially symmetric deformations, any element of the material is subject to a uni-axial, radial compressive stress state $\sigma_\theta - \sigma_r$ together with a hydrostatic tensile stress σ_θ. The latter stress, by itself, produces a positive isotropic, elastic strain of amount $\{(1-2\nu)\sigma_\theta\}/E$. Remembering the sign convention for δ and ε, the appropriate general yield criterion is

$$\sigma_\theta - \sigma_r = \gamma + H\left\{-\varepsilon_r + \frac{(1-2\nu)\sigma_\theta}{E}\right\} \tag{2.25}$$

If there is no Bauschinger effect, then H(ε)=-H(-ε), i.e., H (ε) is an odd function of strain. Thus the general yield criterion for a work-hardening material is

$$\sigma_\theta - \sigma_r = \alpha\gamma + H\left\{-\varepsilon_r + \frac{(1-2\nu)\sigma_\theta}{E}\right\},\qquad \alpha = \pm 1 \tag{2.26}$$

In case of linear work-hardening, H is a linear function of total strain and an analytical discussion is possible. In such a case the rate of work-hardening is constant and we suppose the yield criterion as

$$\sigma_\theta - \sigma_r = \gamma\left(1-\frac{E_t}{E}\right) + E_t\left\{-\frac{\partial u}{\partial r}+\frac{1-2\nu}{E}\sigma_\theta\right\} \tag{2.27}$$

where $H'(\varepsilon) = E_t$ = gradient of the stress-strain curve in the plastic range, supposed constant. Solving (2.22) and (2.27), we find

$$\sigma_r = \frac{K\left(1-\frac{E_t}{3K}\right)}{\left(1-\frac{E_t}{9K}\right)}\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right) - \frac{2\gamma}{3}\frac{\left(1-\frac{E_t}{E}\right)}{1-\frac{E_t}{9K}} + \frac{2E_t\frac{\partial u}{\partial r}}{3\left(1-\frac{E_t}{9K}\right)},$$

$$\sigma_\theta - \sigma_r = \frac{K}{\left(1-\frac{E_t}{9K}\right)}\left(\frac{\partial u}{\partial r}+\frac{2u}{r}\right) - \frac{\gamma}{3}\frac{\left(1-\frac{E_t}{E}\right)}{\left(1-\frac{E_t}{9K}\right)} + \frac{E_t\frac{\partial u}{\partial r}}{3\left(1-\frac{E_t}{9K}\right)}. \tag{2.28ab}$$

Eliminating the stresses σ_r and σ_θ between the equations (2.23) and (2.28ab) we obtain the following ordinary differential equation for u.

$$\frac{d}{dr}\left(\frac{du}{dr}+\frac{2u}{r}\right) = \frac{6\gamma\left(1-\frac{E_t}{E}\right)}{(3K+E_t)}\frac{1}{r} \tag{2.29}$$

The solution of the differential equation (2.29) is

$$U = A_2 r + \frac{B_2}{r^2} + \frac{2}{3}\gamma\frac{\left(1-\frac{E_t}{E}\right)}{3K+E_t} r(3\log r - 1) \tag{2.30}$$

where A_2 and B_2 are arbitrary constants. Therefore, with the help of boundary conditions (2.5abc) and the continuity condition, the stress components and displacement can be obtained from (2.30) and (2.28ab) in the plastic region $a \le r \le c$ and they are

$$\sigma_r = \frac{2\gamma\left(1-\frac{E_t}{E}\right)}{3\left(1+\frac{E_t}{3K}\right)}\log\left(\frac{r}{a}\right)^3 - \frac{4\gamma}{3}\frac{E_t(1-\nu)}{E\left(1+\frac{E_t}{3K}\right)} + \frac{4\gamma}{3}\frac{E_t(1-\nu)}{E\left(1+\frac{E_t}{3K}\right)}\left(\frac{c}{a}\right)^3 - p_1,$$

$$\sigma_\theta = \frac{\gamma\left(1-\frac{E_t}{E}\right)}{\left(1+\frac{E_t}{3K}\right)}\left(3\log\frac{c}{r} - \log\frac{r}{a}\right) - \frac{2\gamma}{3E}\left(\frac{1-\nu}{1+\frac{E_t}{3K}}\right)\left(\frac{c^3r^3-c^3a^3}{a^3r^3}\right) + \frac{\gamma c^3}{b^3} + \frac{p_1}{2},$$

