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Research Article

Degree of Convergence of Iterative Algorithms for Boundedly Lipschitzian Strong Pseudocontractions

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Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to H$ be a boundedly Lipschitzian strong pseudo-contraction with a nonempty fixed point set. Three iterative algorithms are proposed for approximating the unique fixed point of T; one of them is for the self-mapping case, and the others are for the nonself-mapping case. Not only the strong convergence, but also the degree of convergence of the three iterative algorithms is obtained. Some numerical results corresponding to the self-mapping case are given which show advantages of our methods. As an application of our results, adopting the regularization idea, we also propose implicit and explicit algorithms for approximating a fixed point of a boundedly Lipschitzian pseudocontractive self-mapping from C into itself, respectively.

1. Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to H$ is said to be pseudo-contractive if

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \tag{1.1}$$

for every $x, y \in C$. T is said to be strongly pseudo-contractive if there exists a positive constant $\kappa \in [0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \le \kappa\|x - y\|^2, \tag{1.2}$$

for all $x, y \in C$. In this case, we also call T a κ-strong pseudocontraction. Using (1.2), it is easy to see that every strong pseudocontraction has at most one fixed point.

T is said to be Lipschitzian if there exists a positive constant L such that

$$\|Tx - Ty\| \leq L\|x - y\|, \tag{1.3}$$

for all $x, y \in C$. In this case, T is also said to be L-Lipschitzian. In particular, T is said to be nonexpansive if $L = 1$; and it is said to be contractive if $L < 1$. T is said to be boundedly Lipschitzian if, for each bounded subset K of C, there exists a positive constant L_K depending only on K such that

$$\|Tx - Ty\| \leq L_K\|x - y\|, \tag{1.4}$$

for all $x, y \in K$.

We will denote by $F(T)$ the set of fixed points of $T : C \to H$, that is, $F(T) = \{x \in C : Tx = x\}$. Let $\{x_n\}$ be a sequence and x a point in H. Then we use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to x of the sequence $\{x_n\}$, respectively.

Among classes of nonlinear mappings, the class of pseudocontractions is one of the most important classes of mappings. This is mainly due to the fact that there is a precise corresponding relation between the class of pseudocontractions and the class of monotone mappings. A mapping $A : C \to H$ is monotone (i.e., $\langle Ax - Ay,\ x - y\rangle \geq 0$ for all $x, y \in C$) if and only if T is pseudo-contractive, where $T = I - A$ and I denotes the identity mapping on H.

Within the past 40 years or so, mathematicians have been devoting their study to the existence and iterative construction of fixed points for pseudocontractions and of zeros for monotone mappings (see, e.g., [1–18]). However, most of these algorithms have no estimation of degree of convergence even if for strong pseudocontractions in setting of Hilbert spaces. Everyone knows that it is very important to get the degree of convergence for an algorithm in computing science.

The main purpose of this paper is to consider the iterative algorithms for approximating the unique fixed point (if the set of fixed points is not empty) of a boundedly Lipschitzian strong pseudocontraction defined on a nonempty closed convex subset of a real Hilbert space. Three iterative algorithms are proposed; one of them is for the self-mapping case, and the others are for the nonself-mapping case. Not only the strong convergence, but also the degree of convergence of the three iterative algorithms is obtained. Some numerical results corresponding to the self-mapping case are given which show advantages of our methods. As an application of our results, adopting the regularization idea, we also establish implicit and explicit algorithms for approximating a fixed point of a boundedly Lipschitzian pseudo-contractive self-mapping from C into itself, respectively.

In order to give our main results, let us recall a basic existence result for fixed points for continuous strong pseudocontractions which was proved by Deimling [6] in 1974.

Theorem 1.1 (Deimling [6]). *Let D be a closed subset of a real Banach space X, and let $T : D \to X$ be a continuous κ-strong pseudocontraction, and*

$$\rho((1 - \lambda)x + \lambda Tx, D) = o(\lambda) \quad \text{as } \lambda \longrightarrow 0^+ \tag{1.5}$$

for each $x \in D$, where $\rho(z, K)$ denotes the distance from the point $z \in X$ to the subset K of X. Then T has a unique fixed point.

Corollary 1.2 (see [6]). *Let D be a closed convex subset of a real Banach space X, and let $T : D \to D$ be a continuous κ-strong pseudocontraction, then T has a unique fixed point.*

We also need some facts which are listed as lemmas below.

