Internat. J. Math. & Math. Sci.
Vol. 10 No. 1 (1987) 17-33

DIFFEOMORPHISM GROUPS OF CONNECTED SUM OF A PRODUCT OF SPHERES AND CLASSIFICATION OF MANIFOLDS

SAMUEL OMOLOYE AJALA

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540

and

Mathematics Department
University of Lagos
Akoka-Yaba
Lagos-Nigeria, West Africa

(Received April 21, 1986)

ABSTRACT. In [1] and [2] a classification of a manifold M of the type $(n,p,1)$ was given where $H_p(M) = H_{n-p}(M) = \mathbf{Z}$ is the only non-trivial homology groups. In this paper we give a complete classification of manifolds of the type $(n,p,2)$ and we extend the result to manifolds of type (n,p,r) where r is any positive integer and $p = 3,5,6,7 \bmod (8)$.

KEY WORDS AND PHRASES. Pseudo-diffeotopy classes of diffeomorphisms, diffeotopy.
1980 AMS SUBJECT CLASSIFICATION CODE. 57R55

0. INTRODUCTION.

In [1] Edward C. Turner worked on a classification of a manifold M of the type (n,p,r) where this means that M is simply connected smooth n-manifold and $H_p(M) \approx H_{n-1}(M) \approx \mathbf{Z}^r$ the only non-trivial homology groups except for the top and bottom groups. He gave a classification of such manifolds for the case $r = 1$ and $p = 3,5,6,7 \pmod 8$. So Turner gave a classification of M of type $(n,p,1)$ and $p = 3,5,6,7 \pmod 8$. In [2] Hajime Sato independently obtained similar results for M of the type $(n,p,1)$. The question which naturally follows is: Suppose $r = 2,3,4$ and so on, what is the classification of such M ? i.e., what is the classification of M of the type $(n,p,2)$, $(n,p,3)$ and so on? In this paper we will study manifolds for the type $(n,p,2)$ and give its complete classification and then generalize the result to manifolds M of the type (n,p,r) where r is an integer and $p = 3,5,6,7 \pmod 8$.

In §1 we prove the following

THEOREM 1.1 Let M be an n-dimensional oriented, closed, simply connected manifold of the type $(n,p,2)$ with $p = 3,5,6,7 \pmod 8$. Then M is diffeomorphic to $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \cup_h S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ where $n = p+q+1$, $\#_{\partial}$ means connected sum along the boundary as defined by Milnor and Karvaire [3] and $h : S^p \times S^q \# S^p \times S^q \longrightarrow S^p \times S^q \# S^p \times S^q$ is a diffeomorphism.

In §2 we compute the group $\tilde{\pi}_0 \mathrm{Diff}(S^p \times S^q \# S^p \times S^q)$ of pseudo-diffeotopy classes of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$ $p < q$.

Let $GL(2,Z)$ denote the set of 2×2 unimodular matrices and H the subgroup of $GL(2,Z)$ consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ab = cd = 0 \mod 2$ and Z_4 the subgroup of $GL(2,Z)$ of order 4 generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We will adopt the notation $M_{p,q} = \mathrm{Diff}(S^p \times S^q \# S^p \times S^q)$ and $M^+_{p,q}$ the subgroup of $M_{p,q}$ consisting of diffeomorphisms which induce identity map on all homology groups. We will then prove the following

THEOREM 2.1 (i) If $p+q$ is even, then

$$\tilde{\pi}_0(M_{p,q}) \big/ \tilde{\pi}_0(M^+_{p,q}) \approx \begin{cases} Z_4 \oplus Z_4 & \text{if } p \text{ is even, } q \text{ is even} \\ GL(2,Z) \oplus GL(2,Z) & \text{if } p,q = 1,3,7 \\ H \oplus H & \text{if } p,q \text{ odd but } \neq 1,3,7 \\ GL(2,Z) \oplus H & \text{if } p = 1,3,7,\ q \text{ is odd but } \neq 1,3,7 \end{cases}$$

(ii) If $p+q$ is odd then

$$\tilde{\pi}_0(M_{p,q}) \big/ \tilde{\pi}_0(M^+_{p,q}) \approx \begin{cases} Z_4 \oplus H & \text{if } p \text{ is even } q \text{ is odd but } \neq 1,3,7 \\ Z_4 \oplus GL(2,Z) & \text{if } p \text{ is even and } q = 1,3,7 \end{cases}$$

We will further prove the following

THEOREM 2.15 If $p < q$ and $p = 3,5,6,7 \pmod 8$ the order of the group $\tilde{\pi}_0(M^+_{p,q})$ is twice the order of the group $\pi_q(SO(p+1)) \oplus \theta^{p+q+1}$.

In §3 we apply the result in §2 to prove the following

THEOREM 3.7 Let M be an n-dimensional, smooth, closed, oriented manifold such that $n = p+q+1$ and

$$H_i(M) = \begin{cases} \mathbf{Z} & i = 0,n \\ \mathbf{Z} \oplus \mathbf{Z} & i = p, q+1 \\ 0 & \text{elsewhere} \end{cases}$$

then if $p = 3,5,6,7 \pmod 8$ the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to twice the order of the group $\pi_q(SO(p+1)) \oplus \theta^{p+q+1}$. With induction hypothesis and technique used in §1 and §2, one can prove the following

THEOREM 3.8 If M is a smooth, closed simply connected manifold of type (n,p,r) where $n = p+q+1$ and $p = 3,5,6,7 \pmod 8$, then the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to

$$r \text{ times the order of } \pi_q SO(p+1) \oplus \theta^{p+q+1}.$$

1. MANIFOLDS OF TYPE (n,p,r)

DEFINITION: Let M be a closed, simply connected n-manifold. M is said to be of type (n,p,r) if

$$H_i(M) = \begin{cases} \mathbf{Z} & \text{if } i = 0,n \\ \mathbf{Z}^r & \text{if } i = p, q+1 \\ 0 & \text{elsewhere} \end{cases}$$

where $n = p+q+1$

We recall from Milnor and Kervaire [3]

DEFINITION: Let M_1 and M_2 be $(p+q+1)$-manifolds with boundary and H^{p+q+1}

be half-disc, i.e.,

$$H^{p+q+1} = \{x = x_1, x_2, \ldots, x_{p+q+1} \mid |x| \le 1,\ x_1 \ge 0\}$$

Let D^{p+q} be the subset of H^{p+q+1} for which $x_1 = 0$. We can choose embeddings

$$i_\alpha : (H^{p+q+1}, D^{p+q}) \longrightarrow (M_\alpha, \partial M_\alpha) \qquad \alpha = 1,2$$

so that $i_2 \cdot i_1^{-1}$ reverses orientation. We then form the sum $(M_1 - i_1(0)) + (M_2 - i_2(0))$ by identifying $i_1(tu)$ with $i_2((1-t)u)$ for $0 < t < 1$ $u \in S^{p+q} \cap H^{p+q+1}$. This sum is called the connected sum along the boundary and will be denoted by $M_1 \#_\partial M_2$.

REMARK: (1) Notice that the boundary of $M_1 \#_\partial M_2$ is $\partial M_1 \# \partial M_2$.

(2) $M_1 \#_\partial M_2$ has the homotopy type of $M_1 \vee M_2$: the union with a single point in common.

THEOREM 1.1 If M is a smooth manifold of type $(n,p,2)$ where $n = p+q+1$ and $p = 3,5,6,7 \pmod 8$ then there exists a diffeomorphism

$$h : S^p \times S^q \# S^p \times S^q \longrightarrow S^p \times S^q \# S^p \times S^q$$

which induce identity on homology such that M is diffeomorphic to

$$S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} \cup_h S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}.$$

PROOF: Let $\{M, \lambda_1, \lambda_2\}$ be a manifold of type $(n,p,2)$ and λ_1, λ_2 represent the generators of the first and second summands of $H_p(M) \approx \mathbb{Z} \oplus \mathbb{Z}$. We can choose embeddings $\varphi_i : S^p \longrightarrow M$ so as to represent the homology class λ_i $i = 1,2$. Since $p < q$, two homotopic embeddings are isotopic. Let $\alpha_i \in \pi_{p-1} SO(q+1))$ be the characteristic class of the embedded sphere S^p, since $p = 3,5,6,7 \pmod 8$, the normal bundle of the embedded sphere is trivial. It follows that φ_i extends to an embedding $\varphi_i' : S^p \times D^{q+1} \longrightarrow M$ such that its homology class is λ_i. Then we can form a connected sum along the boundary of the two embedded copies of $S^p \times D^{q+1}$ to get $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$. We then have an embedding $i : S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} \longrightarrow M$ such that $i_*[S^p] = \lambda_1 + \lambda_2 \in H_p(M)$. Notice that the boundary of $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$ is $S^p \times S^q \# S^p \times S^q$ and since $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$ has the homotopy type of $S^p \times D^{q+1} \vee S^p \times D^{q+1}$ then it is easy to see that

$$H_i(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } i = p \end{cases}.$$

It is also easy to see that

$$H_i(M - \mathrm{Int}(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } i = p \end{cases}.$$

Now since $S^p \times D^{q+1}$ is a trivial disc bundle over S^p then it has cross sections; hence, there exists orientation reversing diffeomorphism of $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$ onto itself. Thus there exists an orientation reversing embedding

$$j : S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} \longrightarrow M - \mathrm{Int}(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$$

such that $j_*[S^p] = \lambda_1 + \lambda_2$ and in fact this embedding is a homotopy equivalence. It follows by [4, Thm. 4.1] that $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$ is diffeomorphic to $M\text{-Int}(S^p \times D^{q+1} \# S^p \times D^{q+1})$. Consequently, it follows that M is diffeomorphic to $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} \cup_h S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$ for an orientation preserving diffeomorphism $h : S^p \times S^q \#_\partial S^p \times S^q \longrightarrow S^p \times S^q \#_\partial S^p \times S^q$. From the embeddings in the proof, it is clear that h induce identity on homology.

