

TOTALLY REAL SUBMANIFOLDS IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this paper, we establish the following result: Let M be an n -dimensional complete totally real minimal submanifold immersed in CP^n with Ricci curvature bounded from below. Then either M is totally geodesic or $\inf r \leq (3n+1)(n-2)/3$, where r is the scalar curvature of M .

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1. Introduction. Let CP^n be the n -dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c = 4$ and let M be an n -dimensional totally real submanifold of CP^n . Let r be the scalar curvature of M . If M is compact, then many authors studied them and obtained many beautiful results (for example [2, 4, 5]).

In this paper, we make use of Yau's maximum principle to study the complete totally real minimal submanifold with Ricci curvature bounded from below and obtain the following result.

THEOREM 1. *Let M be an n -dimensional complete totally real minimal manifold immersed in CP^n with Ricci curvature bounded from below. Then either M is totally geodesic or $\inf r \leq (3n+1)(n-2)/3$.*

2. Preliminaries. Let M be an n -dimensional totally real minimal submanifold of CP^n . We choose a local field of orthonormal frames $e_1, \dots, e_n, e_{1^*}, \dots, e_{n^*} = Je_1, \dots, e_{n^*} = Je_n$ (J is the complex structure of CP^n), such that, restricted to M , the vectors e_1, \dots, e_n are tangent to M . We make use of the following convention on the range of indices

$$A, B, C, \dots = 1, \dots, n, 1^*, \dots, n^*; \quad i, j, k, \dots = 1, \dots, n. \quad (2.1)$$

With respect to the frame field of CP^n , let w^A be the field of dual frames. Then the structure equations of CP^n are given by

$$dw^A = -\sum w_B^A \wedge w^B, \quad w_A^B + w_B^A = 0, \quad (2.2)$$

$$dw_B^A = -\sum w_C^A \wedge w_B^C + \frac{1}{2} \sum \bar{R}_{BCD}^A w^C \wedge w^D, \quad (2.3)$$

$$\bar{R}_{BCD}^A = \delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC} + J_{AC} J_{BD} - J_{AD} J_{BC} + 2J_{AB} J_{CD}, \quad (2.4)$$

where $J = J_{AB}e_A \otimes e_B$, so that

$$(J_{AB}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad (2.5)$$

where I_n is the identity matrix of order n . We restrict these forms to M . Then from [2], we have

$$w^{i*} = 0, \quad w_i^j = w_{i*}^{j*}, \quad w_j^{i*} = w_i^{j*}, \quad (2.6)$$

$$w_i^{k*} = \sum h_{ij}^{k*} w^j, \quad h_{ij}^{k*} = h_{ji}^{k*} = h_{jk}^{i*} = h_{ik}^{j*}, \quad (2.7)$$

$$dw^i = -\sum w_j^i \wedge w^j, \quad w_i^j + w_j^i = 0, \quad (2.8)$$

$$dw_i^j = -\sum w_i^k \wedge w_k^j + \frac{1}{2} \sum R_{ikl}^j w^k \wedge w^l, \quad (2.9)$$

$$R_{jkl}^i = \bar{R}_{jkl}^i w^k + \sum (h_{ik}^{m*} h_{jl}^{m*} - h_{il}^{m*} h_{jk}^{m*}), \quad (2.10)$$

$$dw_{j*}^{i*} = -\sum w_{k*}^{i*} \wedge w_{j*}^{k*} + \frac{1}{2} \sum R_{j*kl}^{i*} w^k \wedge w^l, \quad (2.11)$$

$$R_{j*kl}^{i*} = \bar{R}_{j*kl}^{i*} + \sum (h_{km}^{i*} h_{ml}^{j*} - h_{ml}^{i*} h_{km}^{j*}). \quad (2.12)$$

The second fundamental form h of M in CP^n is defined as $h = \sum h_{ij}^{k*} w^i \otimes e_{k*}$, whose squared length is $\|h\|^2 = \sum (h_{ij}^{k*})^2$.

If M is minimal in CP^n , i.e., trace $h = 0$, then from (2.4) and (2.10), we have

$$r = n(n-1) - \|h\|^2, \quad (2.13)$$

where r is the scalar curvature of M .

Define h_{ijk}^{m*} and h_{ijkl}^{m*} by

$$\sum h_{ijk}^{m*} w^k = dh_{ij}^{m*} - \sum h_{kj}^{m*} w_i^k - \sum h_{ik}^{m*} w_j^k + \sum h_{ij}^{l*} w_l^{m*}, \quad (2.14)$$

$$\sum h_{ijkl}^{m*} w^l = dh_{ijk}^{m*} - \sum h_{ijk}^{m*} w_i^l - \sum h_{ilk}^{m*} w_j^l - \sum h_{ijl}^{m*} w_k^l + \sum h_{ijk}^{l*} w_l^{m*}, \quad (2.15)$$

respectively.

Let H_{i*} and Δ denote the $(n \times n)$ -matrix (h_{ij}^{l*}) and the Laplacian on M , respectively. By a simple calculation, we have (cf. [2])

$$\begin{aligned} \frac{1}{2} \Delta \|h\|^2 &= \sum (h_{ijk}^{l*})^2 + (n+1) \|h\|^2 + \sum \text{tr} (H_{i*} H_{j*} - H_{j*} H_{i*})^2 \\ &\quad - \sum (\text{tr} H_{i*} \text{tr} H_{j*})^2. \end{aligned} \quad (2.16)$$

The following lemma is important in this paper.