Lemma 1.3 (see [7]). *Let H be a real Hilbert space. Given a closed and convex subset C of H and points $x, y \in H$and given also a real number r such that $0 < r < 1$, then the set*

$$D = \{w \in C : \|w - x\| \leq r\|w - y\|\} \tag{1.6}$$

is closed and convex.

Lemma 1.4 (see, e.g., [9]). *Let K be a closed convex subset of a real Hilbert space H, and let P_K be the (metric or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Given $x \in H$ and $z \in K$, then $z = P_K x$ if and only if there holds the relation*

$$\langle x - z, y - z \rangle \leq 0, \quad \forall y \in K. \tag{1.7}$$

Lemma 1.5 (see [18]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C \to$ a demicontinuous pseudo-contractive self-mapping from C into itself. Then $F(T)$ is a closed convex subset of C and $I - T$ is demiclosed at zero.*

Now we are in a position to prove main results in this paper.

2. Algorithms for Strongly Pseudocontractive Self-Mappings

In this section, we propose an iterative algorithm for boundedly Lipschitzian and strongly pseudo-contractive self-mappings. Since the algorithm has nothing to do with the metric projection, it is easy to realize in practical computing.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a boundedly Lipschitzian and κ-strong pseudocontraction. Take $x_0 \in C$ arbitrarily, and let $C_0 = \{z \in C : \|z - x_0\| \leq (1/(1-\kappa))\|x_0 - Tx_0\|\}$ and $D = \{z \in C : \|z - x_0\| \leq (2/(1-\kappa))\|x_0 - Tx_0\|\}$. Define $\{x_n\}$ recursively by*

$$x_{n+1} = \alpha x_n + (1 - \alpha) T x_n, \quad n \geq 0, \tag{2.1}$$

where α is a constant such that $\max\{(L_D^2 - 1)/(L_D^2 + 1 - 2\kappa), 0\} < \alpha < 1$ and L_D is the bounded Lipschitz constant of T upon D. Then $\{x_n\}$ converges strongly to the unique fixed point x^* of T, and *the estimation of degree of convergence is as follows:*

$$\|x_n - x^*\| \le \frac{\left(\sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\right)^n}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}} \|x_1 - x_0\|. \tag{2.2}$$

In addition, this estimation of degree of convergence is optimal in the sense of ignoring constant factors.

Proof. Firstly, it concludes by using Corollary 1.2 that T has a unique fixed point, denoted by x^*, in C. We also assert that $x^* \in C_0$ holds. Indeed, since T is a κ-strong pseudocontraction, we get that

$$\left\langle (I-T)x - (I-T)y, x - y \right\rangle \ge (1-\kappa)\left\|x - y\right\|^2 \tag{2.3}$$

holds for all $x, y \in C$. Taking $x = x^*$ and $y = x_0$ in (2.3), we have

$$\|x^* - x_0\| \le \frac{1}{1-\kappa}\|x_0 - Tx_0\|, \tag{2.4}$$

That is, $x^* \in C_0$.

Now we prove by mathematical induction that

$$\|x_n - x^*\| \le \left(\sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\right)^n \|x_0 - x^*\| \tag{2.5}$$

and $x_n \in D$ hold for all $n \ge 1$. For $n = 1$, observing that $x_0, \quad x^* \in C_0 \subset D$, T is L_D-Lipschitzian restricted to D, and T is κ-strongly pseudo-contractive, it is easy to get that

$$\begin{aligned} \|x_1 - x^*\|^2 &= \|\alpha(x_0 - x^*) + (1-\alpha)(Tx_0 - x^*)\|^2 \\ &= \alpha^2\|x_0 - x^*\|^2 + (1-\alpha)^2\|Tx_0 - Tx^*\|^2 + 2\alpha(1-\alpha)\langle x_0 - x^*, Tx_0 - Tx^*\rangle \\ &\le \left[\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa\right]\|x_0 - x^*\|^2, \end{aligned} \tag{2.6}$$

hence

$$\|x_1 - x^*\| \le \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\|x_0 - x^*\|. \tag{2.7}$$

Noting that the condition $\max\{(L_D^2 - 1)/(L_D^2 + 1 - 2\kappa), 0\} < \alpha < 1$ implies

$$0 < \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa} < 1, \tag{2.8}$$

we have from (2.4), (2.7), and (2.8) that

$$\|x_1 - x_0\| \leq \|x_1 - x^*\| + \|x_0 - x^*\| \leq 2\|x_0 - x^*\| \leq \frac{2}{1-\kappa}\|x_0 - Tx_0\|, \tag{2.9}$$

That is, $x_1 \in D$.