2. THE GROUP $\tilde{\pi}_0 \mathrm{Diff}(S^p \times S^q \# S^p \times S^q)$

For convenience, we adopt the notation $M_{p,q} = \mathrm{Diff}(S^p \times S^q \# S^p \times S^q)$ and $M^+_{p,q}$ the subset of $M_{p,q}$ consisting of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$ which induce identity on all homology groups.

DEFINITION: Let M be an oriented smooth manifold. $\mathrm{Diff}(M)$ is the group of orientation preserving diffeomorphisms of M. Let $f,g \in \mathrm{Diff}(M)$, f and g are said to be pseudo-diffeotopic if there exists a diffeomorphism H of $M \times I$ such that $H(x,0) = (f(x),0)$ and $H(x,1) = (g(x),1)$ for all $x \in M$. The pseudo-diffeotopy class of diffeomorphisms of M is denoted by $\tilde{\pi}_0(\mathrm{Diff}\, M)$. We wish to compute $\tilde{\pi}_0(M_{p,q})$ for $p < q$. If $f, \in M_{p,q}$ then f induces an automorphism

$$f_* : H_*(S^p \times S^q \# S^p \times S^q) \longrightarrow H_*(S^p \times S^q \# S^p \times S^q)$$

of homology groups of $S^p \times S^q \# S^p \times S^q$. Since pseudo-diffeotopic diffeomorphisms induce equal automorphism on homology then we have a well-defined homomorphism

$$\Phi : \tilde{\pi}_0(M_{p,q}) \longrightarrow \mathrm{Auto}(H_*(S^p \times S^q \# S^p \times S^q))$$

where $\mathrm{Auto}(H_*(S^p \times S^q \# S^p \times S^q)$ denotes the group of dimension preserving automorphisms of $H_*(S^p \times S^q \# S^p \times S^q)$.

THEOREM 2.1 (i) If $p+q$ is even then

$$\Phi(\tilde{\pi}_0(M_{p,q})) = \begin{cases} \mathbb{Z}_4 \oplus \mathbb{Z}_4 & \text{if } p,q \text{ are even} \\ GL(2,Z) \oplus GL(2,Z) & \text{if } p,q \text{ are } 1,3,7 \\ H \oplus H & \text{if } p,q \text{ are odd but } \neq 1,3,7 \\ GL(2,Z) \oplus H & \text{if } p = 1,3,7 \text{ and } q \text{ is odd but } \neq 1,3,7 \end{cases}$$

The following propositions give the proof of Theorem 2.1.

PROPOSITION 2.1 If $p+q$ is even, p is even, then

$$\Phi(\tilde{\pi}_0(M_{p,q})) = Z_4 \oplus Z_4 .$$

PROOF: Since $p+q$ is even and p is even then q must also be even. We have

$$H_i(S^p \times S^q \# S^p \times S^q) = \begin{cases} \mathbb{Z} & \text{if } i = 0\ ,\ p+q \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = p \text{ or } q \\ 0 & \text{elsewhere} \end{cases} .$$

Generators of $H_0(S^p \times S^q \# S^p \times S^q)$ and $H_{p+q}(S^p \times S^q \# S^p \times S^q)$ are mapped to the same generators but $H_p(S^p \times S^q \# S^p \times S^q) = \mathbb{Z} \oplus \mathbb{Z}$. If $f \in M_{p,q}$, we shall denote by $\Phi(f)_p$ the automorphism $f_* : H_p(S^p \times S^q \# S^p \times S^q) \longrightarrow H_p(S^p \times S^q \# S^p \times S^q)$ induced by the image f under Φ in dimension p. Then $\Phi(f)_p = f_* : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is the induced automorphism. If e_1, e_2 are the generators of the first and second summand of $H_p(S^p \times S^q \# S^p \times S^q)$ if $\circ$ denotes the intersection then $e_1 \circ e_1 = 0$, $e_2 \circ e_2 = 0$, $e_1 \circ e_2 = 1$ and $e_2 \circ e_1 = -1$. Let $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL(2,Z)$, if $\Phi(f)_p$ takes e_1, e_2 to e_1', e_2' respectively then $e_1' = a_1e_1 + a_2e_2$ and $e_2' = a_3e_1 + a_4e_2$ then

$$\begin{aligned} e_1' \circ e_1' &= (a_1e_1 + a_2e_2)\cdot(a_1e_1 + a_2e_2) \\ &= a_1a_1e_1\cdot e_1 + a_1a_2e_1\cdot e_2 + a\ a_1e_2\cdot e_1 + a_2a_2e_2\cdot e \\ &= a_1a_2e_1\cdot e_2 + a_2a_1e_2\cdot e_1 = a_1a_2 - a_1a_2 = 0 . \end{aligned}$$

Similarly $e_2'\cdot e_2' = 0$

but $e_1'\cdot e_2' = (a_1e_1 + a_2e_2) \circ (a_3e_1 + a_4e_2)$

$$\begin{aligned} &= a_1a_3e_1 \circ e_1 + a_1a_4e_1 \circ e_2 + a_2a_3e_2 \circ e_1 + a_2a_4e_2 \circ e_2 \\ &= a_1a_4 - a_2a_3 = 1 \quad \text{since } GL(2,Z) \text{ is unimodular.} \end{aligned}$$

$$\begin{aligned} e_2'\cdot e_1' &= (a_3e_1 + a_4e_2)\cdot(a_1e_1 + a_2e_2) = a_3a_1e_1 \circ e_1 + a_3a_2e_1 \circ e_2 + a_4a_1e_2 \circ e_1 \\ &\quad + a_4a_2e_2 \circ e_2 = a_3a_2 - a_4a_1 = -1 \end{aligned}$$

hence for p even $\Phi(f)_p$ is an element of a subgroup of $GL(2,Z)$ generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This subgroup has elements $\left\{\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}\right\} \approx Z_4$. Hence $\Phi(f)_p \in \mathbf{Z}_4$. Similarly for $i = q$ $\Phi(f)_q \in \mathbf{Z}_4$, it then follows that

$$\Phi(\tilde{\pi}_0(M_{p,q})) \subset \mathbf{Z}_4 \oplus \mathbf{Z}_4 .$$

We now show that $\mathbf{Z}_4 \oplus \mathbf{Z}_4 \subset \Phi(\tilde{\pi}_0(M_{p,q}))$. We need to show that the generators of $\mathbf{Z}_4 \oplus \mathbf{Z}_4$ can be realized as the image of Φ. We shall adopt the notation $(S^p \times S^q)_1 \# (S^p \times S^q)_2$ where the subscripts 1 and 2 denote the first and second summands of $S^p \times S^q \# S^p \times S^q$ and let R_p and R_q be reflections of S^p and S^q respectively. If $(x_1,y_1) \in (S^p \times S^q)_1$ and $(x_2,y_2) \in (S^p \times S^q)_2$, we define $f \in M_{p,q}$

$$\begin{aligned} f(x_1,y_1) &= (R_p(x_2),R_q(y_2)) \\ f(x_2,y_2) &= (x_1,y_1) \end{aligned}$$

In other words $f((x_1,y_1)(x_2,y_2)) = ((R_p(x_2),R_q(y_2)), (x_1,y_1))$

$$(x_1,y_1) \in (S^p \times S^q)_1 \quad \text{and} \quad (x_2,y_2) \in (S^p \times S^q)_2 .$$

For $\Phi(f)_p \in \text{Auto}\, H_p(M_{p,q})$, if e_1,e_2 are the generators of the first and second summands of $H_p(S^p \times S^q \# S^p \times S^q) = \mathbf{Z} \oplus \mathbf{Z}$ since f takes x_1 to $R_p(x_2)$ and f takes x_2 to x_1, then it is easily seen that $\Phi(f)_p(e_1) = -e_2$ and $\Phi(f)_p(e_2) = e_1$. Hence $e_1' = -e_2$ and $e_2' = e_1$ and so $e_1' \circ e_1' = -e_2 \circ -e_2 = 0$, $e_2' \circ e_2' = e_1 \circ e_1 = 0$, $e_1' \circ e_2' = -e_2 \circ e_1 = 1$ and $e_2' \circ e_1' = e_1 \circ -e_2 = -1$. Hence Φ maps f in dimension p to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which generates Z_4. Similar argument shows that Φ maps f in dimension q to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which generates $\mathbf{Z}_4$. Then Φ maps onto $Z_4 \oplus Z_4$ hence the proof.