$$u = -\left(\frac{2\gamma}{3}\right)\frac{1-2\nu}{E\left(1+\frac{E_t}{3K}\right)}\left[\frac{3KE(b^3-c^3)-E_t\left(Ec^3+3Kb^3\right)}{3KEb^3} + \left(1-\frac{E_t}{E}\right)\log\frac{c^3}{r^3}\right]r + \frac{\gamma(1-\nu)\left(1-\frac{E_t}{9K}\right)c^3}{E\left(1+\frac{E_t}{3K}\right)}\left(\frac{1}{r^2}\right)$$
$$-\frac{p_e}{q_1}\left[3b^3c^3(4\mu-3K)\left(1-\frac{E_t}{9K}\right)\left\{4E_t r + 9Ka^3\left(1-\frac{E_t}{9K}\right)\frac{1}{r^2}\right\}\right],$$

and

$$p_1 = \left(\frac{2Y}{3}\right)\frac{1-\frac{E_t}{E}}{1+\frac{E_t}{3K}}\left(1+3\log\frac{c}{a}\right) + \left(\frac{4Y}{3}\right)\frac{E_t(1-\nu)}{E\left(1+\frac{E_t}{3K}\right)}\left(\frac{c}{a}\right)^3 - \frac{2Y}{3}\left(\frac{c}{b}\right)^3$$
$$+\frac{p_e}{q_1}\left[36KE_t b^3c^3(4\mu-3K)\left(1-\frac{E_t}{9K}\right)\right], \tag{2.31}$$

where

$$q_1 = 9Ka^3\left(1-\frac{E_t}{9K}\right)\left[12Kb^3(E_t-3\mu)+12\mu c^3(E_t+3K)\right]-4E_t\left[3Kb^3(3K+4\mu)\right]. \tag{2.32}$$

Therefore, the displacement at the internal boundary of the matrix is

$$u_1(a)=[u]_{r=a}=-\left(\frac{2Y}{3}\right)\frac{1-2\nu}{E\left(1+\frac{E_t}{3K}\right)}\left[\frac{3KE\left(b^3-c^3\right)}{3KEb^3}+\left(1+\frac{E_t}{E}\right)\log\frac{c^3}{a^3}\right]a$$
$$+\frac{\gamma(1-\nu)\left(1-\frac{E_t}{9K}\right)}{E\left(1+\frac{E_t}{3K}\right)}\frac{c^3}{a^2}-\frac{p_e}{q_1}\left[3b^3c^3(4\mu-3K)\left(1-\frac{E_t}{9K}\right)\left\{4E_t a+9Ka\left(1-\frac{E_t}{9K}\right)\right\}\right] \tag{2.33}$$

It is noted that this displacement induces the initial elastic part of the displacements.

We consider the inclusion which is under a normal pressure p_1' on its external boundary and is therefore

$$\sigma_r'=-p_1',\quad \sigma_\theta'=-p_1',\quad \tau_{r\theta}'=0.$$

It is not very difficult to mark with the help of the yield criteria of Tresca [11] that the material of the inclusion never yields and it is always in a state of elastic deformation. The displacement at the surface of the inclusion is given by

$$u'(a)=\frac{ap_1'}{3K'} \tag{2.34}$$

By using the condition of continuity of normal stress at the inclusion and matrix boundary, we get

$$3K'(\delta-\varepsilon)=\frac{2\gamma}{3}\,\frac{1-\frac{E}{E}}{1+\frac{E_t}{3K}}\left(1+3\log\frac{c}{a}\right)+\frac{4\gamma}{3}\,\frac{E_t(1-\nu)}{E\left(1+\frac{E_t}{3K}\right)}\frac{c^3}{a^3}$$
$$-\frac{2\gamma}{3}\frac{c^3}{b^3}+\frac{p_e}{q_1}\left[36E_tb^3c^3(4\mu-3K)\left(1-\frac{E_t}{9K}\right)\right] \tag{2.35}$$

