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Research Article

Fixed Points for Multivalued Mappings and the Metric Completeness

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We consider the equivalence of the existence of fixed points of single-valued mappings and multivalued mappings for some classes of mappings by proving some equivalence theorems for the completeness of metric spaces.

1. Introduction

The Banach contraction principle [1] states that for a complete metric space (X, d), every contraction T on X, that is, for some $r \in [0, 1)$, $d(Tx, Ty) \leq rd(x, y)$ for all $x, y \in X$, has a (unique) fixed point.

Connell [2] gave an example of a noncomplete metric space X on which every contraction on X has a fixed point. Thus contractions cannot characterize the metric completeness of X.

Theorem 1.1 (see [3, Kannan]). *Let (X, d) be a complete metric space. Let T be a Kannan mapping on X, that is, for some $\alpha \in [0, 1/2)$, $d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty)$ for all $x, y \in X$. Then T has a (unique) fixed point.*

Subrahmanyam [4] proved that Kannan mappings can be used to characterize the completeness of the metric. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point.

In 2008 Suzuki [5] introduced a new type of mappings and presented a generalization of the Banach contraction principle in which the completeness can also be characterized by the existence of fixed points of these mappings. Define a nonincreasing function θ from $[0, 1)$

onto $(1/2, 1]$ by

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \left(\sqrt{5}-1\right)/2, \\ (1-r)r^{-2} & \text{if } \left(\sqrt{5}-1\right)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases} \tag{1.1}$$

Theorem 1.2 (see [5]). *For a metric space (X, d), the following are equivalent:*

(i) *X is complete;*

(ii) *every mapping T on X such that there exists $r \in [0,1)$, $\theta(r)d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le rd(x,y)$ for all $x, y \in X$ has a fixed point.*

In 2008, Kikkawa and Suzuki [6] partially extended Theorem 1.2 to multivalued mappings.

Theorem 1.3 (see [6]). *Define a strictly decreasing function η from $[0,1)$ onto $(1/2,1]$ by $\eta(r) = 1/(1+r)$. Let (X,d) be a complete metric space and let $T : X \to 2^X \setminus \emptyset$ be a multivalued mapping with bounded and closed values. Assume that there exists $r \in [0,1)$ such that*

$$\eta(r)d(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le rd(x,y), \tag{1.2}$$

for all $x, y \in X$, then there exists $z \in X$ such that $z \in Tz$.

Obviously, the converse of Theorem 1.3 is valid since $1/(1+r) \le \theta(r)$ for all $r \in [0,1)$.

Moţ and Petruşel [7] proved the following theorem which is a generalization of Kikkawa and Suzuki Theorem.

Theorem 1.4 (see [7]). *Let (X,d) be a complete metric space and let $T : X \to 2^X \setminus \emptyset$ be a multivalued mapping with closed values and satisfies the following: if for nonnegative numbers a, b, c with $a+b+c \in [0,1)$ and for each $x, y \in Y$, one has*

$$\frac{1-b-c}{1+a}d(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le ad(x,y) + bd(x,Tx) + cd(y,Ty). \tag{1.3}$$

Then T has a fixed point.

In this paper, we will characterize the completeness of a metric space by the existence of fixed points for both single-valued and multivalued mappings. We first aim to extend, in Section 3, the Suzuki's result (Theorem 1.2) to more general classes of mappings. We then consider multivalued mappings in Section 4. We also show in this section that the converse of Theorem 1.4 is true.

2. Preliminaries

Let (X,d) be a complete metric space and let $f : X \to X$ be a mapping. We say that f is *a Caristi mapping* if there exists a lower semicontinuous function $\varphi : X \to R$ such that φ is bounded below and

$$d(x,f(x)) \leq \varphi(x) - \varphi(f(x)), \quad \text{for } x \in X. \tag{2.1}$$

Recall that a mapping f is *lower semicontinuous* if for each $x_0 \in X$ and for every $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U$.

For a metric space (X,d), let $Cl(X)$ and $CB(X)$ denote, respectively, a collection of all nonempty closed subsets of X and a collection of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric on $CB(X)$. That is, for $A, B \in CB(X)$,

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}, \tag{2.2}$$

where $d(x,D) := \inf\{d(x,y) : y \in D\}$ is the distance from a point x in X to a subset D of X.

The next theorem plays important roles in this paper.

Theorem 2.1 (see cf. [8]). *If* T *is a mapping of a complete metric space* X *into the family of all nonempty closed subsets of* X *and* $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ *is a lower semicontinuous function such that the following condition holds:*

$$\inf\{d(x,y) + \varphi(y) : y \in T(x)\} \leq \varphi(x), \quad \text{for each } x \in X, \tag{2.3}$$

then T has at least one fixed point.