Suppose that $x_{n-1} \in D$ and

$$\|x_{n-1} - x^*\| \leq \left(\sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\right)^{n-1} \|x_0 - x^*\|. \tag{2.10}$$

Similar to (2.7), we have from $x^*, x_{n-1} \in D$ that

$$\|x_n - x^*\| \leq \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\|x_{n-1} - x^*\|. \tag{2.11}$$

Thus (2.11) together with (2.10) leads to (2.5). On the other hand, we have from (2.4), (2.5), and (2.8) that $\|x_n - x_0\| \leq \|x_n - x^*\| + \|x_0 - x^*\| \leq 2\|x_0 - x^*\| \leq (2/(1-\kappa))\|x_0 - Tx_0\|$, that is, $x_n \in D$.

By (2.7), we have

$$\begin{aligned} \|x_0 - x^*\| &\leq \|x_0 - x_1\| + \|x_1 - x^*\| \\ &\leq \|x_0 - x_1\| + \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\|x_0 - x^*\|. \end{aligned} \tag{2.12}$$

Consequently

$$\|x_0 - x^*\| \leq \frac{1}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}}\|x_1 - x_0\|. \tag{2.13}$$

Thus (2.2) is obtained by using (2.5) and (2.13).

Finally, we show that (2.2) is the optimal estimation of degree of convergence in the sense of ignoring constant factors. For this purpose, it suffices to find an example such that

$$\|x_n - x^*\| = O\left\{\left(\sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\right)^n\right\}. \tag{2.14}$$

Indeed, taking $H = R^2$ with the usual inner product and norm and taking $C = H$, let $T : C \to C$ be a rotation operator defined by

$$T\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \forall \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in R^2, \tag{2.15}$$

where $\theta \in (0, \pi/2]$ such that $\kappa = \cos\theta$. Obviously, T has a unique fixed point $x^* = (0,0)^T \in R^2$. Moreover, T is nonexpansive, that is, Lipschitz constant $L = 1$. Since T is a linear operator, using (2.15), we have

$$\langle Tx - Ty, x - y\rangle = \langle T(x-y), x-y\rangle = \|x-y\|^2 \cos\theta, \quad \forall x, y \in R^2, \tag{2.16}$$

hence T is a κ-strong pseudocontraction.

Taking an initial value $x_0 = (1,0)^T$ and a control parameter α such that $0 = (L^2 - 1)/(L^2 + 1 - 2\kappa) < \alpha < 1$, it follows from direct calculating that

$$\|x_n - x^*\| = \left(\sqrt{\alpha^2 + (1-\alpha)^2 L^2 + 2\alpha(1-\alpha)\kappa}\right)^n \|x_0 - x^*\|. \tag{2.17}$$

This shows that the estimation (2.2) cannot be improved. □

Remark 2.2. If $L_D \geq 1$, then $\sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}$ reaches the minimum $\sqrt{(L_D^2 - \kappa^2)/(1 + L_D^2 - 2\kappa)}$ when $\alpha = (L_D^2 - \kappa)/(1 + L_D^2 - 2\kappa)$, so $(L_D^2 - \kappa)/(1 + L_D^2 - 2\kappa)$ is said to be optimal control parameter of process (2.1). If $L_D < 1$, then it is not difficult to verify that the optimal control parameter is zero. The same result also applies to all of the following algorithms.

If T is Lipschitzian on the whole C, that is, there exists a positive constant L such that $\|Tx - Ty\| \leq L\|x - y\|$ for all $x, y \in C$, then we obtain the following result as a special case of Theorem 2.1.

Theorem 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be an L-Lipschitzian and κ-strong pseudocontraction. Take an initial guess $x_0 \in C$ arbitrarily, and define a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = \alpha x_n + (1-\alpha)Tx_n, \quad n \geq 0, \tag{2.18}$$

where α is a constant such that $\max\{(L^2-1)/(L^2+1-2\kappa), 0\} < \alpha < 1$. *Then $\{x_n\}$ converges strongly to the unique fixed point x^*, and the estimation of degree of convergence is obtained as follow:*

$$\|x_n - x^*\| \leq \frac{\left(\sqrt{\alpha^2 + (1-\alpha)^2 L^2 + 2\alpha(1-\alpha)\kappa}\right)^n}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L^2 + 2\alpha(1-\alpha)\kappa}} \|x_1 - x_0\|. \tag{2.19}$$

In addition, this estimation of degree of convergence is optimal in the sense of ignoring constant factors.

