Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 527864, 14 pages
doi:10.1155/2010/527864

Research Article

Bifurcation Analysis for a Delayed Predator-Prey System with Stage Structure

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Received 9 August 2010; Revised 10 October 2010; Accepted 14 October 2010

Academic Editor: Massimo Furi

A delayed predator-prey system with stage structure is investigated. The existence and stability of equilibria are obtained. An explicit algorithm for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions is derived by using the normal form and the center manifold theory. Finally, a numerical example supporting the theoretical analysis is given.

1. Introduction

The age factors are important for the dynamics and evolution of many mammals. The rates of survival, growth, and reproduction almost always depend heavily on age or developmental stage, and it has been noticed that the life history of many species is composed of at least two stages, immature and mature, with significantly different morphological and behavioral characteristics. The study of stage-structured predator-prey systems has attracted considerable attention in recent years (see [1–6] and the reference therein). In [4], Wang considered the following predator-prey model with stage structure for predator, in which the immature predators can neither hunt nor reproduce.

$$\begin{aligned} \dot{x}(t) &= x(t)\left[r - ax(t) - \frac{by_2(t)}{1+mx(t)}\right], \\ \dot{y}_1(t) &= \frac{kbx(t)y_2(t)}{1+mx(t)} - (D+v_1)y_1(t), \\ \dot{y}_2(t) &= Dy_1(t) - v_2y_2(t), \end{aligned} \tag{1.1}$$

where $x(t)$ denotes the density of prey at time t, $y_1(t)$ denotes the density of immature predator at time t, $y_2(t)$ denotes the density of mature predator at time t, b is the search rate, m is the search rate multiplied by the handling time, and r is the intrinsic growth rate. It is assumed that the reproduction rate of the mature predator depends on the quality of prey considered, the efficiency of conversion of prey into newborn immature predators being denoted by k. D denotes the rate at which immature predators become mature predators. v_1 and v_2 denote the mortality rates of immature and mature predators, respectively. All coefficients are positive constants. In [4], he concluded that the system under some conditions has a unique positive equilibrium, which is globally asymptotically stable. Georgescu and Moroşanu [7] generalized the system (1.1) as

$$\begin{aligned}\dot{x}(t) &= n(x(t)) - f(x(t))y_2(t),\\ \dot{y}_1(t) &= kf(x(t))y_2(t) - (D+v_1)y_1(t),\\ \dot{y}_2(t) &= Dy_1(t) - v_2y_2(t),\end{aligned} \tag{1.2}$$

satisfying the following hypotheses:

(H1) (a) $f(x)$ is the predator functional response and satisfies that

$$f \in C^1([0,\infty),[0,\infty)), \quad f(0)=0, \quad f'(x)>0, \quad \lim_{x\to\infty}\frac{f(x)}{x}<\infty. \tag{1.3}$$

(b) $n(x)$ is the growth function and satisfies that $n \in C^1([0,\infty),R), n(x)=0$ if and only if $x \in \{0,x_0\}$, with $x_0>0$ and $n(x)>0$ for $x \in (0,x_0)$, and $n(x)$ is strictly decreasing on $[x_p,\infty), 0<x_p<x_0$.

(c) The prey isocline is given by $h(x) := n(x)/f(x)$ and is assumed to be concave down, that is, $h''(x)<0$ for $x \geqslant 0$.

In [7], they employ the theory of competitive systems and Muldowney's necessary and sufficient condition for the orbital stability of a periodic orbit and obtain the global stability of the positive equilibrium for the general system. It is necessary to forsake some aspects of realism, and one of the features of the real world which is commonly compromised in order to achieve generality is the time delay. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the population to fluctuate. Time delay due to gestation is a common example, because generally the consumption of prey by a predator throughout its past history governs the present birth rate of the predator. Therefore, more realistic models of population interactions should take into account the effect of time delays. So, we introduce the delay τ due to gestation of mature predator into system (1.2) and consider the following system:

$$\begin{aligned}\dot{x}(t) &= n(x(t)) - f(x(t))y_2(t),\\ \dot{y}_1(t) &= kf(x(t-\tau))y_2(t-\tau) - (D+v_1)y_1(t),\\ \dot{y}_2(t) &= Dy_1(t) - v_2y_2(t),\end{aligned} \tag{1.4}$$

where all coefficients are positive constants and the detailed ecological meanings are the same as in system (1.2). Some usual examples of $f(x)$ and $n(x)$ include $f(x) = m\times^c (m > 0, 0 < c \leqslant 1)$, $f(x) = m(1-e^{-cx})$ $(m, c > 0)$, $f(x) = \alpha x e^{-\beta x}(\alpha > 0, \beta > 0)$, $f(x) = bx^p/(1+mx^p)(p > 0)$ and $n(x) = x(r-ax)/(1+\varepsilon x)(\varepsilon > 0)$, $n(x) = rx(1-(x/(r/a))^c)$ $(0 < c \leqslant 1)$, or $n(x) = x(\mathrm{re}^{1-x/k})-d$, and so forth.