3. Completeness and Single-valued Mappings

In 2008, Kikkawa and Suzuki [9] proved fixed point theorems for some generalized Kannan mappings. Let φ be a nonincreasing function defined from [0,1) onto (1/2,1] by

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \leq r < 1. \end{cases} \tag{3.1}$$

Theorem 3.1 (see [9]). *Let (X,d) be a complete metric space and let T be a mapping on X. Let $\alpha \in [0,1/2)$ and put $r := \alpha/(1-\alpha) \in [0,1)$. Assume that*

$$\varphi(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty), \tag{3.2}$$

for all $x, y \in X$, then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

Theorem 3.2 (see [9]). *Let (X,d) be a complete metric space and let T be a mapping on X. Suppose that there exists $r \in [0,1)$ such that*

$$\theta(r)d(x,Tx) \leq d(x,y) \text{ implies } d(Tx,Ty) \leq r\left[d(x,Tx) \vee d(y,Ty)\right], \tag{3.3}$$

for all $x,y \in X$. Then T has a unique fixed point z and $\lim_n T^n x = z$ holds for every $x \in X$.

The above theorems inspire us to present another version of Theorem 1.2. Before doing that we present first the following theorem. The proof of which is a mild modification of the proofs in [5, 9].

Theorem 3.3. *Let (X,d) be a complete metric space and let T be a mapping on X such that there exists $r \in [0,1)$, $\theta(r)d(x,Tx) \leq d(x,y)$ implies $d(Tx,Ty) \leq rd(x,Tx) \vee rd(y,Ty) \vee rd(x,y)$ for all $x,y \in X$, then T has a fixed point.*

Proof. Since $\theta(r) \leq 1$, $\theta(r)d(x,Tx) \leq d(x,Tx)$ holds for every $x \in X$, and thus

$$d\left(Tx,T^2x\right) \leq rd(x,Tx) \vee rd\left(Tx,T^2x\right), \quad \forall x \in X. \tag{3.4}$$

If $d(Tx,T^2x) \leq rd(Tx,T^2x)$ for some $x \in X$, then $Tx = T(Tx)$, and we get a fixed point Tx of T.

Suppose now that

$$d\left(Tx,T^2x\right) \leq rd(x,Tx), \quad \forall x \in X. \tag{3.5}$$

We fix $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_n = T^n x_0$.

Then $d(x_n,x_{n+1}) \leq r^n d(x_0,Tx_0)$, and so $\sum_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$. Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some point $z \in X$.

We show that

$$d(z,Tx) \leq rd(z,x) \vee rd(x,Tx), \quad \text{for each } x \in X \setminus \{z\}. \tag{3.6}$$

Suppose $x \neq z$. Since $x_n \to z$ as $n \to \infty$, there exists $n_0 \in N$ such that $d(x_n,z) \leq (1/3)d(z,x)$ for each $n \geq n_0$. Observe that

$$\begin{aligned} \theta(r)d(x_n,Tx_n) &\leq d(x_n,Tx_n) = d(x_n,x_{n+1}) \leq d(x_n,z) + d(z,x_{n+1}) \\ &\leq \frac{2}{3}d(x,z) \leq d(x,z) - d(x_n,z) \leq d(x_n,x). \end{aligned} \tag{3.7}$$

Hence $d(x_{n+1},Tx) = d(Tx_n,Tx) \leq rd(x_n,Tx_n) \vee rd(x,Tx) \vee rd(x_n,x)$ for each $n \geq n_0$. Letting $n \to \infty$ we get $d(z,Tx) \leq rd(z,x) \vee rd(x,Tx)$, for all $x \neq z$ and we obtain (3.6).

As in the proof of [5, Theorem 1.2], we show that $T^k z = z$ for some k from which it is proved that z is a fixed point of T. For this purpose, we assume $T^k z \neq z$ for all k and find a contradiction. We show, by induction, that

$$d\left(T^{k+1}z, z\right) \leq r^k d(z, Tz), \quad \text{hold } \forall k. \tag{3.8}$$

From (3.6) we have

$$\begin{aligned} d\left(z, T^2 z\right) &\leq r d(z, Tz) \vee r d\left(Tz, T^2 z\right) \\ &\leq r d(z, Tz) \vee r^2 d(z, Tz) = r d(z, Tz). \end{aligned} \tag{3.9}$$

Suppose $d(z, T^{k+1}z) \leq r^k d(z, Tz)$. Thus

$$\begin{aligned} d\left(z, T^{k+2}z\right) &\leq r d\left(T^{k+1}z, T^{k+2}z\right) \vee r d\left(z, T^{k+1}z\right) \quad \text{by (3.6)}, \\ &\leq r \cdot r^{k+1} d(z, Tz) \vee r \cdot r^k d(z, Tz), \\ &= r^{k+1} d(z, Tz) \quad \text{by (3.5)}. \end{aligned} \tag{3.10}$$

Thus (3.8) holds and now we find a contradiction in each of the following cases.