PROPOSITION 2.2 If p+q is even but $p,q = 1,3,7$ then $\Phi(\tilde{\pi}_0(M_{p,q})) = GL(2,Z) \oplus GL(2,Z)$.

PROOF: From [5, Appendix B] and [6] one sees that $GL(2,Z)$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Since $p,q = 1,3,7$ it follows by [7, §1] that there exist maps $f : S^p \longrightarrow SO(p+1)$ and $g : S^q \longrightarrow SO(q+1)$ such that f and g have index +1.

We then define $h \in M_{p,q}$

$$h(x_1,y_1) = (x_1,y_1) \qquad (x_1,y_1) \in (S^p \times S^q)_1$$

$$h(x_2,y_2) = (f(x_1)\cdot x_2\ ,\ g(y_1)\cdot y_2) \qquad (x_2,y_2) \in (S^p \times S^q)_2$$

i.e., $h((x_1,y_1),(x_2,y_2)) = ((x_1,y_1),(f(x_1)\cdot x_2, g(y_1)\cdot y_2))$

Since f has index +1 and h takes x_1 to x_1 and x_2 to $f(x_1)\cdot x_2$ then it follows by an easy application of [7, Prop. 1.2] or [6, Prop. 2.3] that $\Phi(h)_p$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ also since g has index +1 and h takes y_1 to y_1 and y_2 to $g(y_1)\cdot y_2$ then $\Phi(h)_q$ is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence Φ maps h to $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$. We now define $\alpha \in M_{p,q}$ by

$$\alpha(x_1,y_1) = (R_p(x_2),R_q(y_2)) \qquad (x_1,y_1) \in (S^p \times S^q)_1$$

$$\alpha(x_2,y_2) = (x_1,y_1) \qquad (x_2,y_2) \in (S^p \times S^q)_2$$

i.e., $\alpha((x_1,y_1),(x_2,y_2)) = ((R_p(x_2),R_q(y_2)),(x_1,y_1))$

Since α takes x_1 to $R_p(x_2)$ and x_2 to x_1 it follows from Proposition 2.1 that $\Phi(\alpha)_p$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and by similar reasoning $\Phi(\alpha)_q$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This means that Φ maps α to $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$. Since $GL(2,Z)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then it follows that for $p,q = 1,3,7$

$$\Phi(\tilde{\pi}_0(M_{p,q})) \approx GL(2,Z) \oplus GL(2,Z)\ .$$

PROPOSITION 2.3 If $p+q$ is even but p and q are odd but $p,q \neq 1,3,7$, then $\Phi(\tilde{\pi}_0(M_{p,q})) \approx H \oplus H$.

PROOF: By using Proposition 2.1 and [8, Lemma 5] it is enough to produce a diffeomorphism in $M_{p,q}$ whose image under Φ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in each of the dimensions p and q. This is because $\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}, \begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ generate H. However $\begin{pmatrix} \pm1 & 0 \\ 0 & \pm1 \end{pmatrix}$ is trivially the image under Φ of identity map and reflections on each coordinate while $\begin{pmatrix} 0 & \pm1 \\ \pm1 & 0 \end{pmatrix}$ is by Proposition 2.1 the image under Φ of an element of $M_{p,q}$. However, there exists a map $\alpha : S^p \longrightarrow SO(p+1)$ of index 2 by [8] so also is a map $\beta : S^q \longrightarrow SO(q+1)$ of index 2 and then we can define $f \in M_{p,q}$ thus.

$$f(x_1,y_1) = (x_1,y_1) \qquad (x_1,y_1) \in (S^p \times S^q)_1$$

$$f(x_2,y_2) = (\alpha(x_1)\cdot x_2\ ,\ \beta(y_1)\cdot y_2) \qquad (x_2,y_2) \in (S^p \times S^q)_2$$

i.e., $f((x_1,y_1)\ ,\ (x_2,y_2)) = ((x_1,y_1)\ ,\ (\alpha(x_1)\cdot x_2\ ,\ \beta(y_1)\cdot y_2))$.

It easily follows that since f takes x_1 to x_1 and takes x_2 to $\alpha(x_1)\cdot x_2$ with α having index 2 then it follows by applying [7, Lemma 5] that $\Phi(f)_p$ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Similar argument shows that $\Phi(f)_q$ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$; hence $\Phi(\tilde{\pi}_0(M_{p,q})) \approx H \oplus H$.

PROPOSITION 2.4 If $p+q$ is even, $p = 1,3,7$ but q is odd and $\neq 1,3,7$ then $\Phi(\tilde{\pi}_0(M_{p,q})) = GL(2,Z) \oplus H$.

PROOF: $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$ generates $GL(2,Z)$ while $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\}$ generates H, since $q \neq 1,3,7$ and by [8] there exists $\alpha : S^q \longrightarrow SO(q+1)$ of index 2. If R_p is reflection of S^p then we define $h \in M_{p,q}$

$$h(x_1,y_1) = (R_p(x_2),y_1) \qquad (x_1,y_1) \in (S^p \times S^q)_1$$

$$h(x_2,y_2) = (x_1,\alpha(y_1)\cdot y_2) \qquad (x_2,y_2) \in (S^p \times S^q)_2$$

Since h takes x_1 to $R_p(x_2)$ and takes x_2 to x_1 it follows by Proposition 2.1 that $\Phi(h)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; similarly h takes y_1 to y , and y_2 to $\alpha(y_1)\cdot y_2$ and since α has index 2, it follows that $\Phi(h)_q = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Now if R_q is a reflection on S^q and $\beta : S^p \longrightarrow SO_{p+1}$ is of index +1 then we define $f \in M_{p,q}$

$$f(x_1,y_1) = (x_1,R_q(y_2)) \qquad (x_1,y_1) \in (S^p \times S^q)_1$$

$$f(x_2,y_2) = (\beta(x_1)\cdot x_2,y_1) \qquad (x_2,y_2) \in (S^p \times S^q)_2$$

then it is easy to see that $\Phi(f)_p = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\Phi(f)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so the image of h under Φ is $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\}$ and the image of f under Φ is $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$ and since $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\}$ generate H and $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$ generate $GL(2,Z)$ then it follows that $\Phi(\tilde{\pi}_0(M_{p,q})) \approx GL(2,Z) \oplus H$. Hence the proof.

REMARK. For p odd but $\neq 1,3,7$ and $q = 1,3,7$, we have the same result as above using the same method but since by assumption $p < q$ only one dimension (consequently one manifold) comes in here, viz $p = 5$, $q = 7$, i.e., $S^5 \times S^7 \# S^5 \times S^7$.

Combination of Propositions 2.1, 2.2, 2.3, and 2.4 proves Theorem 2.1(i).

PROPOSITION 2.5 Suppose $p+q$ is odd and p is even and q odd $\neq 1,3,7$ then $\Phi(\tilde{\pi}_0(M_{p,q})) \approx \mathbf{Z}_4 \oplus H$.

PROOF: Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate Z_4 and $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\}$ generate H , then we only need to find the diffeomorphism in $M_{p,q}$ that Φ maps to these generators. Similar to Proposition 2.4, we define $f \in M_{p,q}$ by

$$f(x_1,y_1) = (R_p(x_2,y_1) \qquad (x_1,y_1) \in (S^p \times S^q)_1$$

$$f(x_2,y_2) = (x_1,\alpha(y_1)\cdot y_2) \qquad (x_2,y_2) \in (S^p \times S^q)_2$$

where R_p is the reflection on S^p and $\alpha : S^q \longrightarrow SO_{q+1}$ is of index 2 which exists by [3] since $q \neq 1,3,7$. It then follows that $\Phi(f)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Phi(f)_q = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Also we define $g \in M_{p,q}$ thus

$$g(x_1,y_1) = (x_1,R_q(y_2)) \qquad (x_1,y_1) \in (S^p \times S^q)_1 \ , \ (x_2,y_2) \in (S^p \times S^q)_2$$

$$g(x_2,y_2) = (x_2,y_1)$$

i.e., $g((x_1,y_1),(x_2,y_2)) = ((x_1,R_q(y_2)),(x_2,y_1))$
where R_q is the reflection on S^q . Since g takes x_1 to x_1 and x_2 to x_2 then $\Phi(g)_p = \text{identity} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and since g takes y_1 to $R_q(y_2)$ and y_2 to y_1 it follows that by applying Proposition 2.1, $\Phi(g)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence f is mapped by Φ to $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right\}$ while g is mapped by Φ to $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$ and since these matrices generate H and Z_4 respectively then it follows that $\Phi(\tilde{\pi}_0(M_{p,q})) \approx Z_4 \oplus H$.

