Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 782680, 11 pages
doi:10.1155/2010/782680

Research Article

Ordered Non-Archimedean Fuzzy Metric Spaces and Some Fixed Point Results

Ishak Altun[1] and Dorel Miheţ[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey*
[2] *Departament of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, Bv. V. Parvan 4, 300223 Timişoara, Romania*

Correspondence should be addressed to Ishak Altun, ishakaltun@yahoo.com

Received 2 July 2009; Accepted 9 February 2010

Academic Editor: Mohamed A. Khamsi

In the present paper we provide two different kinds of fixed point theorems on ordered nonArchimedean fuzzy metric spaces. First, two fixed point theorems are proved for fuzzy order ψ-contractive type mappings. Then a common fixed point theorem is given for noncontractive type mappings. Kirk's problem on an extension of Caristi's theorem is also discussed.

1. Introduction and Preliminaries

After the definition of the concept of fuzzy metric space by some authors [1–3], the fixed point theory on these spaces has been developing (see, e.g., [4–9]). Generally, this theory on fuzzy metric space is done for contractive or contractive-type mappings (see [2, 10–13] and references therein). In this paper we introduce the concept of fuzzy order ψ-contractive mappings and give two fixed point theorems on ordered non-Archimedean fuzzy metric spaces for fuzzy order ψ-contractive type mappings. Then, using an idea in [14], we will provide a common fixed point theorem for weakly increasing single-valued mappings in a complete fuzzy metric space endowed with a partial order induced by an appropriate function. Some fixed point results on ordered probabilistic metric spaces can be found in [15].

For the sake of completeness, we briefly recall some notions from the theory of fuzzy metric spaces used in this paper.

Definition 1.1 (see [16]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with the unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

A continuous t-norm $*$ is of *Hadžić-type* if there exists a strictly increasing sequence $\{b_n\} \subset (0,1)$ such that $b_n * b_n = b_n$ for all $n \in \mathbb{N}$.

Definition 1.2 (see [3]). A fuzzy metric space (in the sense of Kramosil and Michálek) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$, satisfying the following properties:

(KM-1) $M(x,y,0) = 0$, for all $x,y \in X$,

(KM-2) $M(x,y,t) = 1$, for all $t > 0$ if and only if $x = y$,

(KM-3) $M(x,y,t) = M(y,x,t)$, for all $x,y \in X$ and $t > 0$,

(KM-4) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left continuous, for all $x,y \in X$,

(KM-5) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$, for all $x,y,z \in X$, for all $t,s > 0$.

If, in the above definition, the triangular inequality (KM-5) is replaced by

$$M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s), \quad \forall x,y,z \in X,\ \forall t,s > 0, \tag{NA}$$

then the triple $(X, M, *)$ is called a *non-Archimedean fuzzy metric space*. It is easy to check that the triangular inequality (NA) implies (KM-5), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Example 1.3. Let (X,d) be an ordinary metric space and let θ be a nondecreasing and continuous function from $(0,\infty)$ into $(0,1)$ such that $\lim_{t\to\infty}\theta(t) = 1$. Some examples of these functions are $\theta(t) = t/(t+1)$, $\theta(t) = 1 - e^{-t}$ and $\theta(t) = e^{-1/t}$. Let $a * b \leq ab$ for all $a,b \in [0,1]$. For each $t \in (0,\infty)$, define

$$M(x,y,t) = [\theta(t)]^{d(x,y)} \tag{1.1}$$

for all $x,y \in X$. It is easy to see that $(X, M, *)$ is a non-Archimedean fuzzy metric space.

Definition 1.4 (see [1, 16]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called an M-Cauchy sequence, if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m,n \geq n_0$. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$. A fuzzy metric space $(X, M, *)$ is called M-complete if every M-Cauchy sequence is convergent.

Definition 1.5 (see [7]). Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called *G-Cauchy* if

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \tag{1.2}$$

for all $t > 0$. The space $(X, M, *)$ is called G-complete if every G-Cauchy sequence is convergent.