Case 1 $(0 \leq r < (\sqrt{5}-1)/2)$. We have $r^2 + r - 1 < 0$.

Assume $d(T^2 z, z) < d(T^2 z, T^3 z)$ then

$$\begin{aligned} d(z, Tz) &\leq d\left(z, T^2 z\right) + d\left(T^2 z, Tz\right) < d\left(T^2 z, T^3 z\right) + d\left(T^2 z, Tz\right) \\ &\leq r^2 d(z, Tz) + r d(z, Tz) < d(z, Tz), \end{aligned} \tag{3.11}$$

which is a contradiction. So

$$\begin{aligned} d\left(T^3 z, Tz\right) &\leq r\left(d\left(z, T^2 z\right) \vee d(z, Tz) \vee d\left(T^2 z, T^3 z\right)\right) \\ &\leq r\left(r d(z, Tz) \vee d(z, Tz) \vee r^2 d(z, Tz)\right) = r d(z, Tz). \end{aligned} \tag{3.12}$$

Hence $d(z, Tz) \leq d(z, T^3 z) + d(T^3 z, Tz) \leq r^2 d(z, Tz) + r d(z, Tz) < d(z, Tz)$, which is a contradiction.

Case 2 $((\sqrt{5}-1)/2 \leq r < 2^{-1/2})$. We have $2r^2 < 1$.

We show, by induction, that

$$\theta(r) d\left(T^k z, T^{k+1} z\right) \leq d\left(z, T^k z\right), \quad \forall k \geq 2 \tag{$*$}$$

If $d(z,T^2z) < \theta(r)d(T^2z,T^3z)$, then

$$\begin{aligned} d(z,Tz) &\leq d\left(z,T^2z\right) + d\left(T^2z,Tz\right) < \theta(r)d\left(T^2z,T^3z\right) + rd(z,Tz) \\ &\leq \left(\frac{1-r}{r^2}\right)r^2 d(z,Tz) + rd(z,Tz) = d(z,Tz), \end{aligned} \tag{3.13}$$

which is a contradiction. Therefore $\theta(r)d(T^2z,T^3z) \leq d(z,T^2z)$.

Suppose $\theta(r)d(T^kz,T^{k+1}z) \leq d(z,T^kz)$. Thus

$$\begin{aligned} d\left(Tz,T^{k+1}z\right) &\leq r\left(d\left(z,T^kz\right) \vee d(z,Tz) \vee d\left(T^kz,T^{k+1}z\right)\right) \\ &\leq r\left(r^{k-1}d(z,Tz) \vee d(z,Tz) \vee r^k d(z,Tz)\right) = rd(z,Tz). \end{aligned} \tag{3.14}$$

If $d(z,T^{k+1}z) < \theta(r)d(T^{k+1}z,T^{k+2}z)$, then

$$\begin{aligned} d(z,Tz) &\leq d\left(z,T^{k+1}z\right) + d\left(T^{k+1}z,Tz\right) < \theta(r)d\left(T^{k+1}z,T^{k+2}z\right) + rd(z,Tz) \\ &\leq \left(\frac{1-r}{r^2}\right)r^{k+1}d(z,Tz) + rd(z,Tz) < (1-r+r)d(z,Tz) = d(z,Tz), \end{aligned} \tag{3.15}$$

which is a contradiction. Hence $\theta(r)d(T^{k+1}z,T^{k+2}z) \leq d(z,T^{k+1}z)$, and thus $(*)$ holds.

For $k \geq 2$, $d(z,Tz) \leq d(z,T^{k+1}z) + d(T^{k+1}z,Tz) \leq r^k d(z,Tz) + rd(z,Tz)$. We have $d(z,Tz) \leq rd(z,Tz) < d(z,Tz)$, which is a contradiction.