PROPOSITION 2.6 Suppose $p+q$ is odd and p is even q is odd and $= 1,3,7$. Then $\Phi(\tilde{\pi}_0(M_{p,q})) \approx \mathbf{Z}_4 \oplus GL(2,Z)$.

PROOF: Again since $q = 1,3,7$ by [6, Prop. 2.4] there exists a map $\alpha: S^q \to SO_{q+1}$ of index 1. Since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generates Z_4 and $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$ generate $GL(2,Z)$ we define elements of $M_{p,q}$ that are mapped onto these generators. Let $h \in M_{p,q}$ be defined thus

$$h(x_1,y_1) = (R_p(x_2),y_1) \quad \text{where} \quad (x_1,y_1) \in (S^p \times S^q)_1$$

$$h(x_2,y_2) = (x_1,\alpha(y_1)\cdot y_2) \quad (x_2,y_2) \in (S^p \times S^q)_2$$

i.e., $h(x_1,y_1),(x_2,y_2)) = ((R_p(x_2),y_1),(x_1,\alpha(y_1)-y_2))$

where R_p is the reflection of S^p. Then it is easy to see that $\Phi(h)_p = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ while $\Phi(h)_q = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Also one can define $f \in M_{p,q}$ as

$$f(x_1,y_1) = (x_1,R_q(x_2)) \quad \text{where} \quad (x_1,y_1) \in (S^p \times S^q)_1 \, , \ (x_2,y_2) \in (S^p \times S^q)_2$$

$$f(x_2,y_2) = (x_2,y_1)$$

i.e., $f((x_1,y_1),(x_2,y_2)) = ((x_1,R_q(y_2)),(x_2,y_1))$ where R_q is a reflection of S^q and so it is easily seen that $\Phi(f)_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ while $\Phi(f)_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so h is mapped by Φ to $\left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}$ while f is mapped by Φ to $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}$ and since these sets of matrices generate $GL(2,Z)$ and Z_4 respectively then $\Phi(\tilde{\pi}_0(M_{p,q})) \approx Z_4 \oplus GL(2,Z)$. Combining Propositions 2.5 and 2.6, we obtain Theorem 2.1 (ii).

REMARK. If p is odd but $\neq 1,3,7$ and q is even, we get the same result as in Proposition 2.5 using equivalent method. Also if $p = 1,3,7$ and q is even, we obtain the same result as that of Proposition 2.6.

Since $M^+_{p,q}$ denotes the subgroup of $M_{p,q}$ consisting of diffeomorphisms of $S^p \times S^q \# S^p \times S^q$ which induce identity map on all homology groups, it follows that $M^+_{p,q}$ is the kernel of the homomorphism Φ. We now compute $M^+_{p,q}$. We define a homomorphism

$$G : \tilde{\pi}_0(M^+_{p,q}) \longrightarrow \pi_p SO(q+1)$$

Given an element $\{f\} \in \tilde{\pi}_0(M^+_{p,q})$, since $\Phi(f)$ is identity, it means that if $i(S^p \times \{p_0\})$ is the usual identity embedding of $S^p \times \{p_0\}$ into $S^p \times S^q \# S^p \times S^q$ where p_0 is a fixed point in S^q far away from the connected sum, then the sphere $S^p \times \{p_0\}$ in $S^p \times S^q \# S^p \times S^q$ represents a generator of the homology $H_p(S^p \times S^q \# S^p \times S^q) \approx \mathbf{Z} \oplus \mathbf{Z}$. Since $\Phi(f)$ is identity, it follows that $f(s^p \times p_0)$ is homologous to $i(S^p \times p_0)$ and since $p < q$ and by Hurewicz theorem, f and i are homotopic and in fact with the dimension restriction, they are diffeotopic. By tubular neighborhood theorem, f is diffeotopic to a map say f'' such that $f''(S^p \times D^q) = S^p \times D^q$ where $f''(x,y) = (x,\alpha(f'')(x)\cdot y)$ and $\alpha(f'') : S^p \longrightarrow SO(q)$. Let $i : SO(q) \longrightarrow SO(q+1)$ be the inclusion map and $i_* : \pi_p SO(q) \longrightarrow \pi_p SO(q+1)$ the induced map on the homotopy groups. Then we define

$$G\{f\} = i_*(\alpha(f'')) \, .$$

LEMMA 2.7 G is well-defined.

PROOF: Let $f,h \in M^+_{p,q}$ such that f and h are pseudo-diffeotopic then $f\cdot h^{-1} \in M^+_{p,q}$ is pseudo-diffeotopic to the identity. If $G\{f\} = i_*\alpha(f'')$ and

$G(h) = i_*\alpha(h'')$ where $f(x,y) = (x,\alpha(f'')(x)\cdot y)$ and $h(x,y) = (x,\alpha(h'')(x)\cdot y)$ for $(x,y) \in S^p \times D^q$ then it follows that

$$f\cdot h^{-1}(x,y) = (x,\alpha(f'')\alpha(h'')^{-1}(x)\cdot y) \quad (x,y) \in S^p \times D^q .$$

We wish to show that $i_*\alpha(f'') = i_*\alpha(h'')$. Since $G(f) = i_*\alpha(f'') \in \pi_p SO(q+1)$ and $G(h) = i_*\alpha(h'') \in \pi_p SO(q+1)$ then we can define maps $f_1, h_1 \in \mathrm{Diff}(S^p \times S^q)$ thus $f_1(x,y) = (x, i_*\alpha(f'')(x)\cdot y)$ and $h_1(x,y) = (x, i_*\alpha(h'')(x)\cdot y)$ then consider $f_1h_1^{-1} \in \mathrm{Diff}(S^p \times S^q)$ defined by $f_1h_1^{-1}(x,y) = (x, i_*\alpha(f'')\ i_*\alpha(h'')^{-1}(x)\cdot y)$ $(x,y) \in S^p \times S^q$. Since $f\cdot h^{-1}$ is pseudo-diffeotopic to identity so is $f_1\cdot h_1^{-1}$ by its definition. Hence $f_1\cdot h_1^{-1} \in \mathrm{Diff}(S^p \times S^q)$ is diffeotopic to the identity hence it extends to a diffeomorphism g of $D^{p+1} \times S^q$, i.e., there exists $g \in \mathrm{Diff}(D^{p+1} \times S^q)$ such that $g|\mathrm{Diff}(S^p \times S^q) = f_1\cdot h_1^{-1}$. Let S_β denote the q-sphere bundle over p+1-sphere with characteristic map $\beta : S^p \longrightarrow SO(q+1)$. Then we have

$$S_{i_*\alpha(f'')i_*\alpha(h'')^{-1}} = D^{p+1} \times S^q \bigcup_{f_1h_1^{-1}} D^{p+1} \times S^q$$

so this gives a q-sphere bundle over a p+1-sphere with the characteristic class of the equivalent plane bundle being $i_*\alpha(f'')\cdot i_*\alpha(h'')^{-1}$. However, $f_1h_1^{-1}$ extends to $g \in \mathrm{Diff}(D^{p+1} \times S^q)$ then we have

$$\begin{array}{ccc} S^{p+1} \times S^q = & D_1^{p+1} \times S_1^q \bigcup_{id} D_2^{p+1} \times S_2^q \\ & \downarrow \qquad\qquad \downarrow \\ S_{i_*\alpha(f'')\cdot i_*\alpha(h'')^{-1}} = & D_1^{p+1} \times S_1^q \bigcup_{f_1h_1^{-1}} D_2^{p+1} \times S_2^q \end{array}$$

Hence we define a map $H : S^{p+1} \times S^q \longrightarrow S_{i_*\alpha(f'')\cdot i_*\alpha(h'')^{-1}}$

$$H(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in D_2^{p+1} \times S_2^q \\ g(x,y) & \text{if } (x,y) \in D_1^{p+1} \times S_1^q \end{cases} .$$

H is well-defined and is a diffeomorphism. This means that $S_{i_*\alpha(f'')\cdot i_*\alpha(h'')^{-1}}$ is a trivial q-sphere bundle over S^{p+1} with characteristic class $i_*\alpha(f'')\cdot i_*\alpha(h'')^{-1}$. It then follows from [1, Lemma 3.6(b)] that $i_*\alpha(f'') = i_*\alpha(h'')$. Hence G is well-defined. It is easy to see that G is a homomorphism.

LEMMA 2.8 $G(\tilde{\pi}_0(M^+_{p,q})) = i_*(\pi_p(SO(q)))$.

