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FACTORIAL RATIOS THAT ARE INTEGERS

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1. INTRODUCTION.

The expressions

$$\frac{(2n)!}{n!(n+1)!}, \tag{1.1}$$

$$\frac{(2r+1)!}{r!} \cdot \frac{(2n)!}{n!(n+r+1)!}, \tag{1.2}$$

$$s \cdot \frac{(2n+s-1)!}{n!(n+s)!}, \tag{1.3}$$

$$\frac{(s+2r)!}{(s-1)!r!} \cdot \frac{(2n+s-1)!}{n!(n+s+r)!}, \tag{1.4}$$

are always integers. They are called the Catalan, generalized Catalan, ballot, and the super ballot numbers, respectively [1]. Here we consider two results concerning divisibility by expressions involving factorials, which generalize these and other similar assertions.

For given positive integers $a_1, a_2, \ldots, a_t$, let $\{a_1, a_2, \ldots, a_t\}$ denote the least common multiple of these integers. For integers n and k, $n > k \geq 0$, set

$$L(n,k) = \{n, n-1, \ldots, n-k\}. \tag{1.5}$$

The novel aspect of our approach is the introduction of the function

$$Q(J,B,C) = \prod_{i=0}^{J} (B-i, L(C,i)), \tag{1.6}$$

for $B \geq C > J \geq 0$, where (α, β) denotes the greatest common divisor of the integers α and β.

Our results describe divisibility properties of this function "from above" and "from below". We have

THEOREM 1.1. let m, k, J, be positive integers such that $m \geq k > J \geq 0$, then the number $F(J,m,k)$ given by

$$F(J,m,k) = \frac{Q(J,m,k)}{m(m-1)\ldots(m-J)} \cdot \binom{m}{k} \tag{1.7}$$

is always an integer, where $\binom{m}{k}$ is the binomial coefficient.

THEOREM 1.2. For integers $s \geq 1$, $r \geq 0$, and $n \geq 1$, the integer

$$P(r,s) = \frac{(2r+s)!}{r!(s-1)!} \tag{1.8}$$

is a multiple of

$$Q(r, r+s+2n, r+s+n). \tag{1.9}$$

Applying Theorem 1.1 with $J = 0$, gives that for $m \geq k > 0$,

$$\frac{(m,k)}{m}\binom{m}{k} \tag{1.10}$$

is an integer. (Note that (1.10) holds also for $k = 0$.) Taking $m = 2n + s$, $k = n$, (1.10) yields that

$$\frac{(2n+s,n)}{2n+s}\binom{2n+s}{n} = (2n+s,n)\cdot\frac{(2n+s-1)!}{n!(n+s)!}$$

is an integer. Since $(2n + s, n) = (s, n)$ divides s, we have that (1.3) is an integer. Then (1.1) is the special case $s = 1$.

As for the expression (1.4), we apply Theorem 1.2, with $s \geq 1$, $r \geq 0$ and $n \geq 1$, obtaining that $P(r, s)$ is a multiple of $Q(r, r + s + 2n, r + s + n)$. But by Theorem 1.1,

$$\frac{Q(r,r+s+2n,r+s+n)}{(r+s+2n)(r+s+2n-1)\dots(s+2n)}\cdot\binom{r+s+2n}{r+s+n}$$
$$= Q(r,r+s+2n,r+s+n)\cdot\frac{(s+2n-1)!}{n!(r+s+n)!}$$

is an integer. Thus (1.4) is an integer. Then (1.2) is the special case $s = 1$.

2. PROOF OF THEOREM 1.1.

If not specified otherwise, all letters denote positive integers. Suppose that an integer X is given as a product:

$$X = \prod_{i=1}^{f} X_i. \tag{2.1}$$

For any positive integer A we define

$$N(A, X) = \text{ the number of } X_i \text{ divisible by } A. \tag{2.2}$$

In all applications of this notation, the reference product (2.1) will be uniquely given. For any prime p, let

$$\text{Pow}\,(p, X) = \text{the largest } \alpha \text{ such that } p^\alpha \text{ divides } X \tag{2.3}$$

It is easy to see that

$$\text{Pow}\,(p, X) = \sum_{\tau=1}^{\infty} N(p^\tau, X). \tag{2.4}$$

The following two lemmas are clear.