Case 3 ($2^{-1/2} \leq r < 1$). We claim that $\theta(r)d(x_{2n},x_{2n+1}) \leq d(x_{2n},z)$ or $\theta(r)d(x_{2n+1},x_{2n+2}) \leq d(x_{2n+1},z)$. Suppose not,

$$\begin{aligned} d(x_{2n},x_{2n+1}) &\leq d(x_{2n},z) + d(z,x_{2n+1}) \\ &< \theta(r)(d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})) \\ &\leq \theta(r)(1+r)d(x_{2n},x_{2n+1}) \\ &= d(x_{2n},x_{2n+1}), \end{aligned} \tag{3.16}$$

which is a contradiction. So there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\theta(r)d(x_{n_k},x_{n_{k+1}}) \leq d(x_{n_k},z)$:

$$\begin{aligned} d(z,Tz) &= \lim_k d(x_{n_{k+1}},Tz) \leq \lim_k (rd(x_{n_k},z) \vee rd(x_{n_k},Tx_{n_k}) \vee rd(z,Tz)) \\ &= rd(z,Tz). \end{aligned} \tag{3.17}$$

Thus $Tz = z$, which is a contradiction. □

In fact the following theorem shows that the converse of Theorem 3.3 is valid.

Theorem 3.4. *Let (X, d) be a metric space. Then the following are equivalent:*

(i) *X is complete;*

(ii) *for each $r \in [0,1)$, every mapping T on X such that*

$$\frac{1}{1+r} d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le r d(x, y), \tag{3.18}$$

for all $x, y \in X$ has a fixed point;

(iii) *for each $r := 2\alpha \in [0,1)$, every mapping T on X such that*

$$\theta(r) d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty), \tag{3.19}$$

for all $x, y \in X$ has a fixed point;

(iv) *for each $r \in [0,1)$, every mapping T on X such that*

$$\theta(r) d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le r d(x, Tx) \vee r d(y, Ty) \tag{3.20}$$

for all $x, y \in X$ has a fixed point;

(v) *For nonnegative numbers a, b, c with $a + b + c \in [0,1)$, every mapping T on X such that*

$$\begin{aligned} \frac{1-b-c}{1+a} d(x, Tx) \le d(x, y) \text{ implies } & d(Tx, Ty) \\ & \le a d(x, y) + b d(x, Tx) + c d(y, Ty), \end{aligned} \tag{3.21}$$

for all $x, y \in X$ has a fixed point;

(vi) *for each $r := 2\beta + \gamma \in [0,1)$, every mapping T on X such that*

$$\begin{aligned} \theta(r) d(x, Tx) \le d(x, y) \text{ implies } & d(Tx, Ty) \\ & \le \gamma d(x, y) + \beta [d(x, Tx) + d(y, Ty)], \end{aligned} \tag{3.22}$$

for all $x, y \in X$ has a fixed point;

(vii) *for each $r \in [0,1)$, every mapping T on X such that*

$$\begin{aligned} \theta(r) d(x, Tx) \le d(x, y) \text{ implies } & d(Tx, Ty) \\ & \le r d(x, y) \vee r d(x, Tx) \vee r d(y, Ty), \end{aligned} \tag{3.23}$$

for all $x, y \in X$ has a fixed point.

Proof. The implication (i)⇒(vii) is exactly Theorem 3.3.

(vii)⇒(vi). Let T satisfy (3.22). We show that T satisfies (3.23) to obtain a fixed point for T. Let $\theta(r)d(x,Tx) \le d(x,y)$, $r := 2\beta+\gamma$. Thus $d(Tx,Ty) \le \gamma d(x,y)+\beta[d(x,Tx)+d(y,Ty)] \le (2\beta+\gamma)\max\{d(x,y),d(x,Tx),d(y,Ty)\} = rd(x,y) \vee rd(x,Tx) \vee rd(y,Ty)$, and (3.23) holds.

(vi)⇒(v). Let T satisfy (3.21). To show T satisfies (3.22), let $\theta(r)d(x,Tx) \le d(x,y)$, $a=\gamma, b=c=\beta$ and $r=2\beta+\gamma$. Notice that $\theta(r) \ge 1/(1+r) = 1/(1+2\beta+\gamma) = 1/(1+a+b+c) \ge (1-b-c)/(1+a)$. Thus $((1-b-c)/(1+a))d(x,Tx) \le \theta(r)d(x,Tx) \le d(x,y)$. So we get $d(Tx,Ty) \le ad(x,y)+bd(x,Tx)+cd(y,Ty) = \gamma d(x,y)+\beta d(x,Tx)+\beta d(y,Ty)$, and (3.22) holds.

(v)⇒(ii). Let T satisfy (3.18). To show T satisfies (3.21), let $((1-b-c)/(1+a))d(x,Tx) \le d(x,y)$, $a=r, b=c=0$. Thus $(1/(1+r))d(x,Tx) = ((1-b-c)/(1+a))d(x,Tx) \le d(x,y)$, and so $d(Tx,Ty) \le rd(x,y) = ad(x,y)+bd(x,Tx)+cd(y,Ty)$ and (3.21) holds.

