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RESEARCH NOTES

A REMARK ON THE SLICE MAP PROBLEM

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ABSTRACT. It is shown that there exist a σ-weakly closed operator algebra $\tilde{A}$ generated by finite rank operators and a σ-weakly closed operator algebra $\tilde{B}$ generated by compact operators such that the Fubini product $\tilde{A}\bar{\otimes}_F\tilde{B}$ contains properly $\tilde{A}\bar{\otimes}\tilde{B}$.

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1. INTRODUCTION.

In [6] Kraus initiated the slice map problem for σ-weakly closed operator spaces. By an operator space we mean a norm closed linear subspace of $L(H)$, the operators of a Hilbert space H. As stated in the introduction of [9], the slice map problem is of interest because a number of questions concerning tensor products of σ-weakly closed operator spaces are special cases of the slice map problem [4–9].

A σ-weakly closed operator space A is said to have Property S_σ if $A\bar{\otimes}_F B = A\bar{\otimes}B$ for any σ-weakly closed subspace B [6]. Kraus [9] first gave σ-weakly closed operator spaces not having Property S_σ. Effros et al.[3] also characterized σ-weakly closed operator spaces having Property S_σ. One of useful theorems [7, Theorem 2.1] for the slice map problem says that a σ-weakly closed unital operator algebra generated by finite rank operators has Property S_σ (cf. [10]). In this paper, we show that the condition "unital" is essential in the theorem.

2. MAIN RESULT.

For operator spaces A and B, let $A\check{\otimes}B$ denote the norm closed linear span of $\{a\otimes b : a \in A$ and $b \in B\}$. If A and B are σ-weakly closed, let $A\bar{\otimes}B$ denote the σ-weakly closed linear span of $\{a \otimes b : a \in A$ and $b \in B\}$.

Let X and Y be von Neumann algebras. For $g \in X_*$, the predual of X, the right slice map R_g associated with g is a unique bounded linear map from $X\bar{\otimes}Y$ to Y such that $R_g(x \otimes y) =< x, g > y$. For $h \in Y_*$, the left slice map L_h from $X\bar{\otimes}Y$ to X is a unique bounded linear map such that $L_h(x \otimes y) =< y, h > x$. Let A and B be σ-weakly closed linear subspaces of X and

Y, respectively. We define the Fubini product $A\bar{\otimes}_F B$ of A and B by $A\bar{\otimes}_F B = \{x \in X\bar{\otimes}Y : R_g(x) \in B, L_h(x) \in A$ for every $g \in X_*, h \in Y_*\}$. The space $A\bar{\otimes}_F B$ does not depend on $X\bar{\otimes}Y$ [6, Remark 1.2].

Let A be a C^*-algebra. If we assume that A acts universally on a Hilbert space H, the second dual A^{**} of A can be identified with the σ-weak closure B of A in $L(H)$. In this case, the weak* topology on A^{**} coincides with the σ-weak topology on B.

The following example shows that the condition "containing the identity" is necessary in Theorem 2.1 of [7].

EXAMPLE. There exist a σ-weakly closed operator algebra $\tilde{A}$ generated by finite rank operators on a Hilbert space H and a σ-weakly closed operator algebra $\tilde{B}$ generated by compact operators on H such that $\tilde{A}\bar{\otimes}_F\tilde{B}$ contains properly $\tilde{A}\bar{\otimes}\tilde{B}$.

PROOF. Let c_0 denote the C^*-algebra of all complex sequences that converge to zero. Davie [1] constructed a closed linear subspace A_0 of c_0 satisfying the following properties: (1) A_0 does not have the approximation property in the sense of Grothendieck; (2) A_0 contains a dense linear subspace A_1 with the norm topology such that each element has finite support, where each element of c_0 is identified with a function whose domain is the set of all positive integers.

Since c_0^{**} is *-isomorphic to ℓ^∞, the von Neumann algebra of all bounded sequences, we assume that c_0^{**} acts on the Hilbert space ℓ^2 in the usual way. Let A denote the σ-weak closure of A_0 in c_0^{**}. For a closed linear subspace D_0 of c_0, let D denote the σ-weak closure of D_0 in c_0^{**}. We note that $(c_0\check{\otimes}c_0)^{**} = c_0^{**}\bar{\otimes}c_0^{**}$ and $A\bar{\otimes}_F D \subseteq c_0^{**}\bar{\otimes}c_0^{**}$. Put $F(A_0, D_0, c_0\check{\otimes}c_0) = \{z \in c_0\check{\otimes}c_0 : R_g(z) \in D_0, L_h(z) \in A_0$ for every $g \in c_0^*, h \in c_0^*\}$.