LEMMA 2.1. If X is given by (2.1) and $Y = \prod_{j=1}^{h} Y_j$ is, such that, for all primes p and $\tau > 0$, we have

$$N(p^\tau, Y) \geq N(p^\tau, X), \tag{2.5}$$

then X divides Y.

LEMMA 2.2. For $n \geq 1$, let $n! = \prod_{j=1}^{n} j$ be the reference product for $n!$. Then

$$N(p^\tau, n!) = \left[\frac{n}{p^\tau}\right], \tag{2.6}$$

where $[a]$ is the number of positive integers $\leq a$.

From (1.7) we have

$$F(J,m,k) = Q(J,m,k)\frac{(m-J-1)!}{k!(m-k)!}. \tag{2.7}$$

Write $Q(J,m,k)$ in the form (2.1):

$$Q(J,m,k) = \prod_{i=1}^{J}(m-i, L(k,i)) = \prod_{i=0}^{J} Q_i(m,k). \tag{2.8}$$

By Lemmas 2.1 and 2.2, it is enough to show that

$$N(p^\tau, Q) + \left[\frac{m-J-1}{p^\tau}\right] \geq \left[\frac{m-k}{p^\tau}\right] + \left[\frac{k}{p^\tau}\right]. \tag{2.9}$$

Set

$$\Delta(p^\tau, F) = N(p^\tau, Q) + \left[\frac{m-J-1}{p^\tau}\right] - \left[\frac{m-k}{p^\tau}\right] - \left[\frac{k}{p^\tau}\right], \tag{2.10}$$

so that (2.9) is equivalent to

$$\Delta(p^\tau, F) \geq 0. \tag{2.11}$$

Let

$$\frac{m-k}{p^\tau} = \left[\frac{m-k}{p^\tau}\right] + \frac{d_\tau}{p^\tau}, \qquad \frac{k}{p^\tau} = \left[\frac{k}{p^\tau}\right] + \frac{e_\tau}{p^\tau}, \tag{2.12}$$

where,

$$0 \leq d_\tau \leq p^\tau - 1, \quad 0 \leq \ e_\tau \leq p^\tau - 1. \tag{2.13}$$

Then

$$\frac{m-J-1}{p^\tau} = \left[\frac{m-k}{p^\tau}\right] + \left[\frac{k}{p^\tau}\right] + \frac{d_\tau + e_\tau - J - 1}{p^\tau}, \tag{2.14}$$

implying

$$\left[\frac{m-J-1}{p^\tau}\right] = \left[\frac{m-k}{p^\tau}\right] + \left[\frac{k}{p^\tau}\right] + \left[\frac{d_\tau + e_\tau - J - 1}{p^\tau}\right] \tag{2.15}$$

From (2.10) and (2.15) we have:

$$\Delta(p^\tau, F) = N(p^\tau, Q) + \left[\frac{d_\tau + e_\tau - J - 1}{p^\tau}\right]. \tag{2.16}$$

If $d_\tau + e_\tau - J - 1 \geq 0$, then $\Delta(p^\tau, F) \geq 0$. Suppose that $d_\tau + e_\tau - J - 1 < 0$, then $d_\tau + e_\tau \leq J$. If

$$L = d_\tau + e_\tau, \tag{2.17}$$

$0 \leq L \leq J$. By (2.12) we have that p^τ divides both $m - k - d_\tau$ and $k - e_\tau$, and hence it divides $m - (d_\tau + e_\tau) = m - L$. Then p^τ divides $(m - L, k - e_\tau)$. For $t \geq 0$ we have

$$p^\tau \mid (m - L - tp^\tau, k - e_\tau). \tag{2.18}$$

For each t such that $L + tp^\tau \leq J$, p^τ divides:

$$(m - L - tp^\tau, \{k, k-1, \ldots, k - e_\tau, \ldots, k - L - tp^\tau\}) = Q_{L+tp^\tau}(m,k).$$