Our main purpose of this paper is to investigate the dynamic behaviors of system (1.4) and the frame of this paper is organized as follows. In the next section, we will investigate the stability of equilibria and the existence of local Hopf bifurcation. In Section 3, the direction and stability of the bifurcating periodic solutions are determined by applying the center manifold theorem and normal form theory. In Section 4, a numerical example supporting the theoretical analysis is given.

2. Stability of the Equilibrium and Local Hopf Bifurcations

It is known that time delay does not change the location and number of positive equilibrium. We have the following lemma.

Lemma 2.1. *The system* (1.4) *has two nonnegative equilibria,* $E_0(0,0,0)$, $E_1(x_0,0,0)$, *and a positive equilibrium* $E^*(x^*, y_1^*, y_2^*)$ *if*

(H2) $v_2(D+v_1) < kDf(x_0)$*holds, where* x^*, y_1^* *and* y_2^* *satisfy*

$$\begin{aligned} n(x) &= f(x)y_2, \\ kf(x(t))y_2(t) &= (D+v_1)y_1, \\ Dy_1 &= v_2 y_2. \end{aligned} \tag{2.1}$$

The linear part of (1.4) *at* E_0 *is*

$$\begin{aligned} \dot{x}(t) &= n'(0)x(t), \\ \dot{y}_1(t) &= -(D+v_1)y_1(t), \\ \dot{y}_2(t) &= Dy_1(t) - v_2 y_2(t), \end{aligned} \tag{2.2}$$

and the corresponding characteristic equation is

$$(\lambda - n'(0))\left[\lambda^2 + (D+v_1+v_2)\lambda + v_2(D+v_1)\right] = 0. \tag{2.3}$$

From (H1), one knows that $n'(0) > 0$. *Hence,* (2.3) *has a positive real root and two negative real roots. One has the following lemma.*

Lemma 2.2. *For system* (1.4), E_0 *is a saddle point.*

The linear part of (1.4) *at* E_1 *is*

$$\begin{aligned}\dot{x}(t) &= n'(x_0)x(t) - f(x_0)y_2(t),\\ \dot{y}_1(t) &= kf(x_0)y_2(t-\tau) - (D+v_1)y_1(t),\\ \dot{y}_2(t) &= Dy_1(t) - v_2y_2(t),\end{aligned} \tag{2.4}$$

and the corresponding characteristic equation is

$$(\lambda - n'(x_0))\left[\lambda^2 + (D+v_1+v_2)\lambda + v_2(D+v_1) - kDf(x_0)e^{-\lambda\tau}\right] = 0. \tag{2.5}$$

From (H1), one has that $n'(x_0) < 0$*. Hence, the stability of* E_1 *is decided by the following equation:*

$$\lambda^2 + (D+v_1+v_2)\lambda + v_2(D+v_1) - kDf(x_0)e^{-\lambda\tau} = 0. \tag{2.6}$$

If (H3) $v_2(D+v_1) > kDf(x_0)$*holds, then* $\lambda = 0$ *is not the root of* (2.6)*, and all the roots of* (2.6) *have strictly negative real parts when* $\tau = 0$*. Furthermore, one has the following conclusion.*

Lemma 2.3. *If*

(H4) $v_2(D+v_1) > \max\{D+v_1+v_2/2, kDf(x_0)\}$ *and*

(H5) $\Delta \doteq (D+v_1+v_2)^2 + 4k^2D^2f^2(x_0) - 4v_2(D+v_1)(D+v_1+v_2) > 0$*hold, then* (2.6) *has two pairs of purely imaginary roots noted by* $\pm i\omega_{11}$ *and* $\pm i\omega_{12}$ *when* $\tau = \tau_{1j}^k$*, and the other roots have negative real parts, where* $\omega_{11} = \sqrt{(2v_2(D+v_1) - (D+v_1+v_2) + \sqrt{\Delta})/2}$, $\omega_{12} = \sqrt{(2v_2(D+v_1) - (D+v_1+v_2) - \sqrt{\Delta})/2}$, $\tau_{1j}^k = 1/\omega_{1k}\{\arccos[(-\omega_{1k}^2 + (D+v_1)v_2)/kDf(x_0)] + (2j+1)\pi\}$, $j = 0,1,2,\ldots, k = 1,2$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ *be the root of* (2.6) *satisfying* $\alpha(\tau_{1j}^k) = 0, \omega(\tau_{1j}^k) = \omega_{1k}$*. Thus, the following results hold.*

Lemma 2.4. $\alpha'(\tau_{1j}^k) > 0$.