PROOF: By the definition of G, $G(\tilde{\pi}_0(M^+_{p,q})) \subset i_*(\pi_p SO(q))$ we then show that $i_*(\pi_p SO(q)) \subset G(\tilde{\pi}_0(M^+_{p,q}))$. If $\alpha \in i_*\pi_p(SO(q))$ and $\{a\} = \alpha$ where $a: S^p \to SO(q+1)$ then we can define $f \in M_{p,q}$ by

$$f(x,y) = \begin{cases} (x, a(x)\cdot y) & \text{if } (x,y) \in (S^p \times S^q)_1 \\ (x,y) & \text{if } (x,y) \in (S^p \times S^q)_2 \end{cases}$$

since $a \in i_*(\pi(SO(q)))$ then $f \in \tilde{\pi}_0(M^+_{p,q})$ and so $G(f) = \alpha \in i_*(\pi_p SO(q))$.

In fact since $p < q$, then $\pi_p(S^q) = 0$ hence it follows from the exact sequence $\pi_{p+1}S^q \longrightarrow \pi_p SO_q \xrightarrow{i_*} \pi_p SO_{q+1} \longrightarrow \pi_p S^q \dashrightarrow$ that i_* is an epimorphism and so it is easily seen that G is surjective. Hence the proof.

The next lemma is similar to [6, Lemma 3.3].

LEMMA 2.9 Let $u \in \ker G$, then there exists a representative $f \in M^+_{p,q}$ of u such that f is identity on $S^p \times D^q$.

PROOF: If $p < q-1$, then $\pi_{p+1}(S^q) = 0$ and also $\pi_p(S^q) = 0$ and so it follows from the exact sequence

$$\dashrightarrow \pi_{p+1}(S^q) \longrightarrow \pi_p(SOq) \xrightarrow{i_*} \pi_p(SOq+1) \longrightarrow \pi_p(S^q) \dashrightarrow$$

that i_* is an isomorphism hence if $u = \{f\} \in \ker G$ then $G(u) = i_*\alpha(f'') = 0$ implies $\alpha(f'') = 0$. Since $f(x,y) = (x, \alpha(f'')(x)\cdot y)$ for $(x,y) \in S^p \times D^q$ then it means $f(x,y) = (x,y)$ hence f is identity on $S^p \times D^q$. However, in general let $g \in M_{p,q}$ be defined thus, if $S^p \times D^q_+$, $S^p \times D^q_-$ are subsets of $(S^p \times S^q)_1$, away from the connected sum in $M_{p,q}$, we then define

$$g(x,y) = \begin{cases} (x, \alpha(f'')^{-1}(x)\cdot y) & \text{for } (x,y) \in S^p \times D^q_+ \text{ and } S^p \times D^q_- \subset (S^p \times S^q)_1 \\ (x,y) & (S^p \times S^q)_2 \end{cases}$$

since $i_*\alpha(f'') \in \pi_p(SO(q+1))$ we define $g' \in M_{p,q}$ by

$$g'(x,y) = \begin{cases} (x, i_*\alpha(f'')^{-1}(x)\cdot y) & \text{if } (x,y) \in (S^p \times S^q)_1 \\ (x,y) & \text{if } (x,y) \in (S^p \times S^q)_2 \end{cases}$$

then g and g' are diffeotopic and since $u \in \ker G$, $G(u) = 0 = i_*\alpha(f'')$ then g' is pseudo-diffeotopic to the identity and so follows that g is also pseudo-diffeotopic to the identity in $M_{p,q}$. Then the composition $g \circ f$ is pseudo-diffeotopic to f and clearly by the definition of g, $g \circ f$ keeps $S^p \times D^q_+$ fixed and represents u because it is pseudo-diffeotopic to f. Hence the proof.

We now wish to compute $\ker G$. To do this, we define a homomorphism

$$N : \operatorname{Ker} G \longrightarrow \tilde{\pi}_0(\operatorname{Diff}^+(S^p \times S^q)) \qquad \text{and}$$

show that N is surjective. Here we adopt the notation $\operatorname{Diff}^+(S^p \times S^q)$ to mean the set of all diffeomorphisms of $S^p \times S^q$ to itself which induce identity on all homology groups. Given $u \in \operatorname{Ker} G$, let $f \in M^+_{p,q}$ be its representative then it follows from Lemma 2.9 that we can take f to be identity on $S^p \times D^q$. So we have a map

$$f : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \longrightarrow (S^p \times S^q)_3 \# (S^p \times S^q)_4 \quad \text{such that}$$

f is identity on $S^p \times D^q \subset (S^p \times S^q)_1$.

Using the technique introduced by Milnor [9] and [3], we perform the spherical modification on the domain $(S^p \times S^q)_1 \# (S^p \times S^q)_2$ that removes $S^p \times D^q \subset (S^p \times S^q)_1$ and replaces it with $D^{p+1} \times S^{q-1}$. Clearly we obtain $(S^p \times S^q)_2$ since $S^p \times D^q \bigcup_{id} D^{p+1} \times S^{q-1}$ is diffeomorphic to S^{p+q}. Since f is the identity on $S^p \times D^q$, we can assume that $f(S^p \times D^q) = S^p \times D^q \subset (S^p \times S^q)_3$ and then perform the corresponding spherical modification on the range $(S^p \times S^q)_3 \# (S^p \times S^q)_4$ to obtain $(S^p \times S^q)_4$. After this modification we are then left with a diffeomorphism say f' of $(S^p \times S^q)_1$ onto $(S^p \times S^q)_4$, i.e., $f' \in \operatorname{Diff}(S^p \times S^q)$ since $f \in M^+_{p,q}$ then $f' \in \operatorname{Diff}^+(S^p \times S^q)$. So we define $N\{f\} = \{f'\}$.

LEMMA 2.10 N is well-defined.

PROOF: Let $f, g \in \operatorname{Ker} G$ such that f is pseudo-diffeotopic to g, then f is identity on $S^p \times D^q$ and g is also identity on $S^p \times D^q$. Since f is pseudo-diffeotopic to g then there exists a diffeomorphism

$F \in \operatorname{Diff}((S^p \times S^q \# S^p \times S^q) \times I)$ such that F is identity on $S^p \times D^q \times I$ and $F|(S^p \times S^q \# S^p \times S^q) \times 0 = f$ while $F|(S^p \times S^q \# S^p \times S^q) \times 1 = g$. If we now perform the spherical modification on the domain $(S^p \times S^q)_1 \# (S^p \times S^q)_2 \times I$ of F by removing $S^p \times D^q \times I \subset (S^p \times S^q)_1 \times I$ and replacing it with $D^{p+1} \times S^{q-1} \times I$, then we obtain the manifold $(S^p \times S^q)_2 \times I$ and since F is identity on $S^p \times D^q \times I$, we then perform the corresponding modification on the range $(S^p \times S^q)_3 \# (S^p \times S^q)_4 \times I$ by removing $S^p \times D^q \times I \subset (S^p \times S^q)_3 \times I$ and replacing it with $D^{p+1} \times S^{q-1} \times I$ to obtain $(S^p \times S^q)_4 \times I$. We then obtain a diffeomorphism

$$F' : (S^p \times S^q)_2 \times I \longrightarrow (S^p \times S^q)_4 \times I$$

i.e., $F' \in \operatorname{Diff}^+(S^p \times S^q \times I)$ hence $N(F) = F'$ and $F'|(S^p \times S^q \times 0) = f'$ and $F'|S^p \times S^q \times 1 = g'$ hence f' is pseudo-diffeotopic to g' and so N is well-defined. It is easy to see that N is a homomorphism.

LEMMA 2.11 N is surjective.

PROOF: Let $h' \in \operatorname{Diff}^+(S^p \times S^q)$, we need to find a diffeomorphism $h \in M^+_{p,q}$ such that $N(h) = h'$. If D^{p+q} is a disc in $S^p \times S^q$ then we can assume h' is identity on D^{p+q} then we have $h' \in \operatorname{Diff}^+(S^p \times S^q - D^{p+q})$. We then define $h \in M^+_{p,q}$ thus

$$h(x,y) = \begin{cases} (x,y) & \text{if } (x,y) \in (S^p \times S^q)_1 - D^{p+q} \\ h'(x,y) & \text{if } (x,y) \in (S^p \times S^q)_2 - D^{p+q} \end{cases}$$

where $M^+_{p,q} = \operatorname{Diff}^+(S^p \times S^q)_1 \# (S^p \times S^q)_2$ as earlier stated. h is well-defined and $h \in M^+_{p,q}$. Since h is identity on $(S^p \times S^q)_1$ then it is identity on $S^p \times D^q \subset (S^p \times S^q)_1$ hence $h \in \operatorname{Ker} G$ and clearly $N(h) = h'$ and so N is surjective.