Lemma 1.6 (see [11]). *Each M -complete non-Archimedean fuzzy metric space (X, M, T) with T of Hadžić-type is G-complete.*

Theorem 2.10 in the next section is related to a partial order on a fuzzy metric space under the Łukasiewicz t-norm. We will refer to [14].

Lemma 1.7 (see [14]). *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with $a*b \geq \max\{a+b-1, 0\}$ and $\phi : X \times [0, \infty) \to \mathbb{R}$. Define the relation "$\preceq$" on X as follows:*

$$x \preceq y \Longleftrightarrow M(x, y, t) \geq 1 + \phi(x, t) - \phi(y, t), \quad \forall t > 0. \tag{1.3}$$

Then $\preceq$ is a (partial) order on X, named the partial order induced by ϕ.

2. Main Results

The first two theorems in this section are related to Theorem 2.1 in [17]. We begin by giving the following definitions.

Definition 2.1. Let $\preceq$ be an order relation on X. A mapping $f : X \to X$ is called nondecreasing w.r.t $\preceq$ if $x \preceq y$ implies $fx \preceq fy$.

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set, let $(X, M, *)$ be a fuzzy metric space, and let ψ be a function from $[0, 1]$ to $[0, 1]$. A mapping $f : X \to X$ is called a fuzzy order ψ-contractive mapping if the following implication holds:

$$x, y \in X, \quad x \preceq y \Longrightarrow [M(fx, fy, t) \geq \psi(M(x, y, t)) \ \forall t > 0]. \tag{2.1}$$

Theorem 2.3. *Let $(X, \preceq)$ be a partially ordered set and $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space with $*$ of Hadžić-type. Let $\psi : [0, 1] \to [0, 1]$ be a continuous, nondecreasing function and let $f : X \to X$ be a fuzzy order ψ-contractive and nondecreasing mapping w.r.t $\preceq$. Suppose that either*

$$f \text{ is continuous}, \tag{2.2}$$

or

$$\begin{gathered} x_n \preceq x \quad \forall n, \text{ whenever} \\ \{x_n\} \subset X \text{ is nondecreasing sequence with } x_n \longrightarrow x \in X \end{gathered} \tag{2.3}$$

hold. If there exists $x_0 \in X$ such that

$$x_0 \preceq fx_0, \qquad \lim_{n \to \infty} \psi^n(M(x_0, fx_0, t)) = 1 \tag{2.4}$$

for each $t > 0$, then f has a fixed point.

Proof. Let $x_n = fx_{n-1}$ for $n \in \{1, 2, \ldots\}$. Since $x_0 \preceq fx_0$ and f is nondecreasing w.r.t $\preceq$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{2.5}$$

Then, it immediately follows by induction that

$$M(x_{n+1}, x_{n+2}, t) \geq \varphi(M(x_n, x_{n+1}, t)), \quad (n \in \mathbb{N},\ t > 0), \tag{2.6}$$

hence

$$M(x_n, x_{n+1}, t) \geq \varphi^n(M(x_0, fx_0, t)), \quad (n \in \mathbb{N},\ t > 0). \tag{2.7}$$

By taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \tag{2.8}$$

for all $t > 0$, that is, $\{x_n\}$ is G-Cauchy. Since X is G-complete (Lemma 1.6), then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Now, if f is continuous then it is clear that $fx = x$, while if the condition (2.3) hold then, for all $t > 0$,

$$M(x_{n+1}, fx, t) = M(fx_n, fx, t) \geq \varphi(M(x_n, x, t)) \tag{2.9}$$

and letting $n \to \infty$ it follows

$$M(x, fx, t) \geq \varphi(1) = 1, \tag{2.10}$$

hence $fx = x$. □

Theorem 2.4. *Let $(X, \preceq)$ be a partially ordered set, let $(X, M, *)$ be an M-complete non-Archimedean fuzzy metric space, and let $\varphi : [0,1] \to [0,1]$ be a continuous mapping such that $\varphi(t) > t$ for all $t \in (0,1)$. Also, let $f : X \to X$ be a nondecreasing mapping w.r.t $\preceq$, with the property*

$$M(fx, fy, t) \geq \varphi(M(x, y, t)) \quad \forall t > 0, \text{ whenever } x \preceq y. \tag{2.11}$$

Suppose that either (2.2) or (2.3) holds. If there exists $x_0 \in X$ such that

$$x_0 \preceq fx_0, \qquad M(x_0, fx_0, t) > 0 \tag{2.12}$$

for all $t > 0$, then f has a fixed point.