Proof. By (2.6), we have

$$\lambda'\left(\tau_{1j}^k\right) = \frac{\omega_{1k}^2(D+v_1+v_2) + i\omega_{1k}\left[\omega_{1k}^2 - v_2(D+v_1)\right]}{\left[D+v_1+v_2 - \tau_{1j}^k(\omega_{1k}^2 - v_2(D+v_1))\right] + i\omega_{1k}\left[2 + \tau_{1j}^k(D+v_1+v_2)\right]} \tag{2.7}$$

and $\alpha'(\tau_{1j}^k) = \omega_{1k}^2[(D+v_1)^2 + 2\omega_{1k}^2] > 0$. □

From the above discussion, we have the following.

Theorem 2.5. (i) E_0 *is unstable for any* $\tau \geqslant 0$; (ii) *if (H2) holds, then* E_1 *is unstable and* E^* *exists;* (iii) *if (H4) and (H5) hold, then* E_1 *is asymptotically stable for* $\tau \in [0, \tau_{10})$ *and unstable for* $\tau > \tau_{10}$*, where* $\tau_{10} = \min\{\tau_{10}^1, \tau_{10}^2\}$.

The linear part of (1.4) at E^* is

$$\begin{aligned}\dot{x}(t) &= \left[n'(x^*) - f'(x^*)y_2^*\right]x(t) - f(x^*)y_2(t),\\ \dot{y}_1(t) &= kf'(x^*)y_2^*x(t-\tau) - (D+v_1)y_1(t) + kf(x^*)y_2(t-\tau),\\ \dot{y}_2(t) &= Dy_1(t) - v_2y_2(t),\end{aligned} \tag{2.8}$$

and the corresponding characteristic equation is

$$\begin{aligned}&\lambda^3 + \left[f'(x^*)y_2^* - n'(x^*) + D + v_1 + v_2\right]\lambda^2 + \left[v_2(D+v_1) + (D+v_1+v_2)\left(f'(x^*)y_2^* - n'(x^*)\right)\right]\\ &\quad\times \lambda + v_2(D+v_1)\left[f'(x^*)y_2^* - n'(x^*)\right] + \left[n'(x^*) - \lambda\right]kDf(x^*)e^{-\lambda\tau} = 0.\end{aligned} \tag{2.9}$$

Next, we will investigate the distribution of roots of (2.9). When $\tau = 0$, (2.9) can be reduced to

$$\begin{aligned}&\lambda^3 + \left[f'(x^*)y_2^* - n'(x^*) + D + v_1 + v_2\right]\lambda^2 + (D+v_1+v_2)\left(f'(x^*)y_2^* - n'(x^*)\right)\lambda\\ &\quad + v_2(D+v_1)f'(x^*)y_2^* = 0.\end{aligned} \tag{2.10}$$

By Routh-Hurwitz criteria, if

(H6) $[f'(x^*)y_2^* - n'(x^*) + D + v_1 + v_2](D+v_1+v_2)[f'(x^*)y_2^* - n'(x^*)] > v_2(D+v_1)f'(x^*)y_2^*$ holds, then all roots of (2.10) have strictly negative real parts and $\lambda = 0$ is not the root of (2.9). If the reverse of (2.10) is satisfied, then two characteristic roots have positive real parts. For convenience, we denote (2.9) as follows

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + (b_1\lambda + b_0)e^{-\lambda\tau} = 0, \tag{2.11}$$

where $a_2 = f'(x^*)y_2^* - n'(x^*) + D + v_1 + v_2$, $a_1 = [f'(x^*)y_2^* - n'(x^*)](D+v_1+v_2) + v_2(D+v_1)$, $a_0 = v_2(D+v_1)[f'(x^*)y_2^* - n'(x^*)]$, $b_1 = -v_2(D+v_1)$, $b_0 = v_2(D+v_1)n'(x^*)$. From $a_0 + b_0 > 0$, we have that $\lambda = 0$ is not the root of (2.11). Obviously, $\lambda = i\omega$ $(\omega > 0)$ is a root of (2.11) if and only if

$$i\omega^3 + a_2\omega^2 - ia_1\omega - a_0 - (b_1\omega i + b_0)(\cos\omega\tau - i\sin\omega\tau) = 0. \tag{2.12}$$

Separating the real part and imaginary part, we can obtain

$$\begin{aligned}a_2\omega^2 - a_0 &= b_0\cos\omega\tau + b_1\omega\sin\omega\tau,\\ a_1\omega - \omega^3 &= b_0\sin\omega\tau - b_1\omega\cos\omega\tau,\end{aligned} \tag{2.13}$$

which yields

$$\omega^6 + p\omega^4 + q\omega^2 + s = 0, \tag{2.14}$$

where $p = a_2^2 - 2a_1, q = a_1^2 - 2a_0a_2 - b_1^2, s = a_0^2 - b_0^2$. Set $z = \omega^2$. Then (2.14) takes the following form:

$$G(z) \overset{\text{def}}{=} z^3 + pz^2 + qz + s = 0. \tag{2.15}$$

Lemma 2.6 (see [8]). (a) *If $s < 0$, then* (2.15) *has at least one positive root.*