We recall from [6, §3] the homomorphism

$B : \tilde{\pi}_0 \operatorname{Diff}^+(S^p \times S^q) \longrightarrow \pi_p SO(q+1)$ which is similarly defined as homomorphism G and where Sato gave a computation of $\operatorname{Ker} B$. We will apply this result of $\operatorname{Ker} B$ to the next lemma.

LEMMA 2.12 Ker N is in one-to-one correspondence with Ker B.

PROOF: Let $f \in \operatorname{Ker} B$, we will produce a diffeomorphism $f' \in M^+_{p,q}$ such that $f' \in \operatorname{Ker} N$. Since $f \in \operatorname{Ker} B$ then $f \in \operatorname{Diff}^+(S^p \times S^q)$ and $f|S^p \times D^q = \text{identity}$. We define a diffeomorphism $f' : (S^p \times S^q)_1 \# (S^p \times S^q)_2 \longrightarrow (S^p \times S^q)_3 \# (S^p \times S^q)_4$ by

$$f'(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in (S^p \times S^q)_1 - D^{p+q} \\ (x,y) & \text{if } (x,y) \in (S^p \times S^q)_2 - D^{p+q} \end{cases}$$

f' is well-defined and $f' \in M^+_{p,q}$. Since $f' = f$ on $(S^p \times S^q)$, and since $f|S^p \times D^q \subset (S^p \times S^q)_1$ is identity then it follows that $f'|S^p \times D^q = \text{identity}$ and so $f' \in \operatorname{Ker} G$. However, using $S^p \times D^q \subset (S^p \times S^q)_1$ to perform spherical modification on both sides of the domain and range of f' and the fact that f' is the identity on $(S^p \times S^q)_2$ we clearly see that $N(f') = \text{identity} \in \operatorname{Diff}(S^p \times S^q)_2$ hence $f' \in \operatorname{Ker} N$.

Conversely let $f \in \operatorname{Ker} N$, then $N(f) = f' \in \tilde{\pi}_0\mathrm{Diff}^+(S^p \times S^q)$. We want to show that $f' \in \operatorname{Ker} B$. Since $f \in \operatorname{Ker} N$ then it means the image of f under N is trivial hence $N(f) = f'$ is pseudo-diffeotopic to the identity. We now consider $B(f')$ where $B : \tilde{\pi}_0\mathrm{Diff}^+(S^p \times S^q) \longrightarrow \pi_p(SOq+1)$ is defined in [6] similar to our homomorphism G. Since $f' \in \mathrm{Diff}^+(S^p \times S^q)$ and $p < q$ then $f'|S^p \times D^q = S^p \times D^q$ where $f'(x,y) = (x, b(f')(x)\cdot y)$ for $(x,y) \in S^p \times D^q$ and $b(f') : S^p \longrightarrow SO(q)$. If $i : SO(q) \longrightarrow SO(q+1)$ is the inclusion map and $i_* : \pi_p SO(q) \longrightarrow \pi_p SO(q+1)$ is the induced homomorphism then $B(f') = i_* b(f') \in \pi_p SO(q+1)$.

However since f' is pseudo-diffeotopic to the identity then let $H : S^p \times S^q \times I \longrightarrow S^p \times S^q \times I$ be the pseudo-diffeotopy between f' and identity id. Then

$$\begin{array}{ccccccc}
D^{p+1} \times S^q \underset{f'}{\cup} D^{p+1} \times S^q & = & D^{p+1} \times S^q \underset{f'_1}{\cup} & S^p \times S^q \times I & \underset{id_1}{\cup} & D^{p+1} \times S^q \\
\downarrow \approx & & \downarrow id & \downarrow H & & \downarrow id \\
D^{p+1} \times S^q \underset{id}{\cup} D^{p+1} \times S^q & = & D^{p+1} \times S^q \underset{id'_2}{\cup} & S^p \times S^q \times I & \underset{id_2}{\cup} & D^{p+1} \times S^q
\end{array}$$

is the required diffeomorphism between $D^{p+1} \times S^q \underset{f'}{\cup} D^{p+1} \times S^q$ and $D^{p+1} \times S^q \underset{id}{\cup} D^{p+1} \times S^q = S^{p+1} \times S^q$ where $id_1(x,y) = (x,y,1)$, $id'_2(x,y,0) = (x,y)$, $f'_1(x,y,0) = f'(x,y)$ and $id_2(x,y) = id(x,y,1) = (x,y)$. However, consider $S_{i_* b(f')}$ the q-sphere bundle over a (p+1)-sphere whose characteristic class of the equivalent normal bundle is $i_* b(f') \in \pi_p SO(q+1)$ hence $S_{i_* b(f')} = D^{p+1} \times S^q \underset{f'}{\cup} D^{p+1} \times S^q \approx S^{p+1} \times S^q$ by the above diffeomorphism and since $p < q$ it follows by [1, Prop. 3.6] that $i_* b(f') = 0$. Hence $f' \in \operatorname{Ker} B$ and so $\operatorname{Ker} N$ is in one-to-one correspondence with $\operatorname{Ker} B$. Since N is surjective by Lemma 2.11 then we have

LEMMA 2.13 The order of the group $\operatorname{Ker} G$ equals the order of the direct sum group

$$\operatorname{Ker} B \oplus \tilde{\pi}_0\mathrm{Diff}^+(S^p \times S^q)$$

Also since G is surjective by Lemma 2.8 then it is easily seen that

LEMMA 2.14 The order of $\tilde{\pi}_0(M^+_{p,q})$ is equal to the order of the direct sum group

$$\pi_p SO(q+1) \oplus \operatorname{Ker} B \oplus \tilde{\pi}_0\mathrm{Diff}^+(S^p \times S^q)$$

However one can easily deduce from [6, §4]

LEMMA 2.15 $\ker B \approx \pi_q SO(p+1) \oplus \theta^{p+q+1}$

Also from [6, Thm. II] and [1, Thm. 3.10] we have

LEMMA 2.16 $\tilde{\pi}_0\mathrm{Diff}^+(S^p \times S^q) = \pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus \theta^{p+q+1}$

Combining Lemmas 2.12, 2.13, 2.14, 2.15, and 2.16, we obtain

THEOREM 2.17 For $p < q$, the order of the group $\tilde{\pi}_0(M^+_{p,q})$ equals twice the order of the group $\pi_p SO(q+1) \oplus \pi_q SO(p+1) \oplus \theta^{p+q+1}$.

3. CLASSIFICATION OF MANIFOLDS

Consider the class of manifolds $\{M, \lambda_1, \lambda_2\}$ where M is a manifold of type

$(n,p,2)$ where $n=p+q+1$ and $p=3,5,6,7 \pmod 8$ and λ_1,λ_2 are the generators of $H_p(M) = \mathbf{Z}\oplus\mathbf{Z}$. By the proof of Theorem 1.1 we have an embedding $\varphi_i : S^p\times D^{q+1} \to M$ which represents the homology class λ_i $i=1,2$. If we then take the connected sum along the boundary of the two embedded copies of $S^p\times D^{q+1}$ we have an embedding

$$i : S^p\times D^{q+1} \#_\partial S^p\times D^{q+1} \longrightarrow M \quad \text{such that} \quad i_*[S^p] = \lambda_1+\lambda_2$$

Two of such manifolds $\{M,\lambda_1,\lambda_2\}$ and $\{M',\lambda_1',\lambda_2'\}$ will be said to be equivalent if there is an orientation preserving diffeomorphism of M onto M' which takes λ_1 to λ_1' $i=1,2$. Let $\mathcal{M}_n$ be the equivalent class of manifolds satisfying these conditions. This equivalent class which is also the diffeomorphism class has a group structure. The operation is connected sum along the boundary $S^p\times S^q \# S^p\times S^q$ of $S^p\times D^{q+1}\#_\partial S^p\times D^{q+1}$. For if $\{M,\lambda_1,\lambda_2\}$, $\{M',\lambda_1',\lambda_2'\}\in\mathcal{M}_n$, then let $i_1 : S^p\times D^{q+1}\#_\partial S^p\times D^{q+1} \longrightarrow M$ be an orientation preserving embedding such that $i_{1*}[S^p] = \lambda_1+\lambda_2$ and since there is an orientation reversing diffeomorphism of $S^p\times D^{q+1}\#_\partial S^p\times D^{q+1}$ to itself (because $S^p\times D^{q+1}$ is a trivial $q+1$-disc bundle over S^p) then we have an orientation reversing embedding $i_2 : S^p\times D^{q+1}\#_\partial S^p\times D^{q+1} \longrightarrow M'$ such that $i_{2*}[S^p] = \lambda_1'+\lambda_2'$. We now obtain $M\#_{2p}M'$ from the disjoint sum $(M - \operatorname{Int} i_1(S^p\times D^{q+1}\#_\partial S^p\times D^{q+1})) \cup (M' - \operatorname{Int} i_2(S^p\times D^{q+1}\#_\partial S^p\times D^{q+1}))$ by identifying $i_1(x)$ with $i_2(x)$ for $x\in S^p\times S^q\# S^p\times S^q$. We will call this operation the connected sum along double p-cycle. Where the $2p$ in $M\#_{2p}M'$ means that we are identifying along the boundary of embedded copies of connected sum along the boundary of two copies of $S^p\times D^{q+1}$. It is easy to see that $H_p(M\#_{2p}M') \approx \mathbf{Z}\oplus\mathbf{Z}$. Since we have identified $i_1(S^p\times S^q\# S^p\times S^q)$ with $i_2(S^p\times S^q\# S^p\times S^q)$ we can define $i_{1*}[S^p] = \lambda_1\#\lambda_1'+\lambda_2\#\lambda_2'$ the generators of $H_p(M\#M')$ then we see that $M\#_{2p}M'\in\mathcal{M}_n$.