LEMMA 1 [6]. *Let M^n be a complete Riemannian manifold with Ricci curvature bounded from below and let f be a C^2 -function bounded from above on M^n , then for all $\epsilon > 0$, there exists a point $x \in M^n$ at which*

- (i) $\sup f - \epsilon < f(x)$;
- (ii) $\|\nabla f(x)\| < \epsilon$;
- (iii) $\Delta f(x) < \epsilon$.

PROOF OF THE MAIN THEOREM. By [3], we have $\sum (\text{tr} H_{i^*} H_{j^*})^2 = \sum (\text{tr} H_{i^*}^2)^2$. From [1], we know that $\sum \text{tr} (H_{i^*} H_{j^*} - H_{j^*} H_{i^*})^2 - \sum (\text{tr} H_{i^*}^2)^2 \geq -3/2 \|h\|^4$. So, from (2.16), we obtain

$$\frac{1}{2} \Delta \|h\|^2 \geq \|h\|^2 ((n+1) - 3/2 \|h\|^2). \tag{2.17}$$

We know that $\|h\|^2 = n(n-1) - r$. By the condition of the theorem, we conclude that $\|h\|^2$ is bounded. We define $f = \|h\|^2$ and $F = (f+a)^{1/2}$ (where $a > 0$ is any positive constant number). F is bounded. We have

$$dF = \frac{1}{2} (f+a)^{-1/2} df, \tag{2.18}$$

$$\begin{aligned} \Delta F &= \frac{1}{2} \left(-\frac{1}{2} (f+a)^{-3/2} \|df\|^2 + (f+a)^{-1/2} \Delta f \right) \\ &= \frac{1}{2} (-2 \|dF\|^2 + \Delta f) (f+a)^{-1/2}, \end{aligned} \tag{2.19}$$

i.e.,

$$\Delta F = \frac{1}{2F} (-2 \|dF\|^2 + \Delta f). \tag{2.20}$$

Hence, $F \Delta F = -\|dF\|^2 + 1/2 \Delta f$ or $1/2 \Delta f = F \Delta F + \|dF\|^2$.

Applying Lemma 1 to F , we have for all $\epsilon > 0$, there exists a point $x \in M$ such that at x

$$\|dF(x) < \epsilon\|; \tag{2.21}$$

$$\Delta F(x) < \epsilon; \tag{2.22}$$

$$F(x) > \sup F - \epsilon. \tag{2.23}$$

From (2.21), (2.22), and (2.23), we have

$$\frac{1}{2} \Delta f < \epsilon^2 + F \epsilon = \epsilon(\epsilon + F). \tag{2.24}$$

We take a sequence $\{\epsilon_m\}$ such that $\epsilon_m \rightarrow 0 (m \rightarrow \infty)$ and for all m , there exists a point $x_m \in M$ such that (2.21), (2.22), and (2.23) hold. Therefore, $\epsilon_m (\epsilon_m + F(x_m)) \rightarrow 0 (m \rightarrow \infty)$ (because F is bounded).

From (2.23), we have $F(x_m) > \sup F - \epsilon_m$. Because $\{F(x_m)\}$ is a bounded sequence. So we get $F(x_m) \rightarrow F_0$ (if necessary, we can choose a subsequence). Hence, $F_0 \geq \sup F$. So we have

$$F_0 = \sup F. \tag{2.25}$$

From the definition of F , we get

$$f(x_m) \rightarrow f = \sup f. \tag{2.26}$$

(2.17) and (2.24) imply that

$$f \left((n+1) - \frac{3}{2} f \right) \leq \frac{1}{2} \Delta f \leq \epsilon(\epsilon + F), \tag{2.27}$$

and

$$f(x_m) \left((n+1) - \frac{3}{2} f(x_m) \right) < \epsilon_m^2 + \epsilon_m F(x_m) \leq \epsilon_m^2 + \epsilon_m F_0 \tag{2.28}$$

let $m \rightarrow \infty$, then $\epsilon_m \rightarrow 0$ and $f(x_m) \rightarrow f_0$. Hence,

$$f_0 \left((n+1) - \frac{3}{2}f_0 \right) \leq 0. \quad (2.29)$$

- (i) if $f_0 = 0$, we have $f = \|h\|^2 \equiv 0$. Hence, M is totally geodesic.
 (ii) if $f_0 > 0$, we have $(n+1) - 3/2f_0 \leq 0$ and $f_0 \geq 2/3(n+1)$, that is, $\sup \|h\|^2 \geq 2/3(n+1)$. Therefore, $\inf r \leq (3n+1)(n-2)/3$. This completes the proof. \square

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REFERENCES

- [1] L. An Min and L. Jimin, *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math. (Basel) **58** (1992), no. 6, 582-594. MR 93b:53050. Zbl 767.53042.
- [2] B. Y. Chen and K. Ogiue, *On totally real submanifolds*, Trans. Amer. Math. Soc. **193** (1974), 257-266. MR 49 11433. Zbl 286.53019.
- [3] S. S. Chern, M. do Carmo, and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, 1968) (New York), vol. III, Springer-Verlag, 1970, pp. 59-75. MR 42#8424. Zbl 216.44001.
- [4] G. D. Ludden, M. Okumura, and K. Yano, *A totally real surface in CP^2 that is not totally geodesic*, Proc. Amer. Math. Soc. **53** (1975), no. 1, 186-190. MR 52 1580. Zbl 312.53043.
- [5] Y. B. Shen, *Scalar curvature for totally real minimal submanifolds*, Chinese Ann. Math. Ser. A **12** (1991), no. 5, 573-577 (Chinese). MR 93a:53052. Zbl 777.53055.
- [6] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201-228. MR 55 4042. Zbl 297.31005.

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