(b) *If $s \geqslant 0$ and $\Lambda = p^2 - 3q \leqslant 0$, then* (2.15) *has no positive roots.*

(c) *If $s \geqslant 0$ and $\Lambda = p^2 - 3q > 0$, then* (2.15) *has positive roots if and only if $z_1^* = (1/3)(-p + \sqrt{\Lambda})$ and $G(z_1^*) \leqslant 0$.*

The above Lemma can be seen in [8]. Suppose that (2.15) has positive roots. Without loss of generality, we assume that it has three positive roots z_1, z_2, z_3. Then (2.14) has three positive roots $\omega_1 = \sqrt{z_1}, \omega_2 = \sqrt{z_2}, \omega_3 = \sqrt{z_3}$. By (2.13), we have

$$\cos \omega\tau = \frac{b_1\omega_k^4 + (a_2b_0 - a_1b_1)\omega_k^2 - a_0b_0}{b_0^2 + b_1^2\omega_k^2}. \tag{2.16}$$

Thus, if

$$\tau_j^k = \frac{1}{\omega_k}\left\{\arccos\left(\frac{b_1\omega_k^4 + (a_2b_0 - a_1b_1)\omega_k^2 - a_0b_0}{b_0^2 + b_1^2\omega_k^2}\right) + 2j\pi\right\}, \tag{2.17}$$

where $k = 1, 2, 3, j = 0, 1, 2, \ldots$, then $\pm i\omega_k$ are a pair of purely imaginary roots of (2.11) with $\tau = \tau_j^k$. Suppose that

$$\tau_0 = \tau_0^{k_0} = \min\left\{\tau_0^k\right\}, \quad \omega_0 = \omega_{k_0}, \quad k = 1, 2, 3. \tag{2.18}$$

Thus, by Lemma2.2 and Corollary 2.4 in [9], we can easily get the following results.

Lemma 2.7. (a) *If $s \geqslant 0$ and $\Lambda = p^2 - 3q \leqslant 0$, then for any $\tau \geqslant 0$,* (2.9) *and* (2.10) *have the same number of roots with positive real parts.*

(b) *If either $s < 0$ or $s \geqslant 0, \Lambda = p^2 - 3q > 0, z_1^* > 0$ and $G(z_1^*) \leqslant 0$ is satisfied, then* (2.9) *and* (2.10) *have the same number of roots with positive real parts when $\tau \in [0, \tau_0)$.*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of (2.9) satisfying $\alpha'(\tau_j^k) = 0, \omega(\tau_j^k) = \omega_k$. Thus, the following transversality condition holds.

Lemma 2.8. *If $z_k = \omega_k^2$ and $G(z_k) \neq 0$, then $\alpha'(\tau_j^k) \neq 0$. Furthermore,* $\text{Sign}\{\alpha'(\tau_j^k)\} = \text{Sign}\{G'(z_k)\}$.

Proof. By direct computation to (2.11), we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3\lambda^2 + 2a_2\lambda + a_1)e^{\lambda\tau} + b_1}{\lambda(b_1\lambda + b_0)} + \frac{\tau}{\lambda}. \tag{2.19}$$

By (2.13), we have

$$[\lambda(b_1\lambda + b_0)]|_{\tau=\tau_j^k} = -b_1\omega_k^2 + ib_0\omega_k, \tag{2.20}$$

and

$$\begin{aligned}\left[\left(3\lambda^2 + 2a_2\lambda + a_1\right)e^{\lambda\tau}\right]|_{\tau=\tau_j^k} = &\left[\left(a_1 - 3\omega_k^2\right)\cos\omega_k\tau_j^k - 2a_2\omega_k\sin\omega_k\tau_j^k\right]\\ &+ i\left[2a_2\omega_k\cos\omega_k\tau_j^k + \left(a_1 - 3\omega_k^2\right)\sin\omega_k\tau_j^k\right].\end{aligned} \tag{2.21}$$

From (2.19) to (2.21), we have

$$\alpha'\left(\tau_j^k\right)^{-1} = \frac{z_k}{\Omega}G'(z_k), \tag{2.22}$$

where $\Omega = b_1^2\omega_k^4 + b_0^2\omega_k^2$. Thus $\mathrm{Sign}\{\alpha'(\tau_j^k)\} = \mathrm{Sign}\{\alpha'(\tau_j^k)\}^{-1} = \mathrm{Sign}\{G'(z_k)\} \neq 0$. □

By the above analyses, we can obtain the following theorem.

