Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 397150, 21 pages
doi:10.1155/2010/397150

Research Article

Existence Theorems for Generalized Distance on Complete Metric Spaces

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Received 20 September 2009; Revised 7 May 2010; Accepted 20 May 2010

Academic Editor: L. Górniewicz

We first introduce the new concept of a distance called u-distance, which generalizes w-distance, Tataru's distance, and τ-distance. Then we prove a new minimization theorem and a new fixed point theorem by using a u-distance on a complete metric space. Our results extend and unify many known results due to Caristi, Ćirić, Ekeland, Kada-Suzuki-Takahashi, Kannan, Ume, and others.

1. Introduction

The Banach contraction principle [1], Ekeland's ε-variational principle [2], and Caristi's fixed point theorem [3] are very useful tools in nonlinear analysis, control theory, economic theory, and global analysis. These theorems are extended by several authors in different directions.

Takahashi [4] proved the following minimization theorem. Let X be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x\in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u,v) \leq f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$. Some authors [5–7] have generalized and extended this minimization theorem in complete metric spaces.

In 1996, Kada et al. [5] introduced the concept of w-distance on a metric space as follows. Let X be a metric space with metric d. Then a function $p : X \times X \to [0,\infty)$ is called a w-distance on X if the followings are satisfied.

(1) $p(x,z) \leq p(x,y) + p(y,z)$ for any $x, y, z \in X$.

(2) For any $x \in X, p(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous.

(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

They gave some examples of w-distance and improved Caristi's fixed point theorem [3], Ekeland's variational principle [2], and Takahashi's nonconvex minimization theorem [4]. The fixed point theorems with respect to a w-distance were proved in [8–12].

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers, by $\mathbb{R}$ the set of all real numbers, and by $\mathbb{R}_+$ the set of all nonnegative real numbers.

Recently, Suzuki [6] introduced the concept of τ-distance on a metric space, which generalizes Tataru's distance [13] as follows. Let X be a metric space with metric d.

Then a function p from $X \times X$ into $\mathbb{R}_+$ is called τ-distance on X if there exists a function η from $X \times \mathbb{R}_+$ into $\mathbb{R}_+$ and the followings are satisfied:

(τ1) $p(x,z) \le p(x,y) + p(y,z)$ for all $x, y, z \in X$;

(τ2) $\eta(x,0) = 0$ and $\eta(x,t) \ge t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its second variable;

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w,x) \le \lim_n \inf_n p(w, x_n)$ for all $w \in X$;

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

In this paper, we first introduce the new concept of a distance called u-distance, which generalizes w-distance, Tataru's distance, and τ-distance. Then we prove a new minimization theorem and a new fixed point theorem by using u-distance on a complete metric space. Our results extend and unify many known results due to Caristi [3], Ćirić [14], Ekeland [2], Takahashi [4], Kada et al. [5], Kannan [15], Suzuki [6], and Ume [7, 12] and others.

2. Preliminaries

Definition 2.1. Let X be metric space with metric d. Then a function p from $X \times X$ into $\mathbb{R}_+$ is called u-distance on X if there exists a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ such that

(u1) $p(x,z) \le p(x,y) + p(y,z)$ for all $x, y, z \in X$;

(u2) $\theta(x,y,0,0) = 0$ and $\theta(x,y,s,t) \ge \min\{s,t\}$ for all $x, y \in X$ and $s, t \in \mathbb{R}_+$, and for any $x \in X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in \mathbb{R}_+$ and $y \in X$ imply

$$\left|\theta(x,y,s,t) - \theta(x,y,s_0,t_0)\right| < \varepsilon; \tag{2.1}$$

(u3)

$$\lim_{n\to\infty} x_n = x,$$
$$\lim_{n\to\infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \ge n\} = 0 \tag{2.2}$$

imply

$$p(y,x) \le \lim_{n\to\infty} \inf p(y, x_n) \tag{2.3}$$

for all $y \in X$;

(u4)

$$\lim_{n\to\infty} \sup\{p(x_n, w_m) : m \geq n\} = 0,$$
$$\lim_{n\to\infty} \sup\{p(y_n, z_m) : m \geq n\} = 0,$$
$$\lim_{n\to\infty} \theta(x_n, w_n, s_n, t_n) = 0,$$
$$\lim_{n\to\infty} \theta(y_n, z_n, s_n, t_n) = 0 \quad (2.4)$$

imply

$$\lim_{n\to\infty} \theta(w_n, z_n, s_n, t_n) = 0 \quad (2.5)$$

or

$$\lim_{n\to\infty} \sup\{p(w_m, x_n) : m \geq n\} = 0,$$
$$\lim_{n\to\infty} \sup\{p(z_m, y_n) : m \geq n\} = 0,$$
$$\lim_{n\to\infty} \theta(x_n, w_n, s_n, t_n) = 0,$$
$$\lim_{n\to\infty} \theta(y_n, z_n, s_n, t_n) = 0 \quad (2.6)$$

imply

$$\lim_{n\to\infty} \theta(w_n, z_n, s_n, t_n) = 0; \quad (2.7)$$

(u5)

$$\lim_{n\to\infty} \theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,$$
$$\lim_{n\to\infty} \theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0 \quad (2.8)$$

imply

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \quad (2.9)$$

or

$$\lim_{n\to\infty} \theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0,$$
$$\lim_{n\to\infty} \theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) = 0 \quad (2.10)$$

imply

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \quad (2.11)$$

Remark 2.2. Suppose that $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping satisfying (u2)~(u5). Then there exists a mapping η from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ such that η is nondecreasing in its third and fourth variable, respectively, satisfying (u2)η ~(u5)η, where (u2)η ~(u5)η stand for substituting η for θ in (u2)~(u5), respectively.