By the same argument in the proof of [9, Theorem 5.8] (with a C^*-algebra A replaced by an operator space A), we can choose a closed linear subspace B_0 of c_0 such that $F(A_0, B_0, c_0\check{\otimes}c_0)$ contains properly $A_0\check{\otimes}B_0$. Let B be the σ-weak closure of B_0 in c_0^{**}. Since $A \cap c_0 = A_0$ and $B \cap c_0 = B_0$, we have $F(A_0, B_0, c_0\check{\otimes}c_0) \supseteq (A\bar{\otimes}_F B) \cap (c_0\check{\otimes}c_0)$. The opposite inclusion is trivial. It follows that $F(A_0, B_0, c_0\check{\otimes}c_0) = (A\bar{\otimes}_F B) \cap (c_0\check{\otimes}c_0)$. Since $A\bar{\otimes}B$ is identified with the weak* closure of $A_0\check{\otimes}B_0$ in $(c_0\check{\otimes}c_0)^{**}$, we have $(A\bar{\otimes}B) \cap (c_0\check{\otimes}c_0) = A_0\check{\otimes}B_0$. Hence $A\bar{\otimes}_F B$ contains properly $A\bar{\otimes}B$.

Let $H = \ell^2 \oplus \ell^2$. Put $\tilde{A} = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in A\right\}$ and $\tilde{B} = \left\{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in B\right\}$. Since A_1 consists of finite rank operators on ℓ^2, it is easy to see that $\tilde{A}$ is a σ-weakly closed operator algebra generated by finite rank operators on H. Since c_0 consists of compact operators on ℓ^2, $\tilde{B}$ is a σ–weakly closed operator algebra generated by compact operators on H. Then

$$\tilde{A}\bar{\otimes}_F\tilde{B} \simeq \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes a : a \in A\bar{\otimes}_F B\right\}$$

and

$$\tilde{A}\bar{\otimes}\tilde{B} \simeq \left\{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes a : a \in A\bar{\otimes}B\right\}.$$

Hence $\tilde{A}\bar{\otimes}_F\tilde{B}$ contains properly $\tilde{A}\bar{\otimes}\tilde{B}$. This completes the proof.

Let K be the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space. An operator space A is said to have the operator approximation property if there exists a net $\{\phi_\alpha\}$ of finite rank linear maps from A to itself such that $\phi_\alpha \otimes id_K(z) \to z$ in norm for every $z \in A\check{\otimes}K$ [2].

Using techniques in the proof of Example, we restate Theorem 5.5 of [9] in a slightly different form.

PROPOSITION. Let A_0 be a closed linear subspace of a C^*-algebra D and let A be the weak* closure of A_0 in D^{**}.

Then the following statements are equivalent:

(1) A_0 has the operator approximation property;

(2) $A_0\check{\otimes}B_0 = (A \otimes_F B) \cap (D\check{\otimes}K)$ for any closed linear subspace B_0 of K.

PROOF. We may assmue that D and K act in their universal representations. We note that $D\check{\otimes}K \subseteq D^{**}\check{\otimes}K^{**} = (D\check{\otimes}K)^{**}$. Let B_0 be a closed linear subspace of K and let B be the weak* closure of B_0 in K^{**}. Put $F(A_0, B_0, D\check{\otimes}K) = \{z \in D\check{\otimes}K : R_g(z) \in B_0, L_h(z) \in A_0$ for every $g \in D^*, h \in K^*\}$. Since $A \cap D = A_0$ and $B \cap K = B_0$, we have $F(A_0, B_0, D\check{\otimes}K) \supseteq (A \otimes_F B) \cap (D\check{\otimes}K)$. The opposite inclusion is trivial. It follows that $F(A_0, B_0, D\check{\otimes}K) = (A \otimes_F B) \cap (D\check{\otimes}K)$. Then (2) holds if and only if $F(A_0, B_0, D\check{\otimes}K) = A_0\check{\otimes}B_0$ for any closed subspace B_0 of K. Hence the same argument in the proof of [9, Theorem 5.5] (with a C^*-algebra A replaced by an operator space A) implies that (1) and (2) are equivalent.

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