Theorem 2.9. *If (H2) and (H6) are satisfied, then the following results hold.*

(a) *If $s \geqslant 0$ and $\Lambda = p^2 - 3q \leqslant 0$, then for any $\tau \geqslant 0$, all roots of* (2.11) *have negative real parts. Furthermore, positive equilibrium E^* of* (1.4) *is absolutely stable for $\tau \geqslant 0$;*

(b) *If either $s < 0$ or $z_1^* > 0, G(z_1^*) \leqslant 0, r \geqslant 0$ and $\Lambda = p^2 - 3q > 0$ hold, then $G(z)$ has at least one positive root z_k, and when $\tau \in [0, \tau_0)$, all roots of* (2.11) *have negative real parts. So the positive equilibrium E^* of* (1.4) *is asymptotically stable for $\tau \in [0, \tau_0)$.*

(c) *If the conditions in (b) and $G'(z_k) \neq 0$, then Hopf bifurcation for* (1.4) *occurs at positive equilibrium E^* when $\tau = \tau_j^k$, which means that small amplified periodic solutions will bifurcate from E^*.*

3. Properties of the Hopf Bifurcation

In Section 2, we obtain the conditions which guarantee that system (1.4) undergoes the Hopf bifurcation at the positive equilibrium E^* when $\tau = \tau_j^k$. In this section, we will investigate the direction of the Hopf bifurcation when $\tau = \tau_0$ and the stability of the bifurcating periodic solutions from the equilibrium E^* by using the normal form and the center manifold theory developed by Hassard et al. [10].

Throughout this section, we assume that (b) and (c) of Theorem 2.9 are satisfied. Under the transformation $u_1(t) = x(\tau t) - x^*, u_2(t) = y_1(\tau t) - y_1^*, u_3(t) = y_2(\tau t) - y_2^*, \tau = \tau_0 + \mu$, the system (1.2) is transformed into an FDE in $C = C([-1, 0], R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{3.1}$$

where $u(t) = (u_1(t), u_2(t), u_3(t))^T \in R^3$ and

$$L_\mu(\varphi) = (\tau_0 + \mu)[B_1\varphi(0) + B_2\varphi(-1)], \tag{3.2}$$

where B_1 and B_2 are defined as

$$B_1 = \begin{pmatrix} n'(x^*) - f'(x^*)y_2^* & 0 & -f(x^*) \\ 0 & -(D+v_1) & 0 \\ 0 & D & -v_2 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ kf'(x^*)y_2^* & 0 & kf(x^*) \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.3}$$

$$f(\mu,\varphi) = (\tau_0+\mu)\begin{pmatrix} \frac{1}{2!}\left\{\left[n''(x^*) - f''(x^*)y_2^*\right]\varphi_1^2(0) - 2f'(x^*)\varphi_1(0)\varphi_3(0)\right\} \\ +\frac{1}{3!}\left\{\left[n'''(x^*) - f'''(x^*)y_2^*\right]\varphi_1^3(0) - 3f''(x^*)\varphi_1^2(0)\varphi_3(0)\right\} + O(4) \\ \frac{k}{2!}\left\{\left[n''(x^*) - f''(x^*)y_2^*\right]\varphi_1^2(-1) - 2f'(x^*)\varphi_1(-1)\varphi_3(-1)\right\} \\ +\frac{k}{3!}\left\{\left[n'''(x^*) - f'''(x^*)y_2^*\right]\varphi_1^3(-1) - 3f''(x^*)\varphi_1^2(-1)\varphi_3(-1)\right\} + O(4) \\ 0 \end{pmatrix}. \tag{3.4}$$

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta,\varphi)$ in $\theta \in [-1,0]$ such that

$$L_\mu\varphi = \int_{-1}^{0} d\eta(\theta,\mu)\varphi(\theta), \tag{3.5}$$

where $\varphi \in C$. In fact, we can choose

$$\eta(\theta,\mu) = \tau B_1\delta(\theta) - \tau B_2\delta(\theta+1), \tag{3.6}$$

where

$$\delta(\theta) = \begin{cases} 1, & \theta = 0, \\ 0, & \theta \neq 0. \end{cases} \tag{3.7}$$

For $\varphi \in C^1([-1,0],R^3)$, define

$$A(\mu)\varphi = \begin{cases} \dot{\varphi}(\theta), & \theta \in [-1,0), \\ \int_{-1}^{0} d\eta(s,\mu)\varphi(s), & \theta = 0, \end{cases} \tag{3.8}$$

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-1,0), \\ f(\varphi,\mu), & \theta = 0. \end{cases} \tag{3.9}$$