(ii)⇒(i). Follows the same proof of Theorem 1.2. Notice that, for $0 \le r < 2^{-1/2}, 1/(1+r) \le \theta(r)$.

(vii)⇒(iv). Let T satisfy (3.20). To show T satisfies (3.23), let $\theta(r)d(x,Tx) \le d(x,y)$. Thus $d(Tx,Ty) \le rd(x,Tx) \vee rd(y,Ty) \le rd(x,y) \vee rd(x,Tx) \vee rd(y,Ty)$.

(iv)⇒(iii). Let T satisfy (3.19). We show T satisfies (3.20). Let $\theta(r)d(x,Tx) \le d(x,y)$, $r=2\alpha$. Thus $d(Tx,Ty) \le \alpha d(x,Tx)+\alpha d(y,Ty) \le (2\alpha)\max\{d(x,Tx),d(y,Ty)\} = rd(x,Tx) \vee rd(y,Ty)$.

(iii)⇒(i). We know that every Kannan mapping belongs to the class of mappings in (iii). Thus X is complete by Subrahmanyam [4]. □

4. Completeness and Multivalued Mappings

Inspired by Theorem 1.2 and Theorem 1.3, we prove the following theorem for a larger class of mappings under some certain assumptions.

Theorem 4.1. *Let (X,d) be a metric space. Then the following are equivalent:*

(i) *X is complete;*

(ii) *for each $r \in [0,1)$, every mapping $T : X \to Cl(X)$ such that $\theta(r)d(x,Tx) \le d(x,y)$ implies $H(Tx,Ty) \le rd(x,y) \vee rd(x,Tx) \vee rd(y,Ty)$, $x,y \in X$ and the function $x \mapsto d(x,Tx)$ is lower semicontinuous has a fixed point.*

Observe that Theorem 4.1 is not covered by Theorem 3.4 when considering as single-valued mappings.

Proof of Theorem 4.1. (i)⇒(ii). Let $\varepsilon > 0$ be small enough so that $\varepsilon + r < 1$ and define $\varphi(x) = (1/\varepsilon)d(x,Tx)$. For any $x \in X$, we can find some $f(x) \in Tx$ satisfying $d(x,f(x)) \le (1/(\varepsilon + r))d(x,Tx)$. To apply Theorem 2.1, it remains to show that $d(x,f(x)) \le \varphi(x)-\varphi(f(x))$, $x \in X$. We have $d(x,Tx) \le d(x,f(x)) \le (1/\theta(r))d(x,f(x))$. Thus $H(Tx,Tf(x)) \le rd(x,f(x)) \vee rd(f(x),Tf(x)) \vee rd(x,Tx)$. Note that

$$\begin{aligned} d(f(x),Tf(x)) &\le H(Tx,Tf(x)) \\ &\le rd(x,f(x)) \vee rd(f(x),Tf(x)) \vee rd(x,Tx). \end{aligned} \tag{4.1}$$

Let $K := rd(x,f(x)) \vee rd(f(x),Tf(x)) \vee rd(x,Tx)$.

Case $K = rd(x, f(x)) : d(f(x), Tf(x)) \le rd(x, f(x))$.

Case $K = rd(x, Tx) : d(f(x), Tf(x)) \le rd(x, Tx) \le rd(x, f(x))$.

Case $K = rd(f(x), Tf(x)) : d(f(x), Tf(x)) \le rd(f(x), Tf(x))$ which is impossible.

Hence

$$\begin{aligned} d(x, f(x)) &= \frac{1}{\varepsilon}((\varepsilon + r)d(x, f(x)) - rd(x, f(x))) \\ &\le \frac{1}{\varepsilon}\left((\varepsilon + r) \cdot \frac{1}{\varepsilon + r} d(x, Tx) - d(f(x), Tf(x))\right) = \varphi(x) - \varphi(f(x)). \end{aligned} \tag{4.2}$$

Thus T has a fixed point by Theorem 2.1.

(ii)⇒(i). Suppose X is not complete.

Define a function f as in the proof of Theorem 1.2 and a mapping T as follows:

for each $x \in X$, since $f(x) > 0$ and $\lim_n f(u_n) = 0$, there exists $v \in N$ satisfying $f(u_v) \le (\theta(r)r/(3 + r + \theta(r)r))f(x)$.

We put $Tx = \{u_n : f(u_n) \le (\theta(r)r/(3 + r + \theta(r)r))f(x)\}$ and write $g(x) = \sup_{y \in Tx} f(y)$.