Proof. Suppose that $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a mapping satisfying (u2)~(u5). Define a function $\eta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\eta(x, y, s, t) = s + t + \sup\{\theta(x, y, \alpha, \beta) : 0 \le \alpha \le s, 0 \le \beta \le t\} \quad \text{for all } x, y \in X \text{ and } s, t \in \mathbb{R}_+. \tag{2.12}$$

By (2.12), we have $\eta(x, y, 0, 0) = 0$ and $\eta(x, y, s, t) \ge \min\{s, t\}$ for all $x, y \in X$ and $s, t \in \mathbb{R}_+$. Also it follows from (2.12) that η is nondecreasing in its third and fourth variable, respectively.

We shall prove the following:

$$\begin{aligned}&\text{for any } x \in X \text{ and for every } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that}\\ &|s - s'| < \delta,\ |t - t'| < \delta,\ s, s', t, t' \in \mathbb{R}_+ \text{ and } y \in X \text{ imply}\\ &|\eta(x, y, s, t) - \eta(x, y, s', t')| < \varepsilon.\end{aligned} \tag{2.13}$$

Suppose that (2.13) does not hold. Then

$$\begin{aligned}&\text{there exists } x' \in X,\ \varepsilon' > 0,\ \text{sequences } \{s_n\},\ \{s'_n\},\ \{t_n\}, \text{and } \{t'_n\}\\ &\text{of } \mathbb{R}_+, \text{and sequence } \{y_n\} \text{ of } X \text{ such that } |s_n - s'_n| < \frac{1}{n},\\ &|t_n - t'_n| < \frac{1}{n}, \text{and } |\eta(x', y_n, s_n, t_n) - \eta(x', y_n, s'_n, t'_n)|\\ &\ge \varepsilon' \quad \text{for all } n \in \mathbb{N}.\end{aligned} \tag{2.14}$$

By virtue of (2.12) and (2.14), we have

$$\begin{aligned}0 < \varepsilon' &\le |\eta(x', y_n, s_n, t_n) - \eta(x', y_n, s'_n, t'_n)|\\ &= \big|\{(s_n + t_n) + \sup[\theta(x', y_n, \alpha, \beta) \mid 0 \le \alpha \le s_n, 0 \le \beta \le t_n]\}\\ &\quad - \{(s'_n + t'_n) + \sup[\theta(x', y_n, \alpha, \beta) \mid 0 \le \alpha \le s'_n, 0 \le \beta \le t'_n]\}\big|\\ &\le |s_n - s'_n| + |t_n - t'_n|\\ &\quad + \big|\sup[\theta(x', y_n, \alpha, \beta) \mid 0 \le \alpha \le s_n, 0 \le \beta \le t_n]\\ &\quad\quad - \sup[\theta(x', y_n, \alpha, \beta) \mid 0 \le \alpha \le s'_n, 0 \le \beta \le t'_n]\big|\\ &< \frac{2}{n} + \sup\left[\theta(x', y_n, \alpha, \beta) \mid 0 \le \alpha \le s_n + \frac{1}{n}, 0 \le \beta \le t_n + \frac{1}{n}\right]\\ &\quad - \sup\left[\theta(x', y_n, \alpha, \beta) \mid 0 \le \alpha \le s_n - \frac{1}{n}, 0 \le \beta \le t_n - \frac{1}{n}\right].\end{aligned} \tag{2.15}$$

Combining (u2) and (2.14), we have the following:

$$\begin{aligned}&\text{for some } x' \in X \text{ and for every } \varepsilon > 0, \text{ there exists } \delta > 0, \text{ such that}\\ &|s - s'| < \delta,\ |t - t'| < \delta,\ s, s', t, t' \in \mathbb{R}_+ \text{ and } y \in X \text{ imply}\\ &|\theta(x', y, s, t) - \theta(x', y, s', t')| < \frac{\varepsilon}{4}.\end{aligned} \tag{2.16}$$

Due to (2.16), we get that

$$\text{for this } \delta > 0, \text{ there exists } M \in \mathbb{N} \text{ such that } n \geq M \text{ implies } \frac{2}{n} < \delta. \tag{2.17}$$

From (2.16) and (2.17), we obtain the following.

$$\begin{aligned}&\text{for every } \varepsilon > 0, \text{ there exists } M \in \mathbb{N} \text{ such that } n \geq M \text{ implies}\\ &s_n - \frac{\delta}{2} < s_n - \frac{1}{n} < s_n < s_n + \frac{1}{n} < s_n + \frac{\delta}{2},\\ &t_n - \frac{\delta}{2} < t_n - \frac{1}{n} < t_n < t_n + \frac{1}{n} < t_n + \frac{\delta}{2}.\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\text{For each } n \in \mathbb{N}, \text{ let}\\ &l_{1,n} = \sup\left[\theta(x', y_n, \alpha, \beta) \mid 0 \leq \alpha \leq s_n - \frac{1}{n}, 0 \leq \beta \leq t_n - \frac{1}{n}\right].\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\text{For each } n \in \mathbb{N}, \text{ let}\\ &l_{2,n} = \sup\left[\theta(x', y_n, \alpha, \beta) \mid 0 \leq \alpha \leq s_n + \frac{1}{n}, 0 \leq \beta \leq t_n + \frac{1}{n}\right].\end{aligned} \tag{2.20}$$

In terms of (2.19) and (2.20), we deduce that

$$l_{1,n} \leq l_{2,n} \quad \forall n \in \mathbb{N}. \tag{2.21}$$

In view of (2.21), we get that

$$\liminf_{n \to \infty} l_{1,n} \leq \liminf_{n \to \infty} l_{2,n}. \tag{2.22}$$

On account of (2.20), we know the following:

$$\begin{aligned}&\text{for each } n \in \mathbb{N} \text{ and for every } \varepsilon > 0, \text{ there exists}\\ &\alpha_n \in \left[0, s_n + \frac{1}{n}\right] \text{ and } \beta_n \in \left[0, t_n + \frac{1}{n}\right] \text{ such that}\\ &l_{2,n} - \varepsilon < \theta(x', y_n, \alpha_n, \beta_n).\end{aligned} \tag{2.23}$$