Proof. Let $x_n = fx_{n-1}$ for $n \in \{1, 2, \ldots\}$. Then, as in the proof of the preceding theorem we can prove that

$$M(x_{n+1}, x_{n+2}, t) \geq \varphi(M(x_n, x_{n+1}, t)) \geq M(x_n, x_{n+1}, t), \quad (n \in \mathbb{N},\ t > 0). \tag{2.13}$$

Therefore, for every $t > 0$, $\{M(x_n, x_{n+1}, t)\}_{n\in\mathbb{N}}$ is a nondecreasing sequence of numbers in $(0,1]$. Let, for fixed $t > 0$, $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = l$. Then we have $l \in (0,1]$, since $M(x_0, x_1, t) > 0$. Also, since

$$M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t)) \tag{2.14}$$

and ψ is continuous, we have $l \geq \psi(l)$. This implies $l = 1$ and therefore, for all $t > 0$,

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{2.15}$$

Now we show that $\{x_n\}$ is an M-Cauchy sequence. Supposing this is not true, then there are $\varepsilon \in (0,1)$ and $t > 0$ such that for each $k \in \mathbb{N}$ there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M\left(x_{m(k)}, x_{n(k)}, t\right) \leq 1 - \varepsilon. \tag{2.16}$$

Let, for each k, $m(k)$ be the least integer exceeding $n(k)$ satisfying the inequality (2.16), that is,

$$M\left(x_{m(k)-1}, x_{n(k)}, t\right) > 1 - \varepsilon. \tag{2.17}$$

Then, for each k,

$$\begin{aligned} 1 - \varepsilon &\geq M\left(x_{m(k)}, x_{n(k)}, t\right) \\ &\geq M\left(x_{m(k)-1}, x_{n(k)}, t\right) * M\left(x_{m(k)-1}, x_{m(k)}, t\right) \\ &\geq (1-\varepsilon) * M\left(x_{m(k)-1}, x_{m(k)}, t\right). \end{aligned} \tag{2.18}$$

Letting $k \to \infty$ and using (2.15), we have, for $t > 0$,

$$\lim_{k\to\infty} M\left(x_{m(k)}, x_{n(k)}, t\right) = 1 - \varepsilon. \tag{2.19}$$

Then, since $x_{n(k)} \preceq x_{m(k)}$, we have

$$\begin{aligned} M\left(x_{m(k)}, x_{n(k)}, t\right) &\geq M\left(x_{m(k)}, x_{m(k)+1}, t\right) * M\left(x_{m(k)+1}, x_{n(k)+1}, t\right) * M\left(x_{n(k)+1}, x_{n(k)}, t\right) \\ &\geq M\left(x_{m(k)}, x_{m(k)+1}, t\right) * \psi\left(M\left(x_{m(k)}, x_{n(k)}, t\right)\right) * M\left(x_{n(k)+1}, x_{n(k)}, t\right). \end{aligned} \tag{2.20}$$

Letting $k \to \infty$ and using (2.15) and (2.19), we obtain

$$1 - \varepsilon \geq 1 * \psi(1-\varepsilon) * 1 = \psi(1-\varepsilon) > 1 - \varepsilon, \tag{2.21}$$

which is a contradiction. Thus $\{x_n\}$ is an M-Cauchy sequence. Since X is M-complete, then there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x. \tag{2.22}$$