The displacement of the inner boundary of the matrix will be $a\varepsilon$, which by equation (2.33) is

$$a\varepsilon=-\left(\frac{2\gamma}{3}\right)\frac{1-2\nu}{E\left(1+\frac{Et}{3K}\right)}\left[\frac{3KE\left(b^3-c^3\right)-E_t\left(Ec^3+3Kb^3\right)}{3KEb^3}+\left(1-\frac{E_t}{E}\right)\log\frac{c^3}{a^3}\right]a$$
$$+\frac{\gamma(1-\nu)\left(1-\frac{E_t}{9K}\right)}{E\left(1+\frac{E_t}{3K}\right)}\frac{c^3}{a^2}-\frac{p_e}{q_1}\left[36b^3c^3(4\mu-3K)\left(1-\frac{E_t}{9K}\right)\left\{4E_ta+9Ka\left(1-\frac{E_t}{9K}\right)\right\}\right] \tag{2.36}$$

Solving for δ with the aid of equations (2.35) and (2.36), we obtain

$$\delta=\frac{\gamma(1-\nu)}{E\left(1+\frac{E_t}{3K}\right)}\frac{c^3}{a^3}\left[1+\frac{E_t}{9}\left(\frac{4}{K'}-\frac{1}{K}\right)\right]$$
$$+\frac{2\gamma}{9\left(1+\frac{E_t}{3K}\right)}\left[\left(1-\frac{E_t}{E}\right)3\log\frac{c}{a}\left(\frac{1}{K'}-\frac{1}{K}\right)+\frac{1}{K'}\left(1-\frac{E_t}{E}\right)-\left\{\frac{3KE\left(b^3-c^3\right)-E_t\left(Ec^3+3Kb^3\right)}{3Eb^3}\right\}\right]$$
$$-\frac{2\gamma}{9K'}\frac{c^3}{b^3}+\frac{p_e}{q_1}\,3b^3c^3(4\mu-3K)\left(1-\frac{E_t}{9K}\right)\left[E_t\left(\frac{4K}{K'}-9K\right)\right]. \tag{2.37}$$

If the bulk modulus of the matrix and the inclusion are same, then

$$\delta=\frac{\gamma(1-\nu)}{E}\frac{c^3}{a^3}-\frac{1}{3K'}\frac{p_e}{q_1}b^3c^3(4\mu-3K')(E_t-9K')^2, \tag{2.38}$$

where

$$q_1 = 12a^3(9K' - E_t)\left[K'b^3(E_t - 3\mu) + \mu c^3(E_t + 3K')\right] - 12E_tK'b^3(3\lambda + 2\mu)$$

This result is essentially the same if the work-hardening effect of the plastic material is altogether absent.

Now for the elasto-plastic solids of work-hardening material, the relation between ε and δ is given by

$$\varepsilon = \delta - \left(\frac{2\gamma}{9K}\right)\frac{1-\frac{E_t}{E}}{1+\frac{E_t}{3K}}\left(1+3\log\frac{c}{a}\right) - \frac{4\gamma}{9K}\frac{E_t(1-\nu)}{E\left(1+\frac{E_t}{3K}\right)}\left(\frac{c}{a}\right)^3 + \frac{2Y}{9K}\left(\frac{c}{b}\right)^3$$
$$-12\frac{p_e}{q_1}E_t(bc)^3(4\mu - 3K)\left(1-\frac{E_t}{9K}\right) \tag{2.39}$$

For a particularly plastic material, the corresponding result is given by

$$\varepsilon = \delta - \frac{2\gamma}{3K}\left(\log\frac{c}{a} + \frac{1}{3} - \frac{1}{3}\frac{c^3}{b^3}\right). \tag{2.40}$$

Comparing the above results it follows that the relation between ε and δ for the case of same inclusion and the matrix material depends

(*i*) on Poisson's ratio ν , Young's modulus E, bulk modulus K' of the inclusion and on the yield stress in case of perfectly plastic solid, and (*ii*) on the rate of constant work-hardening factor E_t besides the quantities already mentioned in (*i*).

The equilibrium pressure of the inclusion in the present case is given by (2.31), which may be compared to equation (2.11) for the perfectly plastic case.