Using (2.16), (2.18), (2.19), and (2.23), we have the following:

$$\begin{gathered}\text{for every } \varepsilon > 0, \text{ there exists } M \in \mathbb{N} \text{ such that} \\ l_{2,n} - \varepsilon < l_{1,n} + \frac{\varepsilon}{2}, \quad \text{for all } n \in \mathbb{N} \text{ with } M \leq n.\end{gathered} \tag{2.24}$$

By (2.24), we have

$$\lim_{n\to\infty} \inf l_{2,n} \leq \lim_{n\to\infty} \inf l_{1,n}. \tag{2.25}$$

By virtue of (2.15), (2.19), (2.20), (2.22), and (2.25), we have $0 < \varepsilon' \leq 0$ which is a contradiction. Hence $(\mathrm{u2})_\eta$ holds. From (2.12) and (u2)~(u5), it follows that $(\mathrm{u3})_\eta$~$(\mathrm{u5})_\eta$ are satisfied. □

Remark 2.3. From Remark 2.2, we may assume that θ is nondecreasing in its third and fourth variables, respectively, for a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (u2)~(u5).

We give some examples of u-distance.

Example 2.4. Let $X = [0,\infty)$ be the set of real numbers with the usual metric and let $p : X \times X \to \mathbb{R}_+$ be defined by $p(x,y) = (1/4)x^2$. Then p is a u-distance on X but not a τ-distance on X.

Proof. Define $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\theta(x,y,s,t) = s$ for all $x,y \in X$ and $s,t \in \mathbb{R}_+$. Then p and θ satisfy (u1)~(u5). But for an arbitrary function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and for all sequences $\{z_n\}$, $\{x_n\}$, and $\{y_n\}$ of X such that

$$\begin{gathered} 0 = \lim_{n\to\infty} \eta(z_n, p(z_n, x_n)) = \lim_{n\to\infty} \eta\left(z_n, \frac{1}{4}(z_n)^2\right), \\ 0 = \lim_{n\to\infty} \eta(z_n, p(z_n, y_n)) = \lim_{n\to\infty} \eta\left(z_n, \frac{1}{4}(z_n)^2\right), \end{gathered} \tag{2.26}$$

since the limit of the sequence $\{\eta(z_n, p(z_n, x_n))\}_{n=1}^{\infty}$ and the limit of the sequence $\{\eta(z_n, p(z_n, y_n)\}_{n=1}^{\infty}$ do not depend on $\{x_n\}$ and $\{y_n\}$, the limit of the sequence $\{d(x_n, y_n))\}_{n=1}^{\infty}$ may not be 0. This does not satisfy (τ5). Hence p is not a τ-distance on X. Therefore p is a u-distance on X but not a τ-distance on X. □

Example 2.5. Let p be a τ-distance on a metric space (X,d). Then p is also a u-distance on X.

Proof. Since p is a τ-distance, there exists a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (τ1)~(τ5). Define $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\theta(x,y,s,t) = \left[\frac{2 + \eta(x, p(x,y))}{1 + \eta(x, p(x,y))}\right] \cdot s \quad \forall x,y \in X,\ s,t \in \mathbb{R}_+. \tag{2.27}$$

Then it is easy to see that p and θ satisfy (u2)~(u5). Thus p is a u-distance on X. □

Example 2.6. Let X be a normed space with norm $\|\cdot\|$. Then a function $p : X \times X \to \mathbb{R}_+$ defined by $p(x,y) = \|x\|$ for every $x, y \in X$ is a u-distance on X but not a τ-distance.

Proof. Let $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be as in the proof of Example 2.4. Then it is clear that p satisfies (u1) and θ satisfies (u2)~(u5) on X but p does not satisfy (τ5). Thus p is a u-distance on X but not a τ-distance. □

Example 2.7. Let X be a normed space with norm $\|\cdot\|$. Then a function $p : X \times X \to \mathbb{R}_+$ defined by $p(x,y) = \|y\|$ for every $x, y \in X$ is a u-distance on X.

Proof. Define $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\theta(x,y,s,t) = s + t$ for all $x, y \in X$ and $s, t \in \mathbb{R}_+$. Then p satisfies (u1) and θ satisfies (u2)~(u5). Thus p is a u-distance on X. □

Example 2.8. Let p be a u-distance on a metric space (X, d) and let c be a positive real number. Then a function q from $X \times X$ into $\mathbb{R}_+$ defined by $q(x,y) = c \cdot p(x,y)$ for every $x, y \in X$ is also a u-distance on X.

Proof. Since p is a u-distance on X, there exists a function $\eta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $(\mathrm{u2})_\eta$ ~$(\mathrm{u5})_\eta$ and p satisfies (u1). Define $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\theta(x,y,s,t) = c \cdot \eta(x,y,s,t)$ for all $x, y \in X$ and $s, t \in \mathbb{R}_+$. Then it is clear that q satisfies (u1) and θ satisfies (u2)~(u5). Thus q is a u-distance on X. □

The following examples can be easily obtained from Remark 2.3.

Example 2.9. Let X be a metric space with metric d and let p be a u-distance on X such that p is a lower semicontinuous in its first variable. Then a function $q : X \times X \to \mathbb{R}_+$ defined by $q(x,y) = \max\{p(x,y), p(y,x)\}$ for all $x, y \in X$ is a u-distance on X.

Example 2.10. Let X be a metric space with metric d. Let p be a u-distance on X and let α be a function from X into $\mathbb{R}_+$. Then a function $q : X \times X \to \mathbb{R}_+$ defined by

$$q(x,y) = \max\{\alpha(x), p(x,y)\}, \quad \text{for every } x, y \in X \tag{2.28}$$

is a u-distance on X.