LEMMA 3.1 The connected sum along the double p-cycle is well-defined and associative.

PROOF: We need to show that the operation does not depend on the choice of the embeddings. Suppose there is another embedding $\varphi_i' : S^p\times D^{q+1} \longrightarrow M$ which represents the homology class λ_i $i=1,2$ and gives a corresponding embedding $i_1' : S^p\times D^{q+1}\#_\partial S^p\times D^{q+1} \longrightarrow M$. By the tubular neighborhood theorem $\varphi_i(S^p\times D^{q+1})$ and $\varphi_i'(S^p\times D^{q+1})$ differ only by rotation of their fiber, i.e., by an element of $\pi_p SO(q+1) = 0$ since $p=3,5,6,7 \pmod 8$ hence the two embeddings are isotopic and so the corresponding embeddings

$$i_1 : S^p\times D^{q+1}\#_\partial S^p\times D^{q+1} \longrightarrow M \quad \text{and}$$
$$i_1' : S^p\times D^{q+1}\#_\partial S^p\times D^{q+1} \longrightarrow M \quad \text{are isotopic.}$$

The definition does not therefore depend on the choice of i_1. With similar argument it does not depend on i_2. The connected sum is therefore well-defined. Associativity is easy to check.

LEMMA 3.2 If $\{M,\lambda_1,\lambda_2\}$, $\{M_1,\lambda_{1_1},\lambda_{1_2}\} \in \mathcal{M}_n$, such that they are equivalent. If $\{M',\lambda_1',\lambda_2'\} \in \mathcal{M}_n$ then $(M \underset{2p}{\#} M',\lambda_1 \# \lambda_1',\lambda_2 \# \lambda_2')$ is equivalent to $(M_1 \underset{2p}{\#} M',\lambda_{1_1} \# \lambda_1',\lambda_{1_2} \# \lambda_2')$.

PROOF: Since M,M_1 are equivalent in $\mathcal{M}_n$ then there exists an orientation preserving diffeomorphism $f:M \longrightarrow M_1$ which carries λ_1 to λ_{1_1} and λ_2 to λ_{1_2} hence it carries the embedding $\varphi_i(S^p \times D^{q+1})$ to the corresponding embedding $\varphi_{1_i}(S^p \times D^{q+1})$ $i=1,2$ and so f carries the embedding $i(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \subset M$ to the embedding $i_1(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \subset M_1$ hence f induces a diffeomorphism

$$f':M - \text{Int } i(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \longrightarrow M_1 - \text{Int } i_1(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$$

which carries λ_1 to λ_{1_1} and λ_2 to λ_{1_2} .

Trivially we have the identity map

$$id:M' - \text{Int } i'(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \longrightarrow M' - \text{Int } i(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$$

which carries λ_1' to λ_1' and λ_2' to λ_2' . We then take the connected sum along their boundary $S^p \times S^q \# S^p \times S^q$ to have $M \underset{2p}{\#} M'$ which is disjoint sum of $M\text{-Int } i(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \cup M'\text{-Int } i'(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$ by identifying $i(x)$ and $i'(x)$ for $x \in S^p \times S^q \# S^p \times S^q$. Similarly $M_1 \underset{2p}{\#} M'$ is the disjoint sum of $M\text{-Int } i_1(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \cup M'\text{-Int } i'(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$ by identifying $i_1(x)$ and $i'(x)$ for $x \in S^p \times S^q \# S^p \times S^q$. Clearly we have a diffeomorphism $g:M \underset{2p}{\#} M' \longrightarrow M_1 \underset{2p}{\#} M'$ which is f' on M and identity of M' and g carries $\lambda_1 \# \lambda_1'$ to $\lambda_{1_1} \# \lambda_1'$ and $\lambda_2 \# \lambda_2'$ to $\lambda_{1_2} \# \lambda_2'$. Hence $\{M \underset{2p}{\#} M',\lambda_1 \# \lambda_1',\lambda_2 \# \lambda_2'\}$ is equivalent to $\{M_1 \underset{2p}{\#} M',\lambda_{1_1} \# \lambda_1',\lambda_{1_2} \# \lambda_2'\}$ in $\mathcal{M}_n$. That proves the lemma.

If we now take two copies of $S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}$ and identify the two copies on their common boundaries by the identity map, we will obtain the manifold $S^p \times S^{q+1} \# S^p \times S^{q+1}$, i.e., $S^p \times S^{q+1} \# S^p \times S^{q+1} = (S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \underset{id}{\cup} (S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$ where $id = \text{identity}: S^p \times S^q \# S^p \times S^q \longrightarrow S^p \times S^q \# S^p \times S^q$. If $\lambda_{0_1},\lambda_{0_2}$ are the generators of $H_p(S^p \times S^{q+1} \# S^p \times S^{q+1}) = \mathbf{Z} \oplus \mathbf{Z}$ and $-\lambda_1 + (-\lambda_2) \in H_p(-M) = \mathbf{Z} \oplus \mathbf{Z}$ where $i_*[S^p] = -\lambda_1 + -\lambda_2$ and $i:M \longrightarrow -M$ is the orientation reversing diffeomorphism then we have the following.

LEMMA 3.3 $\mathcal{M}_n$ is a group with identity element $(S^p \times S^{q+1} \# S^p \times S^{q+1},\lambda_{0_1},\lambda_{0_2})$ and for $(M,\lambda_1,\lambda_2) \in \mathcal{M}_n$ $(-M,-\lambda_1,-\lambda_2)$ is the inverse element.

To be able to prove our main theorem later, we need to investigate $\tilde{\pi}_0\text{Diff}^+(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$. As in the case of $\tilde{\pi}_0(M_{p,q})$, we define a homomorphism $\Phi' : \tilde{\pi}_0\text{Diff}(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) \longrightarrow \text{Auto } H_*(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1})$ by induced automorphism of homology groups. Since $S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}$ has the homotopy type of $S^p \times D^{q+1} \vee S^p \times D^{q+1}$ then

$$H_i(S^p \times D^{q+1} \underset{\partial}{\#} S^p \times D^{q+1}) = \begin{cases} Z & \text{if } i=0 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } i=p \end{cases} .$$

Using similar ideas in §2, it is easy to prove the following.

LEMMA 3.4

$$\Phi'(\tilde{\pi}_0(\mathrm{Diff}(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}))) = \begin{cases} Z_4 & \text{if } p \text{ is even} \\ GL(2,Z) & \text{if } p = 1,3,7 \\ H & \text{if } p \text{ is odd but } \neq 1,3,7 \end{cases}$$

Let $\mathrm{Diff}^+(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \subset \mathrm{Diff}(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$ be the set of all diffeomorphisms of $S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ which induce identity automorphisms on its homology. Then it follows that $\tilde{\pi}_0 \mathrm{Diff}^+(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$ is the kernel of Φ'. We define a homomorphism

$$G' : \pi_p SO(q+1) \longrightarrow \tilde{\pi}_0 \mathrm{Diff}^+(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$$

If $\alpha \in \pi_p SOq+1$ and $\alpha = \{a\}$ then we define a map

$$g_a : S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1} \longrightarrow S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$$

by

$$g_a(x,y) = \begin{cases} (x, a(x)\cdot y) & \text{for } (x,y) \in (S^p \times D^{q+1})_1 \\ (x, a(x)\cdot y) & \text{for } (x,y) \in (S^p \times D^{q+1})_2 \end{cases}$$

g_a is clearly well-defined and it is a diffeomorphism and since g_a keeps S^p fixed, it induces identity on all homology groups hence $g_a \in \mathrm{Diff}^+(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$.
We will define $G'\{\alpha\} = \{g_a\}$

LEMMA 3.5 G' is well defined.

PROOF: If $a' \in \pi_p SO(q+1)$ such that a is homotopic to a' and let $H : S^p \times I \longrightarrow SO(q+1)$ be the homotopy such that $H(S^p \times 0) = a$ and $H(S^p \times 1) = a'$ then we construct a diffeomorphism F of $(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \times I$ by

$$F(x,y,t) = \begin{cases} (x, H(x,t)\cdot y) & (x,y) \in S^p \times D^{q+1})_1 \\ (x, H(x,t)\cdot y) & (x,y) \in S^p \times D^{q+1})_2 \end{cases}$$

This is the diffeotopy which connects g_a and $g_{a'}$.