Another important point in the present case is the jump in the hoop stress on the surface of the inclusion given by

$$\sigma_\theta - \sigma'_\theta = -\frac{3\gamma\left(1-\frac{E_t}{E}\right)}{\left(1+\frac{E_t}{3K}\right)}\log\frac{c}{a} + \frac{Yc^3}{b^3} + \frac{3p_1}{2}. \tag{2.41}$$

This result is independent of δ and ε but depends on the constant rate of work-hardening factor E_t.

In the case of perfectly plastic solids the jump in the hoop stress is γ, the yield stress, which can be deduced from the above result by putting $E_t = 0$.

All other important results involved in (2.31), (2.33) and (2.39) are found to agree with the corresponding results for perfectly plastic solids given by Bhargava [12], if we put $b \to \infty$, $p_e = 0$ and $E_t = 0$, in the above equations.

3. CIRCULAR INCLUSION UNDER CONDITIONS OF PLANE STRAIN

Bhargava [12] discussed the problem of circular inclusion in an elasto-plastic material of infinite extent. We consdier here circular cylindrical inclusion in a finite elasto-plastic material with effects of work-hardening. The cylinder is under constant external pressure p_e. As the problem of circular cylindrical inclusion even under the conditions of the plane strain is much more difficult than that of spherical inclusion, we suppose $\nu = \frac{1}{2}$ for compressible material. This assumption greatly simplifies the solution of the problem.

Introducing the cylindrical coordinates (r, θ, z) and the corresponding displacements (*u, v, w)* , we assume for the present problem

$$u=u(r),\ v = w = 0\cdot$$

Proceeding in a similar manner as the case of spherical inclusion, we suppose that the internal circular boundary of the matrix is under a pressure p within the elastic limit. Our discussion is confined to the scope of the infinite theory of strain.

The only equation of equilibrium under no body force is

$$\frac{d}{dr}\left(\frac{du}{dr}+\frac{u}{r}\right)=0 \tag{3.1}$$

and hence the solution is

$$u = Cr + \frac{D}{r} \tag{3.2}$$

where C and D are two arbitrary constants.

The yielding commences when the maximum of $|\sigma_\theta - \sigma_r|$ attains a critical value γ, where γ is a material constant.

Now $\sigma_\theta - \sigma_r = 4\mu\frac{D}{r^2}$ is maximum when $r = a$. Therefore the yielding begins at $r = a$ and the corresponding pressure p_o is determined by the following conditions

$$[\sigma_r]_{r=a} = -p_0,\ [\sigma_r]_{r=b} = -p_e$$

and

$$\gamma = 4\mu\frac{D}{a^2} \tag{3.4}$$

where $\quad p_0 >> p_e$

Therefore yieldings commences when

$$p_0 = \frac{\gamma}{2}\left(1-\frac{a^2}{b^2}\right)+p_e \tag{3.5}$$

and at this stage the displacement of the internal boundary is

$$u_0(a) = \frac{\gamma a^2}{4}\left[\frac{a}{b^2(\lambda+\mu)}+\frac{1}{\mu a}\right]-\frac{a^2}{2(\lambda+\mu)}p_e \tag{3.6}$$

For an increasing pressure beyond $\frac{\gamma}{2}\left(1-\frac{a^2}{b^2}\right)+p_e$, the plastic zone is developed in the matrix, and if c be the radius of elasto-plastic interface, then proceeding as in the case of spherical inclusion under external pressure, the elastic stresses and displacement in the matrix beyond c are given by

$$\sigma_r = \frac{\gamma}{2}c^2\left(\frac{1}{b^2}-\frac{1}{r^2}\right)-p_e,\ \sigma_\theta = \frac{\gamma}{2}c^2\left(\frac{1}{b^2}+\frac{1}{r^2}\right)-p_e,\ \sigma_z = \nu(\sigma_r+\sigma_\theta) = \nu\left[\frac{\gamma c^2}{b^2}-2p_e\right] \tag{3.7abc}$$

and

$$u = \frac{\gamma}{4}c^2\left[\frac{1}{b^2(\lambda+\mu)}r+\frac{1}{\mu}\left(\frac{1}{r}\right)\right]-\frac{r}{2(\lambda+\mu)}p_e \tag{3.8}$$