If f is continuous, then from $x_n = fx_{n-1}$ $(n \in \mathbb{N})$ it follows that $fx = x$. Also, if (2.3) holds, then (since $x_n \preceq x$) we have

$$M(x_{n+1}, fx, t) = M(fx_n, fx, t) \geq \psi(M(x_n, x, t)), \quad (n \in \mathbb{N},\ t > 0). \tag{2.23}$$

Letting $n \to \infty$, we obtain that

$$M(x, fx, t) = 1 \quad \forall t > 0, \tag{2.24}$$

hence $fx = x$. □

Example 2.5. Let $X = (0, \infty)$. Consider the following relation on X:

$$x \preceq y \Longleftrightarrow (x = y \text{ or } x, y \in [1,4],\ x \leq y). \tag{2.25}$$

It is easy to see that $\preceq$ is a partial order on X. Let $a * b = ab$ and

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}}, \quad \forall t > 0. \tag{2.26}$$

Then $(X, M, *)$ is an M-complete non-Archimedean fuzzy metric space (see [18]) satisfying $M(x, y, t) > 0$ for all $t > 0$. Define a self map f of X as follows:

$$fx = \begin{cases} 2x, & 0 < x < 1 \\ \dfrac{x+5}{3}, & 1 \leq x \leq 4 \\ 2x - 5, & x > 4. \end{cases} \tag{2.27}$$

Now, it is easy to see that f is continuous and nondecreasing w.r.t $\preceq$. Also, for $x_0 = 1$ we have $1 = x_0 \preceq fx_0 = 2$. Now we can see that f is fuzzy order ψ-contractive with $\psi(t) = \sqrt{t}$.

Indeed, let $x, y \in X$ with $x \preceq y$. Now if $x = y$, then

$$M(fx, fy, t) = 1 \geq \psi(1) = \psi(M(x, y, t)). \tag{2.28}$$

If $x, y \in [1,4]$ with $x \le y$, then

$$\begin{aligned} M(fx, fy, t) &= \frac{\min\{fx, fy\}}{\max\{fx, fy\}} \\ &= \frac{\min\{(x+5)/3, (y+5)/3\}}{\max\{(x+5)/3, (y+5)/3\}} \\ &= \frac{x+5}{y+5} \\ &\ge \sqrt{\frac{x}{y}} \\ &= \psi(M(x, y, t)). \end{aligned} \tag{2.29}$$

Therefore f is fuzzy order ψ-contractive with $\psi(t) = \sqrt{t}$. Hence all conditions of Theorem 2.4 are satisfied and so f has a fixed point on X.

In order to state our next theorem, we give the concept of weakly comparable mappings on an ordered space.

Definition 2.6. Let $(X, \preceq)$ be an ordered space. Two mappings $f, g : X \to X$ are said to be weakly comparable if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$.

Note that two weakly comparable mappings need not to be nondecreasing.

Example 2.7. Let $X = [0, \infty)$ and $\le$ be usual ordering. Let $f, g : X \to X$ defined by

$$fx = \begin{cases} x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x < \infty, \end{cases} \qquad gx = \begin{cases} \sqrt{x} & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x < \infty. \end{cases} \tag{2.30}$$

Then it is obvious that $fx \le gfx$ and $gx \le fgx$ for all $x \in X$. Thus f and g are weakly comparable mappings. Note that both f and g are not nondecreasing.

Example 2.8. Let $X = [1, \infty) \times [1, \infty)$ and $\preceq$ be coordinate-wise ordering, that is, $(x, y) \preceq (z, w) \Leftrightarrow x \le z$ and $y \le w$. Let $f, g : X \to X$ be defined by $f(x, y) = (2x, 3y)$ and $g(x, y) = (x^2, y^2)$, then $f(x, y) = (2x, 3y) \preceq gf(x, y) = g(2x, 3y) = (4x^2, 9y^2)$ and $g(x, y) = (x^2, y^2) \preceq fg(x, y) = f(x^2, y^2) = (2x^2, 3y^2)$. Thus f and g are weakly comparable mappings.