In order to test the computing effect of the algorithm (2.1), some numerical results for the function

$$\varphi(x) = -\frac{1}{5}x^5 - \frac{1}{3}x^3 + \frac{1}{2}x + \frac{1}{6}, \quad x \in (-\infty, +\infty) \tag{2.20}$$

Table 1: $\alpha = 3/5$.

n	x_n	E_n
10	0.2885	4.8621×10^{-2}
20	0.3104	2.9078×10^{-3}
25	0.3116	7.1819×10^{-4}
30	0.3118	1.7879×10^{-4}
35	0.3119	4.4528×10^{-5}
40	0.3119	1.1090×10^{-5}
42	0.3119	6.3602×10^{-6}

Table 2: $\alpha = 7/9$.

n	x_n	E_n
10	0.2284	2.1176×10^{-1}
20	0.2914	4.2152×10^{-2}
30	0.3071	9.6024×10^{-3}
40	0.3109	1.9356×10^{-3}
45	0.3114	9.3611×10^{-4}
60	0.3119	1.0653×10^{-4}
74	0.3119	1.4022×10^{-5}

Table 3: $\alpha = 8/9$.

n	x_n	E_n
20	0.2248	2.2390×10^{-1}
40	0.2895	4.6446×10^{-2}
60	0.3063	1.1169×10^{-2}
80	0.3105	2.7683×10^{-3}
100	0.3116	6.7323×10^{-4}
120	0.3118	1.6660×10^{-4}
130	0.3119	8.2894×10^{-5}
146	0.3119	2.7133×10^{-5}

are given as follows. Using the mean value theorem, it is easy to verify that φ is a 1/2-strongly pseudo-contractive and boundedly Lipcshitzian function. For each constant $r > 0$, the bounded Lipschitz constant of φ upon the interval $[-r, r]$ is $r^4 + r^2 + 0.5$. Choosing the initial guess $x_0 = 0$ in (2.1), it follows by using Theorem 2.1 that $C_0 = [-1/3, 1/3]$, $D = [-2/3, 2/3]$, $L_D = 185/162$, and the control parameter α such that $7981/34225 = (L_D^2 - 1)/L_D^2 < \alpha < 1$. Since we do not know the exact fixed point x^* of φ, we propose the relative rate of convergence $E_n = |x_n - \varphi x_n|/|x_n|$ $(n \geq 0)$ to test the computing effect of algorithm (2.1) for φ. All the numerical results are in Tables 1–3.

3. Algorithms for Strongly Pseudo-Contractive Nonself-Mappings

In this section, we turn to designing two iterative algorithms for boundedly Lipschitzian and strongly pseudo-contractive nonself-mappings. In this case, a boundedly Lipschitzian strong pseudocontraction may not have a fixed point, so we assume that the mapping has a unique fixed point (noting that each strong pseudocontraction has at most one fixed point). In addition, we will have to use the metric projection in the algorithms.

In fact, the first algorithm is a modification of process (2.1) as follows. We omit its proof, which is very similar to the proof of Theorem 2.1.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to H$ be a boundedly Lipschitzian and κ-strong pseudocontraction with a unique fixed point. Take $x_0 \in C$ arbitrarily, and let $C_0 = \{z \in C : \|z - x_0\| \le (1/(1-\kappa))\ \|x_0 - Tx_0\|\}$ and $D = \{z \in C : \|z - x_0\| \le (2/(1-\kappa))\|x_0 - Tx_0\|\}$. Define a sequence $\{x_n\}$ via the recursive formula*

$$x_{n+1} = P_C(\alpha x_n + (1-\alpha)Tx_n), \quad n \ge 0, \tag{3.1}$$

where α is a constant such that $\max\{(L_D^2 - 1)/(L_D^2 + 1 - 2\kappa), 0\} < \alpha < 1$ and L_D is the bounded Lipschitz constant of T upon D. Then $\{x_n\}$ converges strongly to the unique fixed point x^* of T, and the estimation of degree of convergence is as follows:

$$\|x_n - x^*\| \le \frac{\left(\sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}\right)^n}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L_D^2 + 2\alpha(1-\alpha)\kappa}} \|x_1 - x_0\|. \tag{3.2}$$

In addition, this estimation of degree of convergence is optimal in the sense of ignoring constant factors.