The material in the region $a \le r \le c$ is elasto-plastic and satisfies the work-hardening condition. Presenting an analysis similar to that of a spherical inclusion, the equilibrium equation is

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}(\sigma_r-\sigma_\theta)=0 \tag{3.9}$$

and the yield criterion of linear work-hardening material is

$$\sigma_\theta - \sigma_r = \gamma\left(1-\frac{E_t}{E}\right)+E_t\left(-\frac{\partial u}{\partial r}\right) \tag{3.10}$$

The plastic material satisfies the compressibility equation

$$\frac{\partial u}{\partial r}+\frac{u}{r}=0 \tag{3.11}$$

Since the displacement component u is continuous on the elasto-plastic interface $r=c$, then it is given by

$$u=\frac{\gamma}{4}c^{3}\left[\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{\mu c}\right]\frac{1}{r}-\frac{p_{e}}{2(\lambda+\mu)}\left(\frac{c^{2}}{r}\right) \tag{3.12}$$

Eliminating u and σ_θ from equations (3.9), (3.10) and (3.12), we obtain

$$\frac{\partial\sigma_r}{\partial r}=\frac{\gamma}{r}\left(1-\frac{E_t}{E}\right)+E_t\frac{\gamma}{4}c^{3}\left\{\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{c}\right\}\frac{1}{r^{3}}-E_t\frac{p_e}{2(\lambda+\mu)}\frac{c^{2}}{r^{3}} \tag{3.13}$$

Solving the above differential equation for the normal stress and using the continuity condition on the elasto-plastic interface $r=c$ we have

$$\sigma_r=-\left(1-\frac{E_t}{E}\right)\gamma\log\frac{c}{r}+\frac{\gamma}{2}c^{2}\left[\frac{E_tc}{4}\left\{\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{\mu c}\right\}\left(\frac{1}{c^{2}}-\frac{1}{r^{2}}\right)+\left(\frac{1}{b^{2}}-\frac{1}{c^{2}}\right)\right]$$
$$-p_e\left[\frac{E_tc^{2}}{4(\lambda+\mu)}\left(\frac{1}{c^{2}}-\frac{1}{r^{2}}\right)+1\right]$$

$$\sigma_\theta=\gamma\left(1-\frac{E_t}{E}\right)\left(1-\log\frac{c}{r}\right)+\frac{\gamma}{2}c^{2}\left[\frac{E_tc}{4}\left\{\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{\mu c}\right\}\left(\frac{1}{c^{2}}+\frac{1}{r^{2}}\right)+\left(\frac{1}{b^{2}}-\frac{1}{c^{2}}\right)\right] \tag{3.14ab}$$
$$-p_e\left[\frac{E_tc^{2}}{4(\lambda+\mu)}\left(\frac{1}{c^{2}}+\frac{1}{r^{2}}\right)+1\right]$$

$$\sigma_z=\frac{1}{2}(\sigma_r+\sigma_\theta)$$

$$=\frac{1}{2}\gamma\left[\left(1-\frac{E_t}{E}\right)\left(1-2\log\frac{c}{r}\right)+c^{2}\left\{\frac{E_t}{4c}\left(\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{\mu c}\right)+\left(\frac{1}{b^{2}}-\frac{1}{c^{2}}\right)\right\}-p_e\left[\frac{E_t}{4(\lambda+\mu)}+1\right]\right] \tag{3.15}$$

The pressure p_1 at the internal boundary of the matrix is

$$p_1=\left(1-\frac{E_t}{E}\right)\gamma\log\frac{c}{a}-\frac{\gamma}{2}a^{2}\left[\frac{E_tc}{4}\left\{\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{\mu c}\right\}\left(\frac{1}{c^{2}}-\frac{1}{a^{2}}\right)+\left(\frac{1}{b^{2}}-\frac{1}{c^{2}}\right)\right]$$
$$+p_e\left[\frac{E_tc^{2}}{4(\lambda+\mu)}\left(\frac{1}{c^{2}}-\frac{1}{a^{2}}\right)+1\right] \tag{3.16}$$

As regards the inclusion, if it is under uniform pressure p_1, the principal stress and displacement field is given by

$$\sigma_r^{i}=-p_1\ ,\quad \sigma_\theta^{i}=-p_1\ ,\quad \sigma_z^{i}=-2\nu' p_1\ , \tag{3.17ab}$$

$$\frac{u^{i}}{r}=-p_1\frac{(1+\nu')(1-2\nu')}{E'}, \tag{3.18}$$

where E' and ν' are theYoung's modulus and Poisson's ratio respectively.