Thus each $0 \leq t \leq \left[\frac{J-L}{p^\tau}\right]$ maps onto $Q_{L+tp^\tau}(m,k)$ that is divisible by p^τ. Since this map is 1-1 into the factors $Q_i(m,k)$ in (2.8) that are divisible by p^τ, it follows that

$$N(p^\tau, Q) \geq 1 + \left[\frac{J-L}{p^\tau}\right]. \tag{2.19}$$

From (2.16), (2.17), and (2.19) we have

$$\Delta(p^\tau, F) \geq 1 + \left[\frac{J-L}{p^\tau}\right] + \left[\frac{L-J-1}{p^\tau}\right]. \tag{2.20}$$

It is easy to see that

$$\left[\frac{L-J-1}{p^\tau}\right] = -\left[\frac{J-L}{P^\tau}\right] - 1. \tag{2.21}$$

Since (2.20) and (2.21) imply (2.11), Theorem 1.1 is proved.

3. PROOF OF THEOREM 1.2.

LEMMA 3.1. Let

$$U = \prod_{i=1}^{a} U_i, \quad V = \prod_{j=1}^{b} V_j, \quad W = \prod_{l=1}^{c} W_l, \quad Z = \prod_{k=1}^{d} Z_k. \tag{3.1}$$

For all primes p and integers $\tau > 0$, we assume that

$$N(p^\tau, W) \leq \min(N(p^\tau, U), N(p^\tau, V)), \tag{3.2}$$

and

$$N(p^\tau, Z) \leq \max(N(p^\tau, U), N(p^\tau, V)). \tag{3.3}$$

Then $\frac{UV}{W}$ is an integer divisible by Z.

PROOF. We have for any prime p,

$$\mathrm{Pow}(p, \frac{UV}{W}) = \sum_{\tau=1}^{\infty} (N(p^\tau, U) + N(p^\tau, V) - N(p^\tau, W)). \tag{3.4}$$

Let

$$\lambda(p^\tau) = N(p^\tau, U) + N(p^\tau, V) - N(p^\tau, W).$$

Via (3.2) and (3.3) we have:

$$\begin{aligned} \lambda(p^\tau) &= \max(N(p^\tau, U), N(p^\tau, V)) + \min(N(p^\tau, U), N(p^\tau, V)) - N(p^\tau, W) \\ &\geq N(p^\tau, Z) + N(p^\tau, W) - N(p^\tau, W) = N(p^\tau, Z). \end{aligned}$$

This and (3.4) yield

$$\mathrm{Pow}(p, \frac{UV}{W}) \geq \sum_{\tau=1}^{\infty} N(p^\tau, Z) = \mathrm{Pow}(p, Z),$$

and the lemma follows.

Write (1.9) in the form:

$$Q(r, r+s+2n, r+s+n) = \prod_{k=0}^{r} Q_k, \tag{3.5}$$

where

$$Q_k = (r+s+2n-k, L(r+s+n, k)). \tag{3.6}$$

We also rewrite (1.8) in the form:

$$P(r,s) = \prod_{i=0}^{r-1}(2r+s-i)\prod_{j=0}^{r}(r+s-j) \Big/ r! \tag{3.7}$$

We will obtain Theorem 1.2 by applying Lemma 3.1 with

$$U = \prod_{i=0}^{r-1} U_i = \prod_{i=0}^{r-1} (2r + s - i),$$
$$V = \prod_{j=0}^{r} V_j = \prod_{j=0}^{r} (r + s - j),$$
$$W = \prod_{l=0}^{r-1} W_l = \prod_{l=0}^{r-1} (l + 1) = r\,!,$$
$$Z = \prod_{k=0}^{r} Z_k = \prod_{k=0}^{r} Q_k.$$