If T is Lipschitzian on the whole C, we have the following result as a special case of Theorem 3.1.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to H$ be an L-Lipschitzian and κ-strong pseudocontraction. Let T has a unique fixed point $x^* \in C$. Take an initial guess $x_0 \in C$ arbitrarily, and define $\{x_n\}$ recursively by*

$$x_{n+1} = P_C(\alpha x_n + (1-\alpha)Tx_n), \quad n \ge 0, \tag{3.3}$$

where α is a constant such that $\max\{(L^2-1)/(L^2+1-2\kappa), 0\} < \alpha < 1$. Then $\{x_n\}$ converges strongly to the unique fixed point x^*, and the estimation of degree of convergence is as follows:

$$\|x_n - x^*\| \le \frac{\left(\sqrt{\alpha^2 + (1-\alpha)^2 L^2 + 2\alpha(1-\alpha)\kappa}\right)^n}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L^2 + 2\alpha(1-\alpha)\kappa}} \|x_1 - x_0\|. \tag{3.4}$$

In addition, this estimation of degree of convergence is optimal in the sense of ignoring constant factors.

Now we give the second algorithm for the strongly pseudo-contractive nonself-mapping case.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to H$ be a boundedly Lipschitzian and κ-strong pseudocontraction $(0 \le \kappa < 1)$. Let T have a unique fixed* *point* x^*. *Take* $x_0 \in C$ *arbitrarily, and let* $C_0 = \{z \in C : \|z - x_0\| \le (1/(1-\kappa))\|x_0 - Tx_0\|\}$. *Define a sequence* $\{x_n\}$ *of* C_0 *as follows:*

$$y_n = \alpha x_n + (1-\alpha)Tx_n, \quad n \ge 0,$$
$$C_{n+1} = \left\{ z \in C_0 : \|y_n - z\| \le \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\|x_n - z\| \right\}, \tag{3.5}$$
$$x_{n+1} = P_{C_{n+1}} y_n, \quad n \ge 0,$$

where L_{C_0} *is the bounded Lipschitz constant of* T *upon* C_0 *and* α *is a constant such that* $\max\{(L_{C_0}^2 - 1)/(L_{C_0}^2 + 1 - 2\kappa), 0\} < \alpha < 1$. *Then* $\{x_n\}$ *converges strongly to* x^*. *One also has the estimation of degree of convergence*

$$\|x_n - x^*\| \le \frac{\left(\sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\right)^n}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}} \|x_1 - x_0\|. \tag{3.6}$$

In addition, this estimation of degree of convergence is optimal in the sense of ignoring constant factors.

Proof. By the proof of Theorem 2.1, we have

$$\|x^* - x_0\| \le \frac{1}{1-\kappa}\|x_0 - Tx_0\|, \tag{3.7}$$

that is, $x^* \in C_0$.

Now we verify by mathematical induction that $x^* \in C_n$ and the sequence $\{x_n\}$ generated by (3.5) is well defined for each $n \ge 1$. For $n = 1$, observing that $x_0, x^* \in C_0$, and T is a κ-strong pseudocontraction, we have

$$\begin{aligned} \|y_0 - x^*\|^2 &= \alpha^2\|x_0 - x^*\|^2 + (1-\alpha)^2\|Tx_0 - Tx^*\|^2 + 2\alpha(1-\alpha)\langle x_0 - x^*, Tx_0 - Tx^*\rangle \\ &\le \left[\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa\right]\|x_0 - x^*\|^2. \end{aligned} \tag{3.8}$$

Hence

$$\|y_0 - x^*\| \le \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\|x_0 - x^*\|, \tag{3.9}$$

and this means $x^* \in C_1$. Noting that the condition $\max\{(L_{C_0}^2 - 1)/(L_{C_0}^2 + 1 - 2\kappa), 0\} < \alpha < 1$ implies that $0 < \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa} < 1$, and by using Lemma 1.3, C_1 is nonempty, closed, and convex. Using Lemma 1.4, there exists a unique element $x_1 \in C_1$ such that $x_1 = P_{C_1} y_0$. Suppose that x_k has been obtained and $x^* \in C_k$ for some $k \ge 1$. Likewise, observing that $x_k, x^* \in C_0$, and T is a κ-strong pseudocontraction, we also have

$$\|y_k - x^*\| \le \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\|x_k - x^*\|. \tag{3.10}$$

The definition of C_{k+1} and (3.10) imply that $x^* \in C_{k+1}$. Using Lemma 1.3 again, the condition $\max\{(L_{C_0}^2-1)/(L_{C_0}^2+1-2\kappa),0\} < \alpha < 1$ guarantees that C_{k+1} is nonempty, closed, and convex. So there exists a unique element $x_{k+1} \in C_{k+1}$ such that $x_{k+1} = P_{C_{k+1}} y_k$.

