

## ON THE DEGREE OF APPROXIMATION BY GAUSS WEIERSTRASS INTEGRALS

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**ABSTRACT.** We obtain the degree of approximation of functions belonging to class  $\text{Lip}(\psi(u, v); p)$ ,  $p > 1$  using the Gauss Weierstrass integral of the double Fourier series of  $f(x, y)$ .

**Keywords and phrases.**  $\text{Lip}(\psi(u, v); p)$  class, Gauss Weierstrass integral, arithmetic means, Jackson operator, degree of approximation, Fourier series, Hölder's inequality.

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**1. Introduction and results.** Let the function  $f(x, y)$  be integrable in the sense of Lebesgue over the square  $(0, 2\pi; 0, 2\pi)$  and periodic with period  $2\pi$  in each variable outside the square. Let the double Fourier series of  $f(x, y)$  be

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n}(x, y), \quad (1.1)$$

where

$$\begin{aligned} A_{0,0}(x, y) &= \frac{1}{4} a_{0,0}, \\ A_{m,0}(x, y) &= \frac{1}{2} (a_{m,0} \cos mx + b_{m,0} \sin mx), \\ A_{0,n}(x, y) &= \frac{1}{2} (a_{0,n} \cos nx + b_{0,n} \sin nx), \\ A_{m,n}(x, y) &= a_{m,n} \cos mx \cos ny + b_{m,n} \cos mx \sin ny \\ &\quad + c_{m,n} \sin mx \cos ny + d_{m,n} \sin mx \sin ny. \end{aligned} \quad (1.2)$$

Let

$$\begin{aligned} \phi(u, v) &= \frac{1}{4} \{ f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) \\ &\quad + f(x-u, y-v) - 4f(x, y) \}. \end{aligned} \quad (1.3)$$

The series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{m,n} A_{m,n}(x, y) \quad (1.4)$$

is called the double Fourier series associated with the function  $\phi(u, v)$  such that

$$\lambda_{m,n} = \begin{cases} \frac{1}{4} & \text{for } m = 0, n = 0, \\ \frac{1}{2} & \text{for } m = 0, n > 0; m > 0, n = 0, \\ 1 & \text{for } m > 0, n > 0. \end{cases} \tag{1.5}$$

Also, the coefficients in the series (1.4) are given by

$$a_{m,n} = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(u, v) \cos mu \cos nv \, du \, dv \tag{1.6}$$

and three other similar expressions defining  $b_{m,n}, c_{m,n}, d_{m,n}$ .

We define the Gauss Weierstrass integral of  $f(x, y)$  by

$$\begin{aligned} W_{m,n}(x, y) &= W(x, y; \xi, \eta) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{\exp((k^2\xi/4) + (\ell^2\eta/4))} A_{k,\ell}(x, y) \\ &= \sqrt{\frac{\pi}{(\xi, \eta)}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{f(x+t, y+s)}{\exp((t^2/\xi) + (s^2/\eta))} \, dt \, ds + o(\xi, \eta), \end{aligned} \tag{1.7}$$

where  $o(\xi, \eta) \rightarrow 0$  as  $\xi \rightarrow 0, \eta \rightarrow 0$ , and  $(\xi, \eta)$  is the product of  $\xi$  and  $\eta$ .

A  $2\pi$  periodic function  $f(x, y)$  in each variable  $x$  and  $y$  is said to belong to the class  $\text{Lip}(\psi(u, v); p), p > 1, [2]$  if

$$|f(x + u, y + v) - f(x, y)| \leq M \left( \frac{\psi(u, v)}{(u, v)^{1/p}} \right), \quad 0 < u < \pi, 0 < v < \pi, \tag{1.8}$$

where  $\psi(u, v)$  is a positive increasing function of the variables  $u, v$  and  $M$  is a positive number independent of  $x, u$  and  $v$ .

Yoshimitsu, [1] proved a theorem for obtaining the degree of approximation of class of functions  $\text{Lip}(\alpha, \beta), 0 < \alpha < 1$  and  $0 < \beta < 1$ , by means of the first arithmetic means of double Fourier series. Siddiqi and Mohammadzadeh [3] extended the result in two directions in terms of a positive increasing function of two variables. Recently Khan, [2] extended the result of Yoshimitsu, Siddiqi et al. for the more general operator, the Jackson type operator, and more general class of functions,  $\text{Lip}(\psi(u, v); p), p > 1$ . The object of the present paper is to determine the degree of approximation for the functions belonging to the class  $\text{Lip}(\psi(u, v); p), p > 1$ , by means of Gauss Weierstrass integral of the double Fourier series of  $f(x, y)$ .