Thus $Z = Q(r, r+s+2n, r+s+n) = Q$, and

$$P(r,s) = \frac{UV}{W} = \binom{2r+s}{r}\frac{(r+s)\,!}{(s-1)\,!}$$

is an integer. As for (3.2) we have:

$$N(p^\tau, W) = N(p^\tau, r\,!) = \left[\frac{r}{p^\tau}\right],$$
$$N(p^\tau, U) = \left[\frac{2r+s}{p^\tau}\right] - \left[\frac{r+s}{p^\tau}\right],$$
$$N(p^\tau, V) = \left[\frac{r+s}{p^\tau}\right] - \left[\frac{s-1}{p^\tau}\right],$$

from which the inequalities $N(p^\tau, W) \le N(p^\tau, U)$, $N(p^{\tau, W}) \le N(p^\tau, V)$ are obvious. Thus the proof reduces to establishing (3.3). Consider those Q_k, $0 \le k \le r$, such that p^τ divides Q_k. Since this requires that p^τ divides

$$L(r+s+n, k) = \{r+s+n, r+s+n-1, \ldots, r+s+n-k\},$$

the smallest k for which this occurs is μ^*, where

$$r + s + n - \mu^* \equiv 0 \quad (p^\tau), \quad 0 \le \mu^* < p^\tau, \quad \mu^* \le r. \tag{3.8}$$

(It is the last inequality that constrains, in part, the existence of such a Q_k.) Also, there would be a smallest k^*, $\mu^* \le k^* \le r$, such that

$$r + s + 2n - k^* \equiv 0 \pmod{p^\tau}. \tag{3.9}$$

From (3.8) and (3.9) we have

$$n \equiv k^* - \mu^* \pmod{p^\tau}. \tag{3.10}$$

Thus (3.9) is equivalent to

$$r + s + 2(k^* - \mu^*) - k^* \equiv 0 \pmod{p^\tau}, \quad \mu^* \le k^* \le r. \tag{3.11}$$

If (3.8) and (3.11) are not satisfied then $N(p^\tau, Z) = N(p^\tau, Q) = 0$, and (3.3) certainly holds. Thus we may assume that $N(p^\tau, Q) > 0$. The integers k such that p^τ divides Q_k are precisely those such that

$$k^* \le k \le r, \quad k \equiv k^* \pmod{p^\tau}, \tag{3.12}$$

which gives

$$N(p^\tau, Q) = 1 + \left[\frac{r - k^*}{p^\tau}\right]. \tag{3.13}$$

Consider two cases:

Case I. $\mu^* \le k^* \le 2\mu^*$. Here, for all k satisfying (3.12), we have

$$r + s + 2(k^* - \mu^*) - k \le r + s + 2(k^* - \mu^*) - k^* \le r + s,$$

and

$$r + s + 2(k^* - \mu^*) - k \ge r + s + 2(k^* - \mu^*) - r \ge s.$$

Note that this implies

$$r \ge k - 2(k^* - \mu^*) \ge 0.$$

Thus, in this case, a factor Q_k which is divisible by p^τ maps onto $V_{k-2(k^*-\mu^*)}$, which is divisible by p^τ. Since this map is 1-1 into the set of V_j that are divisible by p^τ, we have

$$N(p^\tau, V) \le N(p^\tau, Q). \tag{3.14}$$

Case II. $k^* > 2\mu^*$. Let

$$q^* = r + s + k^* - 2\mu^*. \tag{3.15}$$

By (3.11), we have $q^* \equiv 0 \pmod{p^\tau}$. Also

$$q^* \ge r + s + 1,$$

and

$$q^* \le r + s + k^* \le 2r + s.$$

Thus q^* is one of the U_i and is divisible by p^τ. Hence the integers of the form $q^* + tp^\tau$ such that

$$q^* + tp^\tau \le 2r + s, \quad t \ge 0, \tag{3.16}$$

are also among the U_i's, which are divisible by p^τ. This yields

$$N(p^\tau, U) \ge 1 + \left[\frac{2r + s - q^*}{p^\tau}\right], \tag{3.17}$$

or inserting (3.15),

$$N(p^\tau, U) \ge 1 + \left[\frac{r - k^* + 2\mu^*}{p^\tau}\right]. \tag{3.18}$$

(Actually equality can be proved in (3.18), but this is not needed). Via (3.13) and (3.18), the inequality

$$N(p^\tau, Q) \le N(p^\tau, U)$$

would be a consequence of

$$\left[\frac{r - k^*}{p^\tau}\right] \le \left[\frac{r - k^* + 2\mu^*}{p^\tau}\right].$$

But the last inequality is obvious since $\mu^* \ge 0$, and the theorem follows.

REFERENCES

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