Example 2.9. Let $X = \mathbb{R}^2$ and $\preceq$ be lexicographical ordering, that is, $(x, y) \preceq (z, w) \Leftrightarrow (x < z)$ or (if $x = z$, then $y \le w$). Let $f, g : X \to X$ be defined by

$$\begin{aligned} f(x, y) &= (\max\{x, y\}, \min\{x, y\}), \\ g(x, y) &= \left(\max\{x, y\}, \frac{x+y}{2}\right), \end{aligned} \tag{2.31}$$

then $f(x,y) \preceq gf(x,y)$ and $g(x,y) \preceq fg(x,y)$ for all $(x,y) \in X$. Thus f and g are weakly comparable mappings. Note that, $(1,4) \preceq (2,3)$ but $f(1,4) = (4,1)(3,2) = f(2,3)$, then f is not nondecreasing. Similarly g is not nondecreasing.

Theorem 2.10. *Let $(X, M, *)$ be an M -complete non-Archimedean fuzzy metric space with $a * b \geq \max\{a+b-1, 0\}$, $\phi : X \times [0,\infty) \to \mathbb{R}$ be a bounded-from-above function, and let $\preceq$ be the partial order induced by ϕ. If $f, g : X \to X$ are two continuous and weakly comparable mappings, then f and g have a common fixed point in X.*

Proof. Let X_0 be an arbitrary point of X and let us define a sequence $\{x_n\}$ in X as follows:

$$x_{2n+1} = fx_{2n}, \quad x_{2n+2n} = gx_{2n+1} \quad \text{for } n \in \{0,1,\ldots\}. \tag{2.32}$$

Note that, since f and g are weakly comparable, we have

$$\begin{aligned} x_1 &= fx_0 \preceq gfx_0 = gx_1 = x_2, \\ x_2 &= gx_1 \preceq fgx_1 = fx_2 = x_3. \end{aligned} \tag{2.33}$$

By continuing this process we get

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots, \tag{2.34}$$

that is, the sequence $\{x_n\}$ is nondecreasing. By the definition of $\preceq$ we have $\phi(x_0,t) \leq \phi(x_1,t) \leq \phi(x_2,t) \leq \cdots$ for all $t > 0$, that is, for even $t > 0$, the sequence $\{\phi(x_n,t)\}$ is a nondecreasing sequence in $\mathbb{R}$. Since ϕ is bounded from above, $\{\phi(x_n,t)\}$ is convergent and hence it is Cauchy. Then, for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ and $t > 0$ we have $|\phi(x_m,t) - \phi(x_n,t)| = \phi(x_m,t) - \phi(x_n,t) < \varepsilon$. Therefore, since $x_n \preceq x_m$, we have

$$\begin{aligned} M(x_n, x_m, t) &\geq 1 + \phi(x_n,t) - \phi(x_m,t) \\ &= 1 - \left[\phi(x_m,t) - \phi(x_n,t)\right] \\ &> 1 - \varepsilon. \end{aligned} \tag{2.35}$$

This shows that the sequence $\{x_n\}$ is M-Cauchy. Since X is M-complete, it converges to a point $z \in X$. As $x_{2n+1} \to z$ and $x_{2n+2} \to z$, by the continuity of f and g we get $fz = gz = z$. □

Corollary 2.11 ([Caristi fixed point theorem in non-Archimedean fuzzy metric spaces]). *Let $(X, M, *)$ be an M -complete non-Archimedean fuzzy metric space with $a * b \geq \max\{a+b-1, 0\}$, let $\phi : X \times [0,\infty) \to \mathbb{R}$ be a bounded-from-above function and $f : X \to X$ be a continuous mapping, such that*

$$M(x, fx, t) \geq 1 + \phi(x,t) - \phi(fx,t) \tag{2.36}$$

for all $x \in X$ and $t > 0$. Then f has a fixed point in X.