LEMMA 3.6 G' is surjective.

PROOF: Let $\{f\} \in \tilde{\pi}_0 \mathrm{Diff}^+(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1})$ then f induces identity on all homology groups. However $H_p(S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}) \approx \mathbf{Z} \oplus \mathbf{Z}$ and so if λ_1 and λ_2 represents the generators of the first and second summand and the embeddings $i_1 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ and $i_2 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} \#_{\partial} S^p \times D^{q+1}$ represents the homology class λ_1 and λ_2 respectively, since f induces identity on homology then $f(S^p \times \{p_0\}$ and $i_1(S^p \times \{p_0\}$ are homologous. Since $p < q$ and by Hurewicz theorem i_1 and $f \circ i_1$ are homotopic, by Haefliger [10] and by the diffeotopy extension theorem and tubular neighborhood theorem, there exists f' in the diffeotopy class of f such that $f'(x,y) = (x, a(x)\cdot y)$ for $(x,y) \in (S^p \times D^{q+1})_1$ where $S^p \times D^{q+1}$ is the tubular neighborhood of $S^p \times \{p_0\}$ and $a : S^p \longrightarrow SOq+1$. Similar argument applies to the embedding $i_2 : S^p \times \{p_0\} \longrightarrow S^p \times D^{q+1} \# S^p \times D^{q+1}$ and so we have a map f'' in the diffeotopy class of f hence in the diffeotopy class of f' and so f'' must be of the form $f''(x,y) = (x, a(x)\cdot y)$ where $(x,y) \in (S^p \times D^{q+1})_2$. It follows that

$$f(x,y) = \begin{cases} (x, a(x)\cdot y) & (x,y) \in (S^p \times D^{q+1})_1 \\ (x, a(x)\cdot y) & (x,y) \in (S^p \times D^{q+1})_2 \end{cases}$$

Hence G' is surjective.

One can easily deduce from Lemma 3.6 that $\tilde{\pi}_0 \mathrm{Diff}^+(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$ is a factor group of $\pi_p(SOq+1)$.

THEOREM 3.7 Let M be an n-dimensional closed simply connected manifold of type $(n,p,2)$ where $n = p+q+1$ with $p = 3,5,6,7 \pmod 8$ then the number of differentiable manifolds satisfying the above conditions up to diffeomorphism is twice the order of the direct sum group $\pi_p SO(p+1) \oplus \theta^n$.

PROOF: We define a map $C : \tilde{\pi}_0(M^+_{p,q}) \longrightarrow \mathscr{M}_n$ and show that C is an isomorphism. Let $\{f\} \in \tilde{\pi}_0(M^+_{p,q})$ then f is a diffeomorphism of $S^p \times S^q \# S^p \times S^q$ which induce identity on homology. We then take two copies $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1$ and $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_2$ of $S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}$ and attach them on the boundary by f to have $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1 \cup_f (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_2$. An orientation is chosen to be compatible with $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1$ and the manifold obtained belongs to the group $\mathscr{M}_n$. The generators of the p-dimensional homology group is fixed to be the one represented by the usual embedding $S^p \times \{p_0\} \longrightarrow (S^p \times D^{q+1})_1 \subset (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1$ and $S^p \times \{p_0\} \longrightarrow (S^p \times D^{q+1})_2 \subset (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1$. We then define $C\{f\} = (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_f (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$. We now show that C is well-defined. Let $f_0, f_1 \in M^+_{p,q}$ such that f_0 is pseudo-diffeotopic to f_1 then there exists $H : (S^p \times S^q \# S^p \times S^q) \times I \longrightarrow (S^p \times S^q \# S^p \times S^q) \times I$ such that $H(x,y,0) = f_0$ and $H(x,y,1) = f_1$ then we wish to show that $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_{f_0} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$ is diffeomorphic to $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_{f_1} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$. We then define a map

$$\begin{array}{ccccccc}
(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_{f_0} S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} & = & S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} & \cup_{f_0'} & (S^p \times S^q \# S^p \times S^q) \times I & \cup_{id_0} & S^p \times D^{q+1} \#_\partial S^p \times D^{q+1} \\
\downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle id} \\
(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_{f_1} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) & = & S^p \times D^{q+1} \#_\partial D^{q+1} & \cup_{id_1} & (S^p \times S^q \# S^p \times S^q) \times I & \cup_{f_1'} & S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}
\end{array}$$

where $id_0(x,y) = (x,y,1)$, $id_1(x,y,0) = (x,y)$, $f_0'(x,y,0) = f_0(x,y)$ and $f_1'(x,y) = f_1(x,y,1)$. This is a well-defined map and is the required diffeomorphism from $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_{f_0} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$ to $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1}) \cup_{f_1} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})$. Hence C is well-defined and it is easy to see that C is a homomorphism. By Theorem 1.1 it follows that C is surjective. We now need to show that C is injective. Suppose $\{f\} \in \tilde{\pi}_0(M^+_{p,q})$ and $C(f) = (M, \lambda_1, \lambda_2)$ is trivial, then it follows that

$M = (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1 \cup_f (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_2$ is diffeomorphic to $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1 \cup_{id} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_2 = S^p \times S^{q+1} \# S^p \times S^{q+1}$ with p-dimensional homology generators $\lambda_{0_1}, \lambda_{0_2}$, by a diffeomorphism d which carries λ_1 to λ_{0_1} and λ_2 to λ_{0_2}, i.e.,

$$\begin{array}{c}(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1 \cup_f (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_2 \\ \downarrow \\ (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1 \cup_{id} (S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_2 = S^p \times S^{q+1} \# S^p \times S^{q+1}\end{array}$$

It is easy to see that since d carries λ_1 to λ_{0_1} and λ_2 to λ_{0_2} and because $p = 3,5,6,7 \pmod 8$ then d is the identity on $(S^p \times D^{q+1} \#_\partial S^p \times D^{q+1})_1$. On the boundary $S^p \times S^q \# S^p \times S^q$, d is just f. Since d is a diffeomorphism it follows that f extends to a diffeomorphism of $(S^p \times D^{q+1} \# S^p \times D^{q+1})_2$ which means $f \in \mathrm{Diff}^+(S^p \times S^q \# S^p \times S^q)$ is extendable to $\mathrm{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1})$, but by Lemma 3.5, $\tilde{\pi}_0 \mathrm{Diff}^+(S^p \times D^{q+1} \# S^p \times D^{q+1})$ is a factor group of $\pi_p(SOq+1)$ but since $p = 3,5,6,7$, mod 8 then $\pi_p(SOq+1) = 0$. Hence f is pseudo-diffeotopic to the identity and so C is injective. It then follows that C is an isomorphism. By Theorem 2.17 and since $p = 3,5,6,7 \pmod 8$ it follows that the order of the group $\tilde{\pi}_0(M^+_{p,q})$ is twice the order of the group $\pi_q SO(p+1) \oplus \theta^n$ and since C is an isomorphism the theorem is proved. The methods used here if carefully applied can be used to obtain a general result.

THEOREM 3.8 If M is a smooth, closed simply connected manifold of type (n,p,r) where $n = p+q+1$ and $p = 3,5,6,7 \pmod 8$ then the number of differentiable manifolds up to diffeomorphism satisfying the above is equal to r times the order of $\pi_q SO(p+1) \oplus \theta^n$.

REFERENCES

1. TURNER, E.C. Diffeomorphisms of a Product of Spheres, Inventiones Math. 8, (1969) 69-82.
2. SATO, H. Diffeomorphism groups and classification of manifolds, J. Math. Soc. Japan, Vol. 21, No. 1, (1969) 1-36.
3. MILNOR, J. and KERVAIRE, M. Groups of Homotopy Spheres I, Ann. of Math. 77 (1963) 504-537.
4. SMALE, S. On the structure of manifolds, Amer. J. Math. 84 (1962) 387-399.
5. KUROSH, A.G. Theory of Groups Vol. II, Chelsea, New York, 1955.
6. SATO, H. Diffeomorphism groups of $S^p \times S^q$ and Exotic Spheres, Quart. J. Math., Oxford (2), 20 (1969) 255-276.
7. LEVINE, J. Self-Equivalences of $S^n \times S^k$: Trans. American Math. Soc. 143 (1969) 523-543.
8. WALL, C.T.C. Killing the middle homotopy groups of odd dimensional manifolds, Trans. American Math. Soc. 103 (1962) 421-433.
9. MILNOR, J. A procedure for killing homotopy groups of differentiable manifolds, Proc. Symposium Tucson, Arizona, 1960.
10. HAEFLIGER, A. Plongements differentiables des variétés. Commentari Helvetici Math. 36, 47-82 (1961).