It is obvious that $g(x) \le (\theta(r)r/(3 + r + \theta(r)r))f(x)$ for all $x \in X$. Since $f(y) < f(x)$, for all $y \in Tx$, for all $x \in X$, thus $x \notin Tx$. That is, T does not have a fixed point. Note that

$$f(x) - f(y) \le d(x, y) \le f(x) + f(y), \quad \forall y \in Tx. \tag{4.3}$$

We have

$$f(x) - g(x) \le d(x, Tx) \le f(x) + g(x), \tag{4.4}$$

$$H(Tx, Ty) \le g(x) + g(y). \tag{4.5}$$

Fix $x, y \in X$ with $\theta(r)d(x, Tx) \le d(x, y)$. To show that the mapping T satisfies the condition in (ii), that is, for all $x, y \in X$,

$$\theta(r)d(x, Tx) \le d(x, y) \text{ implies } H(Tx, Ty) \le rd(x, y) \vee rd(x, Tx) \vee rd(y, Ty). \tag{4.6}$$

Observe that

$$\begin{aligned} d(x, y) &\ge \theta(r)d(x, Tx) \\ &\ge \theta(r)(f(x) - g(x)) \\ &\ge \theta(r)\left(1 - \frac{\theta(r)r}{3 + r + \theta(r)r}\right)f(x) \\ &= \left(\frac{\theta(r)(3 + r)}{3 + r + \theta(r)r}\right)f(x). \end{aligned} \tag{4.7}$$

Case 1 ($f(y) \geq f(x)$).

$$
\begin{aligned}
H(Tx,Ty) &\leq g(x)+g(y) = \frac{3+r}{3}(g(x)+g(y)) - \frac{r}{3}(g(x)+g(y)) \quad \text{by (4.5)} \\
&\leq \frac{3+r}{3}\cdot\frac{\theta(r)r}{3+r+\theta(r)r}(f(x)+f(y)) - \frac{r}{3}(g(x)+g(y)) + \frac{r}{3}(f(y)-f(x)) \\
&\leq \frac{3+r}{3}\cdot\frac{r}{3+r}(f(x)+f(y)) - \frac{r}{3}(g(x)+g(y)) + \frac{r}{3}(f(y)-f(x)) \\
&\leq \frac{r}{3}d(x,Tx) + \frac{r}{3}d(y,Ty) + \frac{r}{3}d(x,y) \quad \text{by (4.4)} \\
&\leq rd(x,Tx) \vee rd(y,Ty) \vee d(x,y).
\end{aligned} \tag{4.8}
$$

Case 2 ($f(y)df(x)$, $d(x,y) < \theta(r)d(y,Ty)$).

$$
\begin{aligned}
H(Tx,Ty) &\leq g(x)+g(y) \leq \frac{\theta(r)r}{3+r+\theta(r)r}(f(x)+f(y)) \quad \text{by (4.5)} \\
&\leq \frac{\theta(r)r}{3+r}d(x,Tx) + \frac{\theta(r)r}{3+r+\theta(r)r}f(x) \\
&\leq \frac{\theta(r)r}{3+r}d(x,Tx) + \frac{\theta(r)r}{3+r+\theta(r)r}\,\frac{3+r+\theta(r)r}{\theta(r)(3+r)}d(x,y) \\
&\leq \frac{r}{3}d(x,Tx) + \frac{r}{3}d(x,y) \leq rd(x,Tx) \vee d(x,y) \vee rd(y,Ty).
\end{aligned} \tag{4.9}
$$

Therefore (4.6) holds.

It remains to show that the mapping $x \mapsto d(x,Tx)$ is lower semicontinuous, that is,

$$
\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d(y,Ty) \geq d(x,Tx) - \varepsilon, \quad \forall y \in B_d(x,\delta). \tag{4.10}
$$

Suppose not, then there exists $\varepsilon > 0$ such that $d(y_k,Ty_k) < d(x,Tx) - \varepsilon$, for all $y_k \in B_d(x,(1/k))$, for each k. Since $d(y_k,Ty_k) = \inf_{u_m^k \in Ty_k} d(y_k,u_m^k), \exists \{u_m^k\} \subseteq Ty_k$ such that $\lim_m d(y_k,u_m^k) = d(y_k,Ty_k)$. We have $d(y_k,u_m^k) - (1/k) \leq d(y_k,Ty_k) < d(x,Tx) - \varepsilon$, for all large m. Thus for each k, $d(x,u_m^k) - d(x,y_k) - (1/k) \leq d(y_k,u_m^k) - (1/k) \leq d(y_k,Ty_k) < d(x,Tx) - \varepsilon$, for all large m. So for those m, $d(x,u_m^k) - (2/k) + \varepsilon < d(x,Tx)$. Consequently,