Our theorem states as follows.

**THEOREM 1.1.** *Let  $f(x, y)$  be a continuous function of period  $2\pi$  with respect to each variable  $x$  and  $y$  belonging to  $\text{Lip}(\psi(u, v); p), p > 1$  class, then*

$$|W(x, y; \xi, \eta) - f(x, y)| = \mathcal{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{(1/p) - (1/2)}} \right), \tag{1.9}$$

provided

$$\left[ \int_0^\xi \int_0^\eta \left( \frac{\psi(t, s)}{(t, s)^{1/p}} \right)^p \, dt \, ds \right]^{1/p} = \mathcal{O}(\psi(\xi, \eta)), \tag{1.10}$$

$$\left[ \int_0^\xi \int_\eta^\pi \left( \frac{\psi(t,s)}{(t,s)^{2+(1/p)}} \right)^p dt ds \right]^{1/p} = \mathcal{O} \left( \frac{\psi(\xi,\eta)}{(\xi,\eta)^2} \right). \tag{1.11}$$

**2. Proof of the theorem.** Using (1.7), we get

$$\begin{aligned} W(x,y;\xi,\eta) - f(x,y) &= \sqrt{\frac{\pi}{(\xi,\eta)}} \int_{-\pi}^\pi \int_{-\pi}^\pi \phi(t,s) e^{-((t^2/\xi)+(s^2/\eta))} dt ds + R(x,y;\xi,\eta) \\ &= 4 \sqrt{\frac{\pi}{(\xi,\eta)}} \int_0^\pi \int_0^\pi \phi(t,s) e^{-((t^2/\xi)+(s^2/\eta))} dt ds + \mathcal{O}(\xi,\eta) \\ &= 4 \sqrt{\frac{\pi}{(\xi,\eta)}} \left( \int_0^\xi \int_0^\eta + \int_0^\xi \int_\eta^\pi + \int_\xi^\pi \int_0^\eta + \int_\xi^\pi \int_\eta^\pi \right) \phi(t,s) e^{-((t^2/\xi)+(s^2/\eta))} dt ds \\ &= I_{1,2}(x,y) + I_{1,3}(x,y) + I_{4,2}(x,y) + I_{4,3}(x,y). \end{aligned} \tag{2.1}$$

Applying Hölder's inequality of two variables and the fact that  $\phi(u,v) \in \text{Lip}(\psi(u,v);p)$ ,  $p > 1$  for  $I_{1,2}(x,y)$ , we get

$$\begin{aligned} |I_{1,2}(x,y)| &\leq 4 \sqrt{\frac{\pi}{(\xi,\eta)}} \left[ \int_0^\xi \int_0^\eta |\phi(t,s)|^p dt ds \right]^{1/p} \\ &\quad \cdot \left[ \int_0^\xi \int_0^\eta (e^{-((t^2/\xi)+(s^2/\eta))})^{p'} dt ds \right]^{1/p'}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \frac{1}{p'} &= \frac{p-1}{p} \\ &\leq 4 \sqrt{\frac{\pi}{(\xi,\eta)}} \left[ \int_0^\xi \int_0^\eta \left( \frac{\psi(t,s)}{(t,s)^{1/p}} \right)^p dt ds \right]^{1/p} \left[ \int_0^\xi \int_0^\eta (e^{-((t^2/\xi)+(s^2/\eta))})^{p'} dt ds \right]^{1/p'} \\ &= \mathcal{O} \left( \frac{\psi(\xi,\eta)}{(\sqrt{\xi},\sqrt{\eta})} \right) \left( \int_0^\xi \int_0^\eta e^{-(t^2 p'/\xi)} e^{-(s^2 p'/\eta)} dt ds \right)^{1/p'} \quad \text{using condition (1.10)} \\ &= \mathcal{O} \left( \frac{\psi(\xi,\eta)}{(\sqrt{\xi},\sqrt{\eta})} \right) \left( \int_0^{\sqrt{\xi p'}} e^{-u^2} \frac{\sqrt{\xi}}{\sqrt{p'}} du \int_0^{\sqrt{\eta p'}} e^{-v^2} \frac{\sqrt{\eta}}{\sqrt{p'}} dv \right)^{1/p'} \\ &= \mathcal{O} \left( \frac{\psi(\xi,\eta)}{(\sqrt{\xi},\sqrt{\eta})} \right) \mathcal{O}(\xi,\eta)^{1/p'} = \mathcal{O} \left( \frac{\psi(\xi,\eta)}{(\xi,\eta)^{(1/p)-(1/2)}} \right), \end{aligned} \tag{2.3}$$

where  $u = t\sqrt{p'/\xi}$  and  $v = s\sqrt{p'/\eta}$ .