Remark 2.11. It follows from Example 2.4 to Example 2.10 that u-distance is a proper extension of τ-distance.

Definition 2.12. Let X be a metric space with a metric d and let p be a u-distance on X. Then a sequence $\{x_n\}$ of X is called p-Cauchy if there exists a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (u2)~(u5) and a sequence $\{z_n\}$ of X such that

$$\lim_{n \to \infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \geq n\} = 0, \tag{2.29}$$

or

$$\lim_{n \to \infty} \sup\{\theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \geq n\} = 0. \tag{2.30}$$

The following lemmas play an important role in proving our theorems.

Lemma 2.13. *Let X be a metric space with a metric d and let p be a u-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence.*

Proof. By assumption, there exists a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (u2)~(u5) and a sequence $\{z_n\}$ of X such that

$$\lim_{n\to\infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \geq n\} = 0, \tag{2.31}$$

or

$$\lim_{n\to\infty} \sup\{\theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \geq n\} = 0. \tag{2.32}$$

Then from (u5), we have $\lim_{n\to\infty} \sup\{d(x_i, x_j) : j > i \geq n\} = 0$. This means that $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.14. *Let X be a metric space with a metric d and let p be a u-distance on X.*

(1) *If sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $\lim_{n\to\infty} p(z, x_n) = 0$ and $\lim_{n\to\infty} p(z, y_n) = 0$ for some $z \in X$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

(2) *If $p(z, x) = 0$ and $p(z, y) = 0$, then $x = y$.*

(3) *Suppose that sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $\lim_{n\to\infty} p(x_n, z) = 0$ and $\lim_{n\to\infty} p(y_n, z) = 0$ for some $z \in X$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

(4) *If $p(x, z) = 0$ and $p(y, z) = 0$, then $x = y$.*

Proof. (1) Let θ be a function from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$ satisfying (u2)~(u5). From Remark 2.3 and hypotheses,

$$\begin{gathered} \lim_{n\to\infty} \theta(z, z, p(z, x_n), p(z, x_n)) = 0, \\ \lim_{n\to\infty} \theta(z, z, p(z, y_n), p(z, y_n)) = 0. \end{gathered} \tag{2.33}$$

By (u5), $\lim_{n\to\infty} d(x_n, y_n) = 0$.

(2) In (1), putting $x_n = x$ and $y_n = y$ for all $n \in \mathbb{N}$, (2) holds.

By method similar to (1) and (2), results of (3) and (4) follow. □

Lemma 2.15. *Let X be a metric space with a metric d and let p be a u-distance on X. Suppose that a sequence $\{x_n\}$ of X satisfies*

$$\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0 \tag{2.34}$$

or

$$\lim_{n\to\infty} \sup\{p(x_m, x_n) : m > n\} = 0. \tag{2.35}$$

Then $\{x_n\}$ is a p-Cauchy sequence and $\{x_n\}$ is a Cauchy sequence.

Proof. Since p is a u-distance on X, there exists a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (u2)~(u5). Suppose $\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0$. Let $\alpha_n = \sup\{p(x_i, x_j) : j > i \geq n\}$. Then we have $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_{f(n)}\}$ be an arbitrary subsequence of $\{x_n\}$. By assumption and (u2), there exists a subsequence $\{x_{f(g(n))}\}$ of $\{x_{f(n)}\}$ such that

$$\lim_{n\to\infty} \theta\big(x_{f(g(n))}, x_{f(g(n+1))}, \alpha_{f(g(n+1))}, \alpha_{f(g(n+1))}\big) = 0,$$
$$\lim_{n\to\infty} \sup\left\{\sup_{m\geq n} p\big(x_{f(g(n))}, x_{f(g(m+1))}\big)\right\} \leq \lim_{n\to\infty} \alpha_{f(g(n))} = 0. \tag{2.36}$$

From (u4), we obtain

$$\begin{aligned}\lim_{n\to\infty} &\theta\big(x_{f(g(n))}, x_{f(g(n))}, \alpha_{f(g(n))}, \alpha_{f(g(n))}\big)\\ &= \lim_{n\to\infty} \theta\big(x_{f(g(n+1))}, x_{f(g(n+1))}, \alpha_{f(g(n+1))}, \alpha_{f(g(n+1))}\big) = 0.\end{aligned} \tag{2.37}$$

Since $\{x_{f(n)}\}$ is an arbitrary sequence of $\{x_n\}$, $\{x_{f(g(n))}\}$ is also an arbitrary sequence of $\{x_n\}$. Hence

$$\lim_{n\to\infty} \theta(x_n, x_n, \alpha_n, \alpha_n) = 0. \tag{2.38}$$

Therefore we get

$$\begin{aligned}\lim_{n\to\infty} \sup_{m\geq n} &\theta\big(x_{n-1}, x_{n-1}, p(x_{n-1}, x_m), p(x_{n-1}, x_m)\big)\\ &\leq \lim_{n\to\infty} \theta(x_{n-1}, x_{n-1}, \alpha_{n-1}, \alpha_{n-1}) = 0.\end{aligned} \tag{2.39}$$

This implies that $\{x_n\}$ is a p-Cauchy sequence. By Lemma 2.13, $\{x_n\}$ is a Cauchy sequence. Similarly, if $\lim_{n\to\infty} \sup\{p(x_m, x_n) : m > n\} = 0$, we can prove that $\{x_n\}$ is also a Cauchy sequence. □

3. Minimization Theorems and Fixed Point Theorems

The following theorem is a generalization of Takahashi's minimization theorem [4].

Theorem 3.1. *Let X be a metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper function which is bounded from below, and let $L : X \times X \times X \times X \to \mathbb{R}_+$ be a function such that,* *one has the following.*

(i) $L(x,y,y,x) \le L(x,z,z,x) + L(z,y,y,z)$ *for all* $x,y,z \in X$.