$$
f(x) - \frac{2}{k} + \varepsilon = \lim_m \left(d\left(x,u_m^k\right) - \frac{2}{k} + \varepsilon \right) \leq d(x,Tx), \quad \forall k, \tag{4.11}
$$

which impliest that

$$f(x)+\varepsilon=\lim_{k}\left(f(x)-\frac{2}{k}+\varepsilon\right)$$
$$\leq d(x,Tx)\leq d(x,u_m),\qquad \forall u_m\in Tx$$
$$\leq \lim_{m} d(x,u_m)$$
$$=\lim_{n} d(x,u_n)=f(x), \tag{4.12}$$

a contradiction. Thus the mapping $x\mapsto d(x,Tx)$ is lower semicontinuous. □

The converse of Theorem 1.4 is also valid by following the same proof of Theorem 1.2. Assuming that X is not complete, we find a fixed point free mapping T satisfying the condition in Theorem 1.4. Following the same proof of Theorem 1.2 by replacing $\eta r/(3+\eta r)$ by $\beta/(1+\beta)$ where $\beta=b\wedge c$, we obtain $f(Tx)\leq \beta/(1+\beta)f(x)$ for all $x\in X$ and T is fixed point free. We now verify the condition in Theorem 1.4 for T.

Fix $x,y\in X$ with $(1-b-c)/(1+a)d(x,Tx)\leq d(x,y)$. We show that

$$H(Tx,Ty)\leq ad(x,y)+bd(x,Tx)+cd(y,Ty). \tag{4.13}$$

Observe that

$$d(x,y)\geq\frac{1-b-c}{1+a}d(x,Tx)\geq\frac{1-b-c}{1+a}(f(x)-f(Tx))$$
$$\geq\frac{1-b-c}{1+a}\left(1-\frac{\beta}{1+\beta}\right)f(x)=\frac{1-b-c}{1+a}\left(\frac{1}{1+\beta}\right)f(x). \tag{4.14}$$

Case 1 ($f(y)\geq f(x)$).

$$H(Tx,Ty)\leq f(Tx)+f(Ty)$$
$$=(1+\beta)(f(Tx)+f(Ty))-\beta(f(Tx)+f(Ty))$$
$$\leq(1+\beta)\cdot\frac{\beta}{1+\beta}(f(x)+f(y))-\beta(f(Tx)+f(Ty))$$
$$\leq\beta d(x,Tx)+\beta d(y,Ty)\leq ad(x,y)+bd(x,Tx)+cd(y,Ty). \tag{4.15}$$

Case 2 ($f(y) \le f(x)$, $d(x,y) < (1-b-c)/(1+a)d(y,Ty)$).

$$\begin{aligned} H(Tx,Ty) &\le f(Tx) + f(Ty) \le \frac{\beta}{1+\beta}(f(x)+f(y)) \\ &\le \beta d(x,Tx) + \frac{\beta}{1+\beta} f(x) \le \beta d(x,Tx) + \frac{\beta(1+\beta)}{1+\beta}\frac{1+a}{1-b-c} d(x,y) \\ &\le \beta d(x,Tx) + \frac{\beta(1+a)}{1-b-c}\frac{1-b-c}{1+a} d(y,Ty) = \beta d(x,Tx) + \beta d(y,Ty) \\ &\le ad(x,y) + bd(x,Tx) + cd(y,Ty). \end{aligned} \tag{4.16}$$

Therefore (4.13) holds, and the proof of the converse of Theorem 1.4 is complete.

Moreover, by following the proof of Theorem 1.4, we can partially extend the class of mappings and still obtain their fixed points. Notice that $(1-2\beta)/(1+\gamma) \le 1/(2\beta+\gamma+1)$.

Theorem 4.2. *Let (X,d) be a metric space. Then the following are equivalent:*

(i) X *is complete.*

(ii) *every mapping $T : X \to Cl(X)$ such that for each $\beta,\gamma \in \mathbb{R}_+$ with $2\beta+\gamma \in [0,1)$ and for each $x,y \in X$,*

$$\frac{1}{2\beta+\gamma+1} d(x,Tx) \le d(x,y) \text{ implies } H(Tx,Ty) \le \beta d(x,Tx) + \beta d(y,Ty) + \gamma d(x,y), \tag{4.17}$$

has a fixed point.