Proof. We take in the above theorem $g = 1_X$ and note that the weak comparability of f and g reduces to (2.36). □

The generalization suggested by Kirk of Caristi's fixed point theorem [19] is well known. A similar theorem in the setting of non-Archimedean fuzzy metric spaces is stated in the final part of our paper.

In what follows $\nu : [0,1] \to [0,1]$ is nondecreasing, subadditive mapping (i.e., $\nu(a+b) \le \nu(a) + \nu(b)$ for all $a, b \in [0,1]$), with $\nu(0) = 0$.

Theorem 2.12. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space with $a*b \ge \max\{a+b-1, 0\}$ and $\phi : X \times [0,\infty) \to \mathbb{R}$. Define the relation "$\preceq$" on X through*

$$x \preceq y \Longleftrightarrow \phi(y,t) - \phi(x,t) \ge \nu(1 - M(x,y,t)), \quad \forall t > 0. \tag{2.37}$$

Then "$\preceq$" is a (partial) order on X.

Proof. Since $\nu(0) = 0$, then for all $x \in X$ and $t > 0$,

$$0 = \phi(x,t) - \phi(x,t) \ge \nu(1 - M(x,x,t)) = 0, \tag{2.38}$$

that is, "$\preceq$" is reflexive.

Let $x, y \in X$ be such that $x \preceq y$ and $y \preceq x$. Then for all $t > 0$,

$$\begin{aligned} &\phi(y,t) - \phi(x,t) \ge \nu(1 - M(x,y,t)), \\ &\phi(x,t) - \phi(y,t) \ge \nu(1 - M(x,y,t)), \end{aligned} \tag{2.39}$$

implying that $M(x,y,t) = 1$ for all $t > 0$, that is, $x = y$. Thus "$\preceq$" is antisymmetric.

Now for $x, y, z \in X$, let $x \preceq y$ and $y \preceq z$. Then, for given $t > 0$,

$$\phi(y,t) - \phi(x,t) \ge \nu(1 - M(x,y,t)), \tag{2.40}$$

$$\phi(z,t) - \phi(y,t) \ge \nu(1 - M(z,y,t)). \tag{2.41}$$

By using (2.40) and (2.41) we get

$$\begin{aligned} \phi(z,t) - \phi(x,t) &\ge \nu(1 - M(x,y,t)) + \nu(1 - M(y,z,t)) \\ &\ge \nu(1 - M(x,y,t) + 1 - M(y,z,t)). \end{aligned} \tag{2.42}$$

On the other hand, from the triangular inequality (NA), the inequality

$$M(x,z,t) \ge M(x,y,t) + M(y,z,t) - 1 \tag{2.43}$$

holds. This implies

$$1 - M(x,y,t) + 1 - M(y,z,t) \ge 1 - M(x,z,t). \tag{2.44}$$

As ν is nondecreasing, it follows that

$$\nu(1 - M(x, y, t) + 1 - M(y, z, t)) \geq \nu(1 - M(x, z, t)) \tag{2.45}$$

and therefore

$$\phi(z, t) - \phi(x, t) \geq \nu(1 - M(x, z, t)). \tag{2.46}$$

This shows that $x \preceq z$, that is, "$\preceq$" is transitive. □

From the above theorem we can immediately obtain the following generalization of Corollary 2.11.

Corollary 2.13. *Let* $(X, M, *)$ *be an* M *-complete non-Archimedean fuzzy metric space with* $a * b \geq \max\{a + b - 1, 0\}$, *let* $\phi : X \times [0, \infty) \to \mathbb{R}$ *be a bounded-from-above function and* $f : X \to X$ *be a continuous mapping, such that*

$$\phi(fx, t) - \phi(x, t) \geq \nu(1 - M(x, fx, t)) \tag{2.47}$$

for all $x \in X$ *and* $t > 0$. *If* ν *satisfies the property*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \nu(x) < \delta \Longrightarrow x < \varepsilon, \tag{2.48}$$

then f *has a fixed point in* X.

The reader is referred to the nice paper [20] for some discussion of Kirk's problem on an extension of Caristi's fixed point theorem.

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