Leting $u = (u_1, u_2, u_3)^T$, then system (3.1) can be rewritten as

$$\dot{u}_t = A(\varphi)u_t + R(\varphi)u_t. \tag{3.10}$$

For $\psi \in C^1([0,1],(R^3)^*)$, define

$$A^*\alpha(s) = \begin{cases} -\dot{\alpha}(s), & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\alpha(-t), & s = 0 \end{cases} \tag{3.11}$$

and a bilinear form

$$\langle \psi, \varphi \rangle = \overline{\psi}(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \overline{\psi}(\xi - \theta)d\eta(\theta)\rho(\xi)d\xi, \tag{3.12}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then A^* and A are adjoint operators. In addition, from Section 2 we know that $\pm i\omega_0\tau_0$ are eigenvalues of $A(0)$. Thus they are also eigenvalues of A^*. By direct computation, we conclude that

$$q(\theta) = (1, \beta, \gamma)^T e^{i\omega_0\tau_0\theta} \tag{3.13}$$

is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_0$, and

$$q^*(s) = \overline{B}(1, \beta^*, \gamma^*)e^{i\omega_0\tau_0 s} \tag{3.14}$$

is the eigenvector A^* corresponding to $-i\omega_0\tau_0$. Moreover,

$$\langle q^*(s), q(\theta) \rangle = 1, \qquad \langle q^*(s), \overline{q}(\theta) \rangle = 0, \tag{3.15}$$

where

$$\begin{gathered} \beta = \frac{(i\omega_0\tau_0 + v_2)\gamma}{D}, \quad \gamma = \frac{n'(x^*) - f'(x^*)y_2^* - i\omega_0\tau_0}{f(x^*)}, \\ \beta^* = \frac{f'(x^*)y_2^* - n'(x^*) - i\omega_0\tau_0}{kf'(x^*)y_2^* e^{i\omega_0\tau_0}}, \quad \gamma^* = \frac{(D + v_1 - i\omega_0\tau_0)\beta^*}{D}, \quad B = \frac{1}{\Gamma}, \end{gathered} \tag{3.16}$$

where $\Gamma = 1 + \beta\overline{\beta^*} + \gamma\overline{\gamma^*} + k\tau_0\overline{\beta^*}e^{-i\omega_0\tau_0}[f'(x^*)y_2^* + \gamma f(x^*)]$.

Using the same notations as in Hassard et al. [10], let u_t be the solution of (3.1) when $\tau = \tau_0$. Defining $z(t) = \langle q^*, u_t \rangle, u_t = (x_t, y_t)$, then

$$\dot{z}(t) = \langle q^*, \dot{u}_t \rangle = i\omega_0 z(t) + \overline{q}^*(0)\widehat{f}(z, \overline{z}), \tag{3.17}$$

where

$$\hat{f} = f\left(\tau_0, W(z,\overline{z}) + 2\,\mathrm{Re}\{zq\}\right), \quad W(z,\overline{z}) = u_t - 2\,\mathrm{Re}\{zq\},$$
$$W(z,\overline{z}) = W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots. \tag{3.18}$$

Notice that W is real if u_t is real. We consider only real solutions. Rewrite (3.19) as

$$\dot{u}_t = i\omega_0\tau_0 z(t) + g(z,\overline{z}), \tag{3.19}$$

where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2}\cdots. \tag{3.20}$$

Substituting (3.10) and (3.17) into $\dot{W} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}q$, we have

$$\dot{W} = \begin{cases} AW - 2\mathrm{Re}\left\{\overline{q}^*(0)\hat{f}q(\theta)\right\}, & \theta \in [-\tau, 0) \\ AW - 2\mathrm{Re}\left\{\overline{q}^*(0)\hat{f}q(\theta)\right\} + \hat{f}, & \theta = 0, \end{cases} \overset{\text{def}}{=} AW + H(z,\overline{z},\theta), \tag{3.21}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots. \tag{3.22}$$

Expanding the above series and comparing the coefficients, we obtain

$$(A - 2i\omega_0\tau_0 I)W_{20}(\theta) = -H_{20}(\theta), \qquad AW_{11} = -H_{11}(\theta). \tag{3.23}$$

For $u_t = u(t+\theta) = W(z,\overline{z},\theta) + zq(\theta) + \overline{z}q(\overline{\theta})$, we have

$$(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots = \overline{q}^*(0)\hat{f}(z,\overline{z}). \tag{3.24}$$