(ii) *For any sequence* $\{v_n\}_{n=1}^{\infty}$ *in* X *satisfying*

$$\lim_{n\to\infty} \sup\{L(v_n,v_m,v_m,v_n) : m > n\} = 0, \tag{3.1}$$

there exists $x_0 \in X$ *such that* $\lim_{n\to\infty} v_n = x_0$,

$$\begin{aligned} f(x_0) &\le \lim_{n\to\infty} \sup f(v_n), \\ L(v_n,x_0,x_0,v_n) &\le \lim_{m\to\infty} \inf L(v_n,v_m,v_m,v_n). \end{aligned} \tag{3.2}$$

(iii) $L(x,y,y,x) = L(x,z,z,x) = 0$ *imply* $y = z$.

(iv) *For every* $x \in X$ *with* $\inf_{v\in X} f(v) < f(x)$, *there exists* $y \in X - \{x\}$ *such that*

$$h(x,y) \le f(x) - f(y), \tag{3.3}$$

where a function $h : X \times X \to \mathbb{R}_+$ *is defined by*

$$h(v,w) = L(v,w,w,v) \tag{3.4}$$

for all $v,w \in X$. *Then, there exists* $x_0 \in X$ *such that*

$$f(x_0) = \inf_{v\in X} f(v). \tag{3.5}$$

Proof. Suppose $\inf_{v\in X} f(v) < f(x)$ for all $x \in X$. For each $x \in X$, let

$$S(x) = \{v \in X \mid h(x,v) \le f(x) - f(v)\}. \tag{3.6}$$

Then, by condition (iv) and (3.6), $S(x)$ is nonempty for each $x \in X$. From condition (i) and (3.6), we obtain

$$S(v) \subseteq S(x), \quad \text{for each } v \in S(x). \tag{3.7}$$

For each $x \in X$, let

$$c(x) = \inf\{f(v) \mid v \in S(x)\}. \tag{3.8}$$

Choose $x \in X$ with $f(x) < \infty$. Then, from (3.7) and (3.8), there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that

$$x_1 = x, \quad x_{n+1} \in S(x_n), \quad S(x_n) \subseteq S(x),$$
$$f(x_{n+1}) < c(x_n) + \frac{1}{n} \tag{3.9}$$

for all $n \in \mathbb{N}$.

From (3.6), (3.8) and (3.9), we have

$$h(x_n, x_{n+1}) \leq f(x_n) - f(x_{n+1}), \tag{3.10}$$

$$f(x_{n+1}) - \frac{1}{n} < c(x_n) \leq f(x_{n+1}). \tag{3.11}$$

By (3.10), $\{f(x_n)\}_{n=1}^{\infty}$ is a nonincreasing sequence of real numbers and so it converges. Therefore, from (3.11) there is some $\beta \in \mathbb{R}$ such that

$$\beta = \lim_{n \to \infty} c(x_n) = \lim_{n \to \infty} f(x_n). \tag{3.12}$$

From condition (i) and (3.10), we get

$$h(x_n, x_m) \leq f(x_n) - f(x_m) \tag{3.13}$$

for all $m > n$. From (3.12) and (3.13), we have

$$\lim_{n \to \infty} \sup\{L(x_n, x_m, x_m, x_n) : m > n\} = 0. \tag{3.14}$$

Thus, by condition (ii), (3.12), and (3.13), there exists $x_0 \in X$ such that

$$\lim_{n \to \infty} x_n = x_0, \tag{3.15}$$

$$f(x_0) \leq \lim_{n \to \infty} f(x_n) = \beta, \tag{3.16}$$

$$h(x_n, x_0) \leq \lim_{m \to \infty} \inf h(x_n, x_m). \tag{3.17}$$

From (3.13), (3.16), and (3.17), we have

$$\begin{aligned} f(x_0) \le \beta &= \lim_{m\to\infty}\sup f(x_m) \\ &\le \lim_{m\to\infty}\sup\{f(x_n) - h(x_n, x_m)\} \\ &= f(x_n) + \lim_{m\to\infty}\sup\{-h(x_n, x_m)\} \\ &= f(x_n) - \lim_{m\to\infty}\inf h(x_n, x_m) \\ &\le f(x_n) - h(x_n, x_0). \end{aligned} \tag{3.18}$$

From (3.6), (3.8), and (3.18), it follows that

$$x_0 \in S(x_n) \text{ and hence } c(x_n) \le f(x_0), \quad \forall n \in \mathbb{N}. \tag{3.19}$$

Taking the limit in inequality (3.19) when n tends to infinity, we have

$$\lim_{n\to\infty} c(x_n) \le f(x_0). \tag{3.20}$$

From (3.12), (3.16), and (3.20), we have

$$\beta = f(x_0). \tag{3.21}$$

On the other hand, by condition (iv) and (3.6), we have the following property:

$$\text{there exists } v_1 \in X - \{x_0\}, \quad \text{satisfying } v_1 \in S(x_0). \tag{3.22}$$

From (3.7), (3.8), (3.19), and (3.22), we have

$$\begin{gathered} v_1 \in S(x_n), \quad \forall n \in \mathbb{N}, \\ c(x_n) \le f(v_1). \end{gathered} \tag{3.23}$$

From (3.6), (3.12), (3.21), (3.22), (3.23), it follows that

$$\beta = f(v_1). \tag{3.24}$$

From (3.21), (3.22), and (3.24), we have

$$L(x_0, v_1, v_1, x_0) = 0. \tag{3.25}$$

By method similar to (3.22)~(3.25),

$$\text{there exists } v_2 \in X - \{v_1\}, \quad \text{such that } L(v_1, v_2, v_2, v_1) = 0. \tag{3.26}$$