According to Bhargava [9] the inclusion never yields for infinitesimal strains. The circular inclusion of radius of radius a spontaneously undergoes dimensional change to a circle of radius $a(1+\delta)$ in the absence of matrix and attains the radius $a(1+\varepsilon)$, when it is in equilibrium in the presence of the matrix. Presenting an analysis similar to that of a spherical inclusion, we obtain

$$\varepsilon=\frac{\gamma}{4}c^{3}\left[\frac{1}{b^{2}(\lambda+\mu)}c+\frac{1}{\mu c}\right]\frac{1}{a^{2}}-\frac{p_e}{2(\lambda+\mu)}\left(\frac{c}{a}\right)^{2} \tag{3.19}$$

$$\delta-\varepsilon=p_1\frac{(1+\nu')(1-2\nu')}{E'} \tag{3.20}$$

where p_1 is given by (3.16).

These results (3.15) - (3.20) for elasto-plastic solids with work-hardening effects are in good agreement with the corresponding results of a perfectly plastic solid if we put $p_e=0$ and $E_t=0$ in the above results.

The jump in the hoop stress as we cross the common boundary of the inclusion and the matrix is

$$\sigma_\theta - \sigma'_\theta = \gamma\left(1-\frac{E_t}{E}\right)+\left(\frac{Y}{4}\right)\frac{E_t c^3}{a^2}\left\{\frac{1}{b^2(\lambda+\mu)}c+\frac{1}{\mu c}\right\}-\frac{E_t}{2(\lambda+\mu)}\left(\frac{c}{a}\right)^2 p_e \tag{3.21}$$

If $E_t = 0$, this result reduces to that of perfectly elastic solid. It is important to point out that in case of perfectly plastic solid the hoop stress is independent of the external boundary of the matrix.

Finally, all the results obtained in the above analysis depend on the constant rate of work-hardening factor E_t which plays an important role in plasticity.

REFERENCES

1. Mott, N. F., and Nabarro, F. R. N., Proc. Roy. Soc. A 52 (1940) p. 25.
2. Eshelby, J. D., Phil. Trans. Roy. Soc. A. 244 (1951) p. 87.
3. Eshelby, J. D., Proc. Roy. Soc. (London) A. 241 , 376 (1957) p. 376.
4. Eshelby J. D., Proc. Roy. Soc. (London) A. 252 (1959b) p. 561.
5. Eshelby, J. D., Progress in solid Mechanics Vol. II (Ed: Sneddon and Hill) (1961).
6. Jawson, M. A., and Bhargava, R. D., Proc. Camb. Phil., Soc. 57 (1961) p. 669.
7. Willis, J. R., Jour. Mech. Phy. Solids Vol. 13 (1965) p. 377.
8. Willis, J. R., Quart. Jour. Mech. Appld. Math., Vol. 17 (1964) p. 156.
9. Bhargava, R.D., Appl. Sci. Res A, Vol. II, (1962) p. 80.
10. Sengupta, P. R., Ind. J. Theo. Phys. Vol. XI, No. 1(1963) pp. 27-37.
11. Tresca, H., Compt. Rend. 59 (1864) p. 754.
12. Bhargava, R. D., Ind. Jour. Math. Vol. 5, No. 2 (1963)
13. Hopkins, H. G., Progress in solid Mechanics, Vol. I (Ed: Sneddon and Hill) (1961) p. 83.
14. Sengupta, P. R., Ind. J. Mech. Math. Part II. Special issue, (1969) pp. 80 - 89.
15. Sengupta, P. R., and Kundu, H. N., Ind. J. Theo. Phys. Vol. 18, No. 4 (1970) pp. 109 - 121.
16. Tokuoka, Tatsuo, Indiana Univ. J. 20 (1970/71) pp. 193 - 206.
17. Hill, R., The Mathematical theory of plasticity, Oxford Univ. Press (1950).