Finally, we prove that (3.6) holds and $\{x_n\}$ converges strongly to x^*. Observing process (3.5), $\{x_n\} \subset C_0$, and $x^* \in C_n$ for all $n \geq 0$, we have from an argument similar to getting (3.10) that

$$\begin{aligned} \|x_n - x^*\| &= \left\|P_{C_n} y_{n-1} - x^*\right\| \\ &= \left\|P_{C_n} y_{n-1} - P_{C_n} x^*\right\| \\ &\leq \left\|y_{n-1} - x^*\right\| \\ &\leq \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\,\|x_{n-1} - x^*\|. \end{aligned} \tag{3.11}$$

By induction step, we have

$$\|x_n - x^*\| \leq \left(\sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\right)^n \|x_0 - x^*\|. \tag{3.12}$$

By (3.11) and triangular inequality, we have

$$\begin{aligned} \|x_0 - x^*\| &\leq \|x_0 - x_1\| + \|x_1 - x^*\| \\ &\leq \|x_1 - x_0\| + \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}\,\|x_0 - x^*\|, \end{aligned} \tag{3.13}$$

hence

$$\|x_0 - x^*\| \leq \frac{1}{1 - \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa}} \|x_1 - x_0\|. \tag{3.14}$$

Thus the combination of (3.12) and (3.14) leads to (3.6), and $\{x_n\}$ converges strongly to x^* due to the fact that $0 < \sqrt{\alpha^2 + (1-\alpha)^2 L_{C_0}^2 + 2\alpha(1-\alpha)\kappa} < 1$.

By the same argument in the proof of Theorem 2.1, we assert that (3.6) is the optimal estimation of degree of convergence. □

Remark 3.4. The formulation of process (3.1) is simpler than that of process (3.5). But process (3.5) is believed to have faster rate of convergence than that of process (3.1) due to the fact that $L_{C_0} \leq L_D$, in general.

4. Algorithms for Pseudo-Contractive Self-Mappings

Let C be a nonempty closed convex subset of a real Hilbert space H, and let $V : C \to C$ be a boundedly Lipschitzian pseudocontraction with a nonempty fixed point set S, that is,

$S = F(V) \neq \emptyset$. It follows from Lemma 1.5 that S is closed and convex, so the metric projection operator P_S is well defined.

In this section, adopting the regularization idea, we propose implicit and explicit algorithms for approximating a fixed point of V, respectively. More precisely, given an arbitrary element $u \in C$, for each $t \in (0,1)$, it is easy to show that $V_t : C \to C$ defined by

$$V_t : x \longmapsto tu + (1-t)Vx, \qquad x \in C \tag{4.1}$$

is a boundedly Lipschitzian $(1-t)$-strong pseudocontraction. Then we have from Corollary 1.2 that V_t has a unique fixed point. Denote by y_t the unique fixed point of V_t. Namely, y_t is the only solution of the fixed point equation

$$y_t = tu + (1-t)Vy_t. \tag{4.2}$$

Firstly, we prove that (y_t) converges strongly to a fixed point of V, as $t \to 0$. Next, we give our explicit method based on this implicit method and Theorem 2.1.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $V : C \to C$ be a boundedly Lipschitzian pseudocontraction with a nonempty fixed point set S. Let $t \in (0,1)$, and (y_t) is determined by* (4.2). *Then (y_t) is bounded, and $\lim_{t\to 0} y_t = x^* \in S$. Moreover, $x^* = P_S u$.*

Proof. First we show that (y_t) is bounded. Take $p \in S$ arbitrarily; noting that V is a pseudocontraction, we have from (4.2) and the fact $p = Vp$ that

$$\begin{aligned} \|y_t - p\|^2 &= \langle t(u-p) + (1-t)(Vy_t - Vp), y_t - p\rangle \\ &\le t\langle u-p, y_t-p\rangle + (1-t)\|y_t - p\|^2. \end{aligned} \tag{4.3}$$

Hence

$$\|y_t - p\|^2 \le \langle u-p, y_t - p\rangle. \tag{4.4}$$

Clearly, $\|y_t - p\| \le \|u-p\|$. This says that (y_t) is bounded. Since V is boundedly Lipschitzian, so it is not difficult to show that (Vy_t) is also bounded. Thus we can assert that the set of weak cluster points $\omega\{(y_t)\} \neq \emptyset$, where $\omega\{(y_t)\} = \{y : y_{t_n} \rightharpoonup y$ for some sequence $\{t_n\}$ in $(0,1)$ such that $t_n \to 0\}$.