From (3.25), (3.26), and condition (i), we obtain

$$L(x_0, v_2, v_2, x_0) = 0. \tag{3.27}$$

From (3.25), (3.27), and condition (iii), we obtain

$$v_1 = v_2. \tag{3.28}$$

This is a contradiction from (3.26). □

Corollary 3.2. *Let X be a complete metric space with metric d, and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below. Assume that there exists a u-distance p on X such that for each $u \in X$ with $f(u) > \inf\{f(x) \mid x \in X\}$, there exists $v \in X$ with $v \neq u$ and $f(v) + p(u,v) \leq f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf\{f(x) \mid x \in X\}$.*

Proof. Let $L : X \times X \times X \times X \to \mathbb{R}_+$ be a mapping such that

$$L(x, y, v, w) = \max\{p(x,v), p(w,y)\} \tag{3.29}$$

for all $x, y, v, w, \in X$. It follows easily from Definition 2.12, Lemmas 2.13, 2.14, and 2.15, and (u3) that conditions of Corollary 3.2 satisfy all conditions of Theorem 3.1. Thus, we obtain result of Corollary 3.2. □

Remark 3.3. Corollary 3.2 is a generalization of Kadaet al. [5, Theorem 1] and Suzuki [6, Theorem 5].

From Lemmas 2.13, 2.14, and 2.15, we have the following fixed point theorem.

Theorem 3.4. *Let X be a complete metric space with metric d, let p be a u-distance on X, and let T be a selfmapping of X. Suppose that there exists $r \in [0,1)$ such that*

$$\begin{aligned} p(Tx, Ty) \leq r \cdot \max\{&p(x,y), p(x,Tx), p(y,Ty), p(x,Ty), p(y,Tx), \\ &p(y,x), p(Tx,x), p(Ty,y), p(Ty,x), p(Tx,y)\} \end{aligned} \tag{3.30}$$

for all $x, y \in X$ and

$$\inf\{p(x,y) + p(x,Tx) : x \in X\} > 0 \tag{3.31}$$

for every $y \in X$ with $y \neq Ty$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$. Moreover, if $v = Tv$, then $x_0 = v$, $p(v,v) = 0$.

Proof. By method similar to [12, Lemma 2.4], for every $x \in X$,

$$\alpha(x) := \sup\left\{p\left(T^i x, T^j x\right) \mid i, j \in \mathbb{N} \cup \{0\}\right\} < \infty. \tag{3.32}$$

Define $q : X \times X \to \mathbb{R}_+$ by

$$q(x,y) = \max\{\alpha(x), p(x,y)\} \tag{3.33}$$

for every $x, y \in X$. By Example 2.10, q is a u-distance on X. Then we get

$$\begin{aligned} q\left(Tx, T^2x\right) &= \max\left\{\alpha(Tx), p\left(Tx, T^2x\right)\right\} \\ &= \alpha(Tx) \le r \cdot \alpha(x) = r \cdot q(x, Tx), \\ q\left(T^2x, Tx\right) &= \max\left\{\alpha\left(T^2x\right), p\left(T^2x, Tx\right)\right\} \\ &\le \alpha(Tx) \le r \cdot \alpha(x) = r \cdot q(x, Tx), \\ q(Tx, Tx) &= \max\{\alpha(Tx), p(Tx, Tx)\} \\ &= \alpha(Tx) \le r \cdot \alpha(x) = r \cdot q(x, x) \end{aligned} \tag{3.34}$$

for all $x \in X$. Thus we have

$$\begin{aligned} q(T^n x, T^m x) &\le \sum_{k=n}^{m-1} q\left(T^k x, T^{k+1} x\right) \\ &\le \sum_{k=n}^{m-1} r^k \cdot q(x, Tx) \le \frac{r^n}{1-r} q(x, Tx) \end{aligned} \tag{3.35}$$

for all $m > n$. Now we have

$$\lim_{n\to\infty} \sup\{q(T^n x, T^m x) : m > n\} \le \lim_{n\to\infty} \frac{r^n}{1-r} q(x, Tx) = 0. \tag{3.36}$$

Thus

$$\lim_{n\to\infty} \sup\{q(T^n x, T^m x) : m > n\} = 0. \tag{3.37}$$

By Lemma 2.15, $\{T^n x\}$ is a q-Cauchy and hence $\{T^n x\}$ is a Cauchy from Lemma 2.13. Since X is complete and $\{T^n x\}$ is a q-Cauchy, there exists $x_0 \in X$ such that

$$\begin{aligned} &\lim_{n\to\infty} T^n x = x_0, \\ q(T^n x, x_0) &\le \lim_{m\to\infty} \inf q(T^n x, T^m x) \le \frac{r^n}{1-r} q(x, Tx). \end{aligned} \tag{3.38}$$

Suppose $x_0 \neq Tx_0$. Then, by hypothesis, we have

$$\begin{aligned}
0 &< \inf\{p(x,x_0)+p(x,Tx) : x \in X\} \\
&\le \inf\{q(x,x_0)+q(x,Tx) : x \in X\} \\
&\le \inf\left\{q(T^n x,x_0)+q\left(T^n x,T^{n+1}x\right) : n \in \mathbb{N}\right\} \\
&\le \inf\left\{\frac{2r^n}{1-r}q(x,Tx) : n \in \mathbb{N}\right\} \\
&= 0.
\end{aligned} \tag{3.39}$$

This is a contradiction. Therefore we have $x_0 = Tx_0$. If $v = Tv$, we have $p(v,v) = p(Tv,Tv) \le rp(v,v)$ and hence $p(v,v) = 0$. To prove unique fixed point of T, let $x_0 = Tx_0$ and $v = Tv$. Then, by hypothesis, we have

$$\begin{aligned}
p(x_0,v) = p(Tx_0,Tv) &\le r\cdot\max\{p(x_0,v),p(v,x_0),p(x_0,x_0),p(v,v)\}, \\
p(v,x_0) = p(Tv,Tx_0) &\le r\cdot\max\{p(x_0,v),p(v,x_0),p(x_0,x_0),p(v,v)\}, \\
p(x_0,x_0) = p(Tx_0,Tx_0) &\le r\cdot\max\{p(x_0,v),p(v,x_0),p(x_0,x_0),p(v,v)\}, \\
p(v,v) = p(Tv,Tv) &\le r\cdot\max\{p(x_0,v),p(v,x_0),p(x_0,x_0),p(v,v)\}.
\end{aligned} \tag{3.40}$$

Thus

$$p(x_0,v) = p(v,x_0) = p(x_0,x_0) = p(v,v) = 0. \tag{3.41}$$

By Lemma 2.14, we have $x_0 = v$. □

From Theorem 3.4, we have the following corollary which generalizes the results of Ćirić [14], Kannan [15], and Ume [12].