Applying Hölder's inequality of two variables to  $I_{1,3}(x,y)$  and using the fact that  $\phi(u,v) \in \text{Lip}(\psi(u,v);p)$ ,  $p > 1$ , we get

$$\begin{aligned}
|I_{1,3}(x, y)| &\leq \frac{4\sqrt{\pi}}{(\sqrt{\xi}, \sqrt{\eta})} \\
&\cdot \left[ \left( \int_0^\xi \int_\eta^\pi \left| \frac{\phi(t, s)}{(t, s)^2} \right|^p dt ds \right)^{1/p} \left( \int_0^\xi \int_\eta^\pi \frac{e^{-(t^2/\xi) + (s^2/\eta)p'}}{(t, s)^{-2p'}} dt ds \right)^{1/p'} \right] \\
&\leq \frac{4\sqrt{\pi}}{(\sqrt{\xi}, \sqrt{\eta})} \left\{ \left[ \int_0^\xi \int_\eta^\pi \left( \frac{\psi(t, s)}{(t, s)^{2+(1/p)}} \right)^p dt ds \right]^{1/p} \right. \\
&\quad \cdot \left. \left( \int_0^\xi e^{-t^2 p'/\xi} \cdot t^{2p'} dt \int_\eta^\pi e^{-s^2 p'/\eta} s^{2p'} ds \right)^{1/p'} \right\} \\
&= \mathbb{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{5/2}} \right) \left[ \int_0^{\sqrt{\xi p'}} e^{-u^2} \left( \frac{\sqrt{\xi}}{\sqrt{p'}} u \right)^{2p} \left( \frac{\sqrt{\xi}}{\sqrt{p'}} \right) du \right]^{1/p} \\
&\quad \cdot \left[ \int_{\frac{\sqrt{\eta p'}}{\sqrt{\eta p'}}}^{(\pi/\sqrt{\eta p'})} e^{-v^2} \left( \frac{\sqrt{\eta}}{\sqrt{p'}} v \right)^{2p'} \left( \frac{\sqrt{\eta}}{\sqrt{p'}} \right) dv \right]^{1/p'} \quad \text{using condition (1.11)} \\
&= \mathbb{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{5/2}} \right) \mathbb{O}(\xi, \eta)^{2+(1/p')} = \mathbb{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{(1/p)-(1/2)}} \right).
\end{aligned} \tag{2.4}$$

Similarly, we can prove that

$$|I_{4,2}(x, y)| = \mathbb{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{(1/p)-(1/2)}} \right), \quad |I_{4,3}(x, y)| = \mathbb{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{(1/p)-(1/2)}} \right). \tag{2.5}$$

Finally, we get

$$W(x, y; \xi, \eta) - f(x, y) = \mathbb{O} \left( \frac{\psi(\xi, \eta)}{(\xi, \eta)^{(1/p)-(1/2)}} \right). \tag{2.6}$$

**REMARK 2.1.** It may also be remarked that by giving different values to  $\psi(u, v)$ , we get some interesting results:

(i) If  $\psi(u, v) = u^\alpha * v^\beta$ , [1],  $0 < \alpha < 1$ ,  $0 < \beta < 1$ , then we have

$$|W(x, y; \xi, \eta) - f(x, y)| = \mathbb{O}(\xi^{\alpha+(1/2)-(1/p)} * \eta^{\beta+(1/2)-(1/p)}). \tag{2.7}$$

(ii) If  $\psi(u, v) = J(u, v)(u, v)^{1/p}$ , [3], where  $J(u, v)$  is a positive increasing function of variables  $u$  and  $v$ , then

$$|W(x, y; \xi, \eta) - f(x, y)| = \mathbb{O} \left( \frac{J(\xi, \eta)}{(\xi, \eta)^{-1/2}} \right). \tag{2.8}$$

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