Next, we prove $\omega\{(y_t)\} \subset F(V)$; namely, if (t_j) is a null sequence in $(0,1)$ such that $y_{t_j} \rightharpoonup \widehat{y}$ as $j \to \infty$, then $\widehat{y} \in S$. To see this, using (4.2), we get

$$y_{t_j} - Vy_{t_j} = t_j\left(u - Vy_{t_j}\right). \tag{4.5}$$

Clearly, this together with the boundedness of (Vy_t) implies that $y_{t_j} - Vy_{t_j} \to 0$ as $j \to \infty$. Using Lemma 1.5, we have $V\widehat{y} = \widehat{y}$, that is, $\widehat{y} \in S$. Taking $t = t_j$ and $p = \widehat{y}$ in (4.4), we have

$$\left\| y_{t_j} - \widehat{y} \right\|^2 \leq \left\langle u - p, y_{t_j} - \widehat{y} \right\rangle. \tag{4.6}$$

Taking the limit as $j \to \infty$, we see that $y_{t_j} \to \widehat{y}$.

Finally, we turn to proving that $\lim_{j\to\infty} y_t = x^* = P_S u$. Since $I - V$ is monotone, for any $\widetilde{y} \in S$, we have

$$\left\langle (I - V)y_{t_j} - (I - V)\widetilde{y}, y_{t_j} - \widetilde{y} \right\rangle \geq 0. \tag{4.7}$$

Observing $V\widetilde{y} = \widetilde{y}$, it follows from (4.2) that

$$\left\langle t_j \left(u - Vy_{t_j} \right), y_{t_j} - \widetilde{y} \right\rangle \geq 0. \tag{4.8}$$

Thus, we have

$$\left\langle u - Vy_{t_j}, \widetilde{y} - y_{t_j} \right\rangle \leq 0. \tag{4.9}$$

Since V is continuous and $\lim_{j\to\infty} y_{t_j} = \widehat{y}$, we obtain by taking the limit that

$$\langle u - \widehat{y}, \widetilde{y} - \widehat{y} \rangle \leq 0, \quad \widetilde{y} \in S. \tag{4.10}$$

By Lemma 1.4, we get $\widehat{y} = P_S u = x^*$. This means that $\omega\{(y_t)\} = \{x^*\}$. Thus we have proved that $\lim_{j\to\infty} y_t = x^* = P_S u$. □

Our following explicit method is motivated by Theorem 4.1, Theorem 2.1, and Zhou's iterative method in [17]. Given a sequence $\{t_n\} \subset (0,1)$ $(n = 0, 1, \ldots)$ such that $t_n \to 0$ as $n \to \infty$. Denote by $\{y_n\}$ the unique fixed point of the mapping $V_n = t_n u + (1 - t_n)V$. Namely,

$$y_n = t_n u + (1 - t_n)Vy_n. \tag{4.11}$$

Theorem 4.1 says that $\lim_{n\to\infty} y_n = x^* = P_S u$. Observe that V_n is boundedly Lipschitzian and $(1 - t_n)$-strongly pseudo-contractive.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $V : C \to C$ be a boundedly Lipschitzian pseudocontraction with a nonempty fixed point set S. Let $\{t_n\}$, $\{\varepsilon_n\} \subset (0,1)$ such that $t_n, \varepsilon_n \to 0$ as $n \to \infty$. For arbitrary initial datum $x_0 = x_0^0 \in C$. Define iteratively a sequence $\{x_n\}$ in an explicit manner as follows:*

$$\begin{gathered} x_n^{m+1} = \alpha_n x_n^m + (1 - \alpha_n)V_n x_n^m, \quad n \geq 0;\ m = 0, 1, \ldots, N(n) - 1, \\ x_{n+1} = x_{n+1}^0 = t_n u + (1 - t_n)Vx_n^{N(n)}, \quad n \geq 0, \end{gathered} \tag{4.12}$$

where $N(n)$ is the least positive integer satisfying

$$\frac{s_n^{N(n)}}{1-s_n}\left\|x_n^1 - x_n^0\right\| \le \frac{\varepsilon_n}{L_n}. \tag{4.13}$$