Corollary 3.5. *Let X be a complete metric space with metric d, let p be a τ-distance on X, and let T be a selfmapping of X. Suppose that there exists $r \in [0,1)$ such that*

$$\begin{aligned}
p(Tx,Ty) \le r\cdot\max\{&p(x,y),p(x,Tx),p(y,Ty),p(x,Ty),p(y,Tx), \\
&p(y,x),p(Tx,x),p(Ty,y),p(Ty,x),p(Tx,y)\}
\end{aligned} \tag{3.42}$$

for all $x, y \in X$ and

$$\inf\{p(x,y)+p(x,Tx) : x \in X\} > 0 \tag{3.43}$$

for every $y \in X$ with $y \neq Ty$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0,x_0) = 0$. Moreover, if $v = Tv$, then $v = x_0$ and $p(v,v) = 0$.

Proof. Since a τ-distance is a u-distance, Corollary 3.5 follows from Theorem 3.4. □

The following corollary is a generalization of Suzuki's fixed point theorem [6].

Corollary 3.6. *Let X, T, and p be as in Corollary 3.5. Suppose that there exists $r \in [0,1)$ such that*

$$p\left(Tx, T^2x\right) \leq r \cdot \max\{p(x,x), p(x,Tx), p(Tx,x)\} \tag{3.44}$$

for all $x, y \in X$. Assume that if

$$\begin{aligned} \lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0, \\ \lim_{n\to\infty} p(x_n, Tx_n) = 0, \\ \lim_{n\to\infty} p(x_n, z) = 0, \end{aligned} \tag{3.45}$$

then $Tz = z$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$. Moreover, if $Tv = v$, then $v = x_0$ and $p(v,v) = 0$.

Proof. Let q and T be as in Theorem 3.4. Then from Theorem 3.4 and hypotheses of Corollary 3.6, we have the following properties.

(1) $\{T^n x\}$ is a Cauchy sequence.

(2) There exists $x_0 \in X$ such that $\lim_{n\to\infty} T^n x = x_0$.

(3) One has

$$\begin{aligned} \lim_{n\to\infty} p(T^n x, x_0) &\leq \lim_{n\to\infty} q(T^n x, x_0) \\ &\leq \lim_{n\to\infty} \frac{r^n}{1-r} \max\{q(x,Tx), q(x,x)\}. \end{aligned} \tag{3.46}$$

(4) There exists

$$\lim_{n\to\infty} p\left(T^n x, T^{n+1}x\right) = \lim_{n\to\infty} p\left(T^{n+1}x, T^n x\right) = 0. \tag{3.47}$$

(5) One has

$$\lim_{n\to\infty} \sup\{p(T^n x, T^m x) : m > n\} = 0. \tag{3.48}$$

By (1)~(5) and hypotheses, we have $Tx_0 = x_0$. The remainders are same as Theorem 3.4. □

The following theorem is a generalization of Caristi's fixed point theorem [3].

Theorem 3.7. *Let X be a metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper function which is bounded from below, and let $L : X \times X \times X \times X \to \mathbb{R}_+$ be a function satisfying (i), (ii), and (iii) of Theorem 3.1. Let T be a selfmapping of X such that*

$$f(Tx) + h(x, Tx) \leq f(x), \quad \forall x \in X, \tag{3.49}$$

where a function $h : X \times X \to \mathbb{R}_+$ is defined by

$$h(v, w) = L(v, w, w, v) \tag{3.50}$$

for all $v, w \in X$. Then, there exists $x_0 \in X$ such that

$$Tx_0 = x_0, \qquad L(x_0, Tx_0, Tx_0, x_0) = 0. \tag{3.51}$$

Proof. Suppose $x \neq Tx$ for all $x \in X$. Then, by Theorem 3.1, there exists $x_0 \in X$ such that

$$f(x_0) = \inf_{v \in X} f(v). \tag{3.52}$$

Since

$$f(Tx_0) + h(x_0, Tx_0) \leq f(x_0), \tag{3.53}$$

we have

$$\begin{aligned} f(Tx_0) &= f(x_0) = \inf_{v \in X} f(v), \\ L(x_0, Tx_0, Tx_0, x_0) &= 0. \end{aligned} \tag{3.54}$$

By hypothesis, we obtain

$$f\left(T^2 x_0\right) + h\left(Tx_0, T^2 x_0\right) \leq f(Tx_0). \tag{3.55}$$

Hence

$$\begin{aligned} f\left(T^2 x_0\right) &= f(Tx_0), \\ L\left(Tx_0, T^2 x_0, T^2 x_0, Tx_0\right) &= 0. \end{aligned} \tag{3.56}$$