Comparing the coefficients, we obtain

$$
\begin{aligned}
g_{20} &= 2\tau_0 B\left\{\frac{1}{2}\left[n''(x^*) - f''(x^*)y_2^*\right]\left(1 + k\overline{\beta^*}e^{-2i\omega_0\tau_0}\right) - f'(x^*)\gamma - k\overline{\beta^*}\gamma f'(x^*)e^{-2i\omega_0\tau_0}\right\},\\
g_{11} &= \tau_0 B\left(1 + k\overline{\beta^*}\right)\left[n''(x^*) - f''(x^*)y_2^* - f'(x^*)(\gamma + \overline{\gamma})\right],\\
g_{02} &= 2\tau_0 B\left\{\frac{1}{2}\left[n''(x^*) - f''(x^*)y_2^*\right]\left(1 + k\overline{\beta^*}e^{2i\omega_0\tau_0}\right) - f'(x^*)\gamma - k\overline{\beta^*}\gamma f'(x^*)e^{2i\omega_0\tau_0}\right\},\\
g_{21} &= 2\tau_0 B\left\{\frac{1}{2}\left[n'''(x^*) - f'''(x^*)y_2^*\right]\left[1 + k\overline{\beta^*}e^{-i\omega_0\tau_0}\right]\right.\\
&\quad + \frac{1}{2}\left[W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right]\left[n''(x^*) - f''(x^*)y_2^*\right] - \frac{1}{2}f''(x^*)(\overline{\gamma} + 2\gamma)\left(1 + k\overline{\beta^*}e^{-i\omega_0\tau_0}\right)\\
&\quad - f'(x^*)\left[W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0) + \frac{1}{2}\overline{\gamma}W_{20}^{(1)}(0) + \gamma W_{11}^{(1)}(1)\right]\\
&\quad + \frac{k}{2}\overline{\beta^*}\left[n''(x^*) - f''(x^*)y_2^*\right]\left[W_{20}^{(1)}(-1)e^{i\omega_0\tau_0} + 2W_{11}^{(1)}(-1)e^{-i\omega_0\tau_0}\right] - k\overline{\beta^*}f'(x^*)\\
&\quad \left.\times\left[W_{11}^{(3)}(-1)e^{-i\omega_0\tau_0} + \frac{1}{2}W_{20}^{(3)}(-1)e^{i\omega_0\tau_0} + \frac{1}{2}\overline{\gamma}W_{20}^{(1)}(-1)e^{i\omega_0\tau_0} + \gamma W_{11}^{(1)}(-1)e^{-i\omega_0\tau_0}\right]\right\}.
\end{aligned}
\tag{3.25}
$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. For $\theta \in [-1, 0)$, we have

$$H(z, \overline{z}, \theta) = -2\,\mathrm{Re}\left\{\overline{q}^*(0)\hat{f}q(\theta)\right\} = -\overline{q}^*\hat{f}q(\theta) - \overline{q}^*(0)\hat{f}q(\theta) = -gq(\theta) - \overline{gq}(\theta). \tag{3.26}$$

Comparing the coefficients with (3.22) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \qquad H_{11}(\theta) = -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \tag{3.27}$$

It follows from the definition of W that

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) - H_{20}(\theta) = 2i\omega_0\tau_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{20}\overline{q}(\theta). \tag{3.28}$$

Solving for $W_{20}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g}_{20}}{3\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_1 e^{2i\omega_0\tau_0\theta}, \tag{3.29}$$

and similarly

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega_0\tau_0}q(0)e^{i\omega_0\tau_0\theta} + \frac{i\overline{g}_{11}}{\omega_0\tau_0}\overline{q}(0)e^{-i\omega_0\tau_0\theta} + E_2, \tag{3.30}$$

where E_1 and E_2 are both 3-dimensional vectors and can be determined by setting $\theta = 0$ in H. Hence combining the definition of A, we can get

$$\begin{gathered}\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = AW_{20}(0) = 2i\omega_0\tau_0 W_{20}(0) - H_{20}(0),\\ \int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0).\end{gathered} \tag{3.31}$$

Noticing that $(i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta} d\eta(\theta))q(0) = 0$, $(-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta} d\eta(\theta))\overline{q}(0) = 0$, we have

$$\left(2i\omega_0\tau_0 I - \int_{-1}^{0} e^{2i\omega_0\tau_0\theta} d\eta(\theta)\right)E_1 = \widehat{f}_{z^2}. \tag{3.32}$$

Similarly, we have

$$\left(\int_{-1}^{0} d\eta(\theta)\right)E_2 = -\widehat{f}_{z\overline{z}}. \tag{3.33}$$

Hence, we get

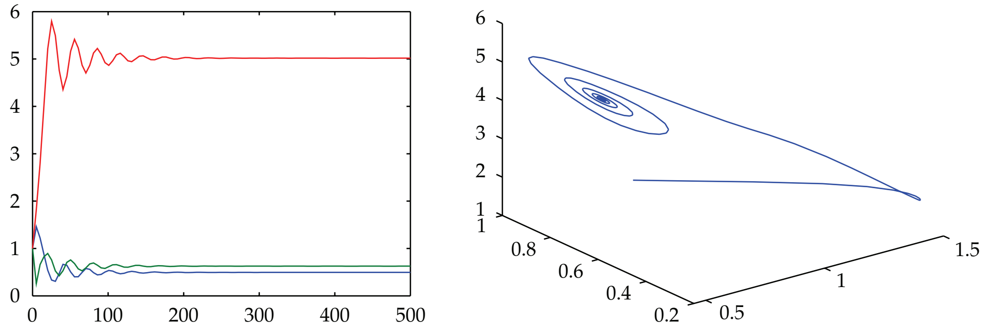
$$\begin{aligned}&\begin{pmatrix} 2i\omega_0 - n'(x^*) + f'(x^*)y_2^* & 0 & f(x^*) \\ -kf'(x^*)y_2^* e^{-2i\omega_0\tau_0} & 2i\omega_0 + D + v_1 & -kf(x^*)e^{-2i\omega_0\tau_0} \\ 0 & -D & 2i\omega_0 + v_2 \end{pmatrix} E_1\\ &\quad = \left[\frac{1}{2}(n''(x^*) - f''(x^*)y_2^*) - \gamma f'(x^*)\right]\begin{pmatrix} 1 \\ ke^{-2i\omega_0\tau_0} \\ 0 \end{pmatrix},\end{aligned} \tag{3.34}$$