L_n is the bounded Lipschitz constant of V_n upon $D_n = \{z \in C : \|z - x_n\| \le 2\|x_n - V_n x_n\|/t_n\}$,

$$s_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 L_n^2 + 2\alpha_n(1-\alpha_n)(1-t_n)} \tag{4.14}$$

and the control parameter sequence $\{\alpha_n\}$ such that $\max\{(L_n^2-1)/(L_n^2+1-2(1-t_n)),\ 0\} < \alpha_n < 1$. *Then $\{x_n\}$ converges strongly to $P_S u$.*

Proof. Recalling that, for each $n \ge 0$, V_n is a boundedly Lipschitzian $(1-t_n)$-strong pseudocontraction, we have by using Theorem 2.1 that

$$\lim_{m\to\infty} x_n^m = y_n, \tag{4.15}$$

$$\left\|x_n^m - y_n\right\| \le \frac{s_n^m}{1-s_n}\left\|x_n^1 - x_n^0\right\|, \quad m \ge 1. \tag{4.16}$$

Since the condition $\max\{(L_n^2-1)/(L_n^2+1-2(1-t_n)),\ 0\} < \alpha_n < 1$ implies $0 < s_n < 1$, so there exists a least positive integer $N(n)$ satisfying condition (4.13). Using Theorem 4.1, we have

$$\lim_{n\to\infty} y_n = P_S u. \tag{4.17}$$

In order to complete the proof, it suffices to show that $x_{n+1} - y_n \to 0$ as $n \to \infty$. To this end, we estimate $\|x_{n+1} - y_n\|$. By the proof of Theorem 2.1, we assert that $y_n \in D_n$ and $x_n^m \in D_n$ for all $m \ge 1$. Thus we have from (4.11)–(4.16) that

$$\begin{aligned} \left\|x_{n+1} - y_n\right\| &\le (1-t_n)\left\|Vx_n^{N(n)} - Vy_n\right\| \\ &= \left\|V_n x_n^{N(n)} - V_n y_n\right\| \\ &\le L_n\left\|x_n^{N(n)} - y_n\right\| \\ &\le \frac{L_n s_n^{N(n)}}{1-s_n}\left\|x_n^1 - x_n^0\right\| \\ &\le \varepsilon_n. \end{aligned} \tag{4.18}$$

Hence $x_{n+1} - y_n \to 0$ as $n \to \infty$, since $\varepsilon_n \to 0$ as $n \to \infty$. Consequently, $x_n \to P_S u \in S$. □

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References

[1] F. E. Browder, "Existence of periodic solutions for nonlinear equations of evolution," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1100–1103, 1965.
[2] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.
[3] R. E. Bruck, Jr., "A strongly convergent iterative solution of $0 \in Ux$ for a maximal monotone operator U in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 48, pp. 114–126, 1974.
[4] R. Chen, Y. Song, and H. Zhou, "Convergence theorems for implicit iteration process for a finite family of continuous pseudocontractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 701–709, 2006.
[5] C. E. Chidume and H. Zegeye, "Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps," *Proceedings of the American Mathematical Society*, vol. 132, no. 3, pp. 831–840, 2004.
[6] K. Deimling, "Zeros of accretive operators," *Manuscripta Mathematica*, vol. 13, pp. 365–374, 1974.
[7] S. He, J. Zhao, and Z. Li, "Degree of convergence of modified averaged iterations for fixed points problems and operator equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 4098–4104, 2009.
[8] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, pp. 147–150, 1974.
[9] G. Marino and H.-K. Xu, "Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 336–346, 2007.
[10] C. Morales, "Pseudocontractive mappings and the Leray-Schauder boundary condition," *Commentationes Mathematicae*, vol. 20, no. 4, pp. 745–756, 1979.
[11] C. H. Morales and J. S. Jung, "Convergence of paths for pseudocontractive mappings in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3411–3419, 2000.
[12] M. O. Osilike, "Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps," *Journal of Mathematical Analysis and Applications*, vol. 294, no. 1, pp. 73–81, 2004.
[13] S. Reich, "Strong convergence theorems for resolvents of accretive operators in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 75, no. 1, pp. 287–292, 1980.
[14] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, Japan, 2000.
[15] H.-K. Xu and R. G. Ori, "An implicit iteration process for nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 22, no. 5-6, pp. 767–773, 2001.
[16] Y. Yao, Y.-C. Liou, and R. Chen, "Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 12, pp. 3311–3317, 2007.
[17] H. Zhou, "Strong convergence of an explicit iterative algorithm for continuous pseudo-contractions in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 11, pp. 4039–4046, 2009.
[18] H. Zhou, "Convergence theorems of fixed points for Lipschitz pseudo-contractions in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 546–556, 2008.