By conditions (i) and (iii) of Theorem 3.1, it follows that

$$Tx_0 = T^2 x_0. \tag{3.57}$$

This is a contradiction. □

Corollary 3.8. *Let X be a complete metric space with metric d and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below. Let p be a u-distance on X. Suppose that T is a selfmapping of X such that*

$$f(Tx) + p(x, Tx) \leq f(x) \tag{3.58}$$

for all $x \in X$. Then there exists $x_0 \in X$ such that

$$Tx_0 = x_0, \qquad p(x_0, x_0) = 0. \tag{3.59}$$

Proof. Define $L : X \times X \times X \times X \to \mathbb{R}_+$ by

$$L(v, w, x, y) = \max\{p(v, x), p(y, w)\} \tag{3.60}$$

for all $v, w, x, y \in X$. Then, by Definition 2.12 and Lemmas 2.13, 2.14, and 2.15, we can easily show that conditions of Corollary 3.8 satisfy all conditions of Theorem 3.7. Thus, Corollary 3.8 follows from Theorem 3.7. □

Remark 3.9. Since a w-distance and a τ-distance are a u-distance, Corollary 3.8 is a generalization of Kada-Suzuki-Takahashi [5, Theorem 2] and Suzuki [6, Theorem 3].

The following theorem is a generalization of Ekeland's ε-variational principle [2].

Theorem 3.10. *Let X be a complete metric space with metric d, let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function which is bounded from below, and let $L : X \times X \times X \times X \to \mathbb{R}_+$ be a function satisfying (i), (ii), and (iii) of Theorem 3.1. Then the following (1) and (2) hold.*

(1) *For each $x \in X$ with $f(x) < \infty$, there exists $v \in X$ such that $f(v) \leq f(x)$ and*

$$f(m) > f(v) - h(v, m) \tag{3.61}$$

for all $m \in X$ with $m \neq v$, where a function $h : X \times X \to \mathbb{R}_+$ is defined by

$$h(v, w) = L(v, w, w, v) \tag{3.62}$$

for all $v, w \in X$.

(2) *For each $\varepsilon > 0$ and $x \in X$ with $h(x, x) = 0$, and*

$$f(x) < \inf_{a \in X} f(a) + \varepsilon, \tag{3.63}$$

there exists $v \in X$ such that $f(v) \leq f(x)$,

$$\begin{gathered} h(x, v) \leq 1, \\ f(m) > f(v) - \varepsilon \cdot h(v, m) \end{gathered} \tag{3.64}$$

for all $m \in X$ with $m \neq v$.

Proof. (1) Let $x \in X$ be such that $f(x) < \infty$, and let

$$Z = \{s \in X \mid f(s) \leq f(x)\}. \tag{3.65}$$

Then, by hypotheses, Z is nonempty and closed. Thus Z is a complete metric space. Hence we may prove that there exists an element $v \in Z$ such that $f(m) > f(v) - h(v,m)$ for all $m \in X$ with $m \neq v$. Suppose not. Then, for every $v \in Z$, there exists $m \in Z$ such that $m \neq v$ and $f(m) + h(v,m) \leq f(v)$. By Theorem 3.1, there exists $x_0 \in Z$ such that

$$f(x_0) = \inf_{a \in Z} f(a). \tag{3.66}$$

Again for $x_0 \in Z$, there exists $x_1 \in Z$ such that $x_1 \neq x_0$ and

$$f(x_1) + h(x_0, x_1) \leq f(x_0). \tag{3.67}$$

Hence we have $f(x_1) = f(x_0)$ and $L(x_0, x_1, x_1, x_0) = 0$. Similarly, there exists $x_2 \in Z$ such that $x_2 \neq x_1$ and

$$f(x_2) + h(x_1, x_2) \leq f(x_1). \tag{3.68}$$

Thus we have $f(x_2) = f(x_1)$ and $L(x_1, x_2, x_2, x_1) = 0$. From conditions (i) and (iii) of Theorem 3.1, we obtain

$$x_1 = x_2. \tag{3.69}$$

This is a contradiction. The proof of (1) is complete.

(2) Let

$$Y = \{a \in X \mid f(a) \leq f(x) - \varepsilon \cdot h(x,a)\}. \tag{3.70}$$

Then Y is nonempty and closed. Hence Y is complete. As in the proof of (1), we have that there exists $v \in Y$ such that

$$f(m) > f(v) - \varepsilon \cdot h(v,m) \tag{3.71}$$

for every $m \in X$ with $m \neq v$. On the other hand, since $v \in Y$, we have

$$\begin{gathered} f(v) \leq f(x) - \varepsilon \cdot h(x,v) \leq f(x), \\ h(x,v) \leq \frac{1}{\varepsilon}\{f(x) - f(v)\} \leq \frac{1}{\varepsilon}\left\{f(x) - \inf_{a \in X} f(a)\right\} \leq \frac{1}{\varepsilon} \cdot \varepsilon = 1. \end{gathered} \tag{3.72}$$

This completes the proof of (2). □

Corollary 3.11. *Let X, f, and p be as in Corollary 3.8. Then the following (1) and (2) hold.*

(1) *For each $x \in X$ with $f(x) < \infty$, there exists $v \in X$ such that $f(v) \leq f(x)$ and*

$$f(m) > f(v) - p(v, m) \tag{3.73}$$

for all $m \in X$ with $m \neq v$.

(2) *For each $\varepsilon > 0$ and $x \in X$ with $p(x, x) = 0$, and*

$$f(x) < \inf_{a \in X} f(a) + \varepsilon, \tag{3.74}$$

there exists $v \in X$ such that $f(v) \leq f(x)$,

$$p(x, v) \leq 1, \qquad f(m) > f(v) - \varepsilon \cdot p(v, m) \tag{3.75}$$

for all $m \in X$ with $m \neq v$.

Proof. By method similar to Corollary 3.8, Corollary 3.11 follows from Theorem 3.10. □

Remark 3.12. Corollary 3.11 is a generalization of Suzuki [6, Theorem 4].

Acknowledgments

The author would like to thank the referees for useful comments and suggestions. This work was supported by the Korea Research Foundation (KRF) Grant funded by the Korea government (MEST) (2009-0073655).

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