$$\begin{pmatrix} n'(x^*) - f'(x^*)y_2^* & 0 & -f(x^*) \\ kf'(x^*)y_2^* & D + v_1 & kf(x^*) \\ 0 & D & -v_2 \end{pmatrix} E_2 = \left[(\gamma + \overline{\gamma})f'(x^*) - n''(x^*) + f''(x^*)y_2^*\right]\begin{pmatrix} 1 \\ k \\ 0 \end{pmatrix}. \tag{3.35}$$

Then g_{21} can be expressed by the parameters. Based on the above analysis, we can see that each g_{ij} can be determined by the parameters. Thus we can compute the following quantities:

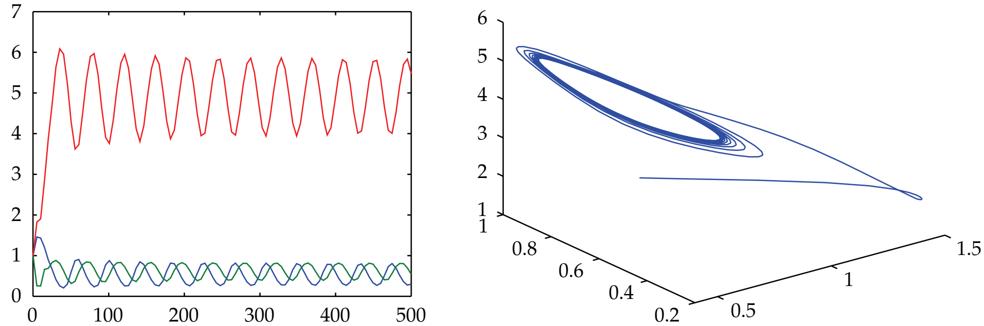
$$\begin{gathered}C_1(0) = \frac{i}{2\omega_0\tau_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2}, \quad \beta_2 = 2\operatorname{Re}\{C_1(0)\},\\ T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\lambda'(\tau_0)}{\omega_0}, \quad \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\lambda'(\tau_0)}.\end{gathered} \tag{3.36}$$

Hence, we have following theorem.



(a) (b)

Figure 1: When $\tau = 5$, the positive equilibrium E^* of system (4.1) is asymptotically stable.



(a) (b)

Figure 2: When $\tau = 10$, the positive equilibrium E^* of system (4.1) is unstable, and small amplified periodic solutions exist.

Theorem 3.1. *μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(< 0)$, the Hopf bifurcation is supercritical (subcritical); β_2 determines the stability of the bifurcation periodic solutions: the bifurcation periodic solutions are orbitally stable (unstable) if $\beta_2 < 0(> 0)$; T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(< 0)$.*

4. Numerical Examples

In this section, we give a numerical example:

$$\begin{aligned}
\dot{x}(t) &= x(t)\left[2 - x(t) - 0.3y_2(t)\right], \\
\dot{y}_1(t) &= 0.24x(t-\tau)y_2(t-\tau) - 0.95y_1(t), \\
\dot{y}_2(t) &= 0.8y_1(t) - 0.1y_2(t).
\end{aligned} \tag{4.1}$$

Then we can conclude that the system (4.1) has a unique positive equilibrium $E^*(0.4948, 0.6272, 5.0174)$. When $\tau = 5$, the dynamics behaviors of system (4.1) areshown in

Figure 1. From Section 2, we can obtain $\omega_0 = 0.4867$, $\tau_0^0 = 7.9367$, $\tau_j^0 = 7.9367 + 2j\pi/\omega_0 (j = 1, 2, 3, \ldots)$. From the formulas in Section 3, when $\tau_0 = 10$, it follows that $G'(\omega_0^2) = 0.9401 > 0$, $\Omega = 0.0027$, $\mathrm{Re}\, C(0) = -34.9505 < 0$, $\mu_2 = 0.4220 > 0$, and $\beta_2 = -69.9010 < 0$. Therefore, the Hopf bifurcation is supercritical, and the bifurcating periodic solution is orbitally asymptotically stable. The plots are shown in Figure 2.

Acknowledgments

The authors wish to thank the reviewers for their valuable comments and suggestions that led to a truly significant improvement of the paper. This research is supported by the Natural Science Research Project from Institutions of Higher Education of Hebei Province (no. Z 2010105).

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