Proof. (i)⇒(ii). Following the same proof of Theorem 1.4 by replacing $\eta := (1-b-c)/(1+a)$ in its proof by $\eta := 1/(2\beta+\gamma+1)$. Thus we obtain a sequence $\{x_n\}$ such that

(1) $x_{n+1} \in T(x_n)$, for each $n \in N$ and;

(2) $d(x_n, x_{n+1}) \le (k(\beta+\gamma)/(1-k\beta))^n d(x_0,x_1)$, for $n \in N$.

Choose k so that $1 < k < 1/(\gamma+2\beta)$ and therefore $0 \le k(\beta+\gamma)/(1-k\beta) < 1$. We see that the sequence $\{x_n\}$ is Cauchy in X, and so $\{x_n\}$ converges to some $z \in X$. We show $d(z,Tx) \le \gamma d(z,x) + \beta d(x,Tx)$, for each $x \in X \setminus \{z\}$.

Suppose $x \ne z$. Since $x_n \to z$ as $n \to \infty$, there exists $n_0 \in N$ such that $d(x_n,z) \le (1/3)d(z,x)$ for each $n \ge n_0$. We have $\eta d(x_n,Tx_n) \le d(x_n,Tx_n) \le d(x_n,x_{n+1}) \le d(x_n,z) + d(z,x_{n+1}) \le (2/3)d(x,z) \le d(x,z) - d(x_n,z) \le d(x_n,x)$. Hence $d(x_{n+1},Tx) \le H(Tx_n,Tx) \le \beta d(x_n,Tx_n) + \beta d(x,Tx) + \gamma d(x_n,x)$ for each $n \ge n_0$. Letting $n \to \infty$, we get $d(z,Tx) \le \gamma d(z,x) + \beta d(x,Tx)$, for all $x \ne z$ as desired.

Next, we show $H(Tx,Tz) \le \beta d(x,Tx) + \beta d(z,Tz) + \gamma d(x,z)$, for all $x \in X$. For $x \ne z$, we obtain for each $n \in N$, $y_n \in Tx$ such that $d(z,y_n) \le d(z,Tx) + (1/n)d(x,z)$. Clearly $d(x,Tx) \le d(x,y_n) \le d(x,z) + d(z,y_n) \le d(x,z) + d(z,Tx) + (1/n)d(x,z) \le (1+\gamma+(1/n))d(x,z) + \beta d(x,Tx)$, for all $n \in N$. Hence, as $n \to \infty$ we get $(1-\beta)d(x,Tx) \le (1+\gamma)d(x,z)$ and so $\eta d(x,Tx) \le d(x,z)$ implying that $H(Tx,Tz) \le \beta d(x,Tx) + \beta d(z,Tz) + \gamma d(x,z)$ for $x \in X$.

Finally, we obtain

$$d(z,Tz) = \lim_n d(x_{n+1},Tz) \le \lim_n H(Tx_n,Tz), \\ \le \lim_n \left(\beta d(x_n,Tx_n) + \beta d(z,Tz) + \gamma d(x_n,z)\right) = 0. \tag{4.18}$$

Thus $z = Tz$ and T has a fixed point.

(ii)⇒(i). Let $\beta = 0$, and $\gamma = r$, we have $(1/(r+1))d(x,Tx) \le d(x,y)$ implying $H(Tx,Ty) \le rd(x,y)$. Hence X is complete by the converse of Theorem 1.4. □

5. Caristi Set-Valued Mappings

In 2008, Ćirić [10] proved the following fixed point theorems.

Theorem 5.1 ([10]). *Let (X,d) be a complete metric space and let $T : X \to Cl(X)$. If there exist constants $b,c \in (0,1), c < b$, such that for any $x \in X$ there is $y \in Tx$ satisfying the following two conditions:*

$$bd(x,y) \le d(x,Tx), \qquad d(y,Ty) \le cd(x,y). \tag{5.1}$$

Then T has a fixed point in X provided a function $f(x) = d(x,Tx)$ is lower semicontinuous.

Theorem 5.2 ([10]). *Let (X,d) be a complete metric space and $T : X \to Cl(X)$. If there exists a function $\varphi : [0,\infty) \to [0,1)$ satisfying*

$$\limsup_{r\to t^+} \varphi(r) < 1, \quad \text{for each t} \in [0,\infty), \tag{5.2}$$

and such that for any $x \in X$ there is $y \in Tx$ satisfying the following two conditions:

$$d(x,y) \le (2-\varphi(d(x,y)))d(x,Tx), \qquad d(y,Ty) \le \varphi(d(x,y))d(x,y). \tag{5.3}$$

Then T has a fixed point in X provided a function $f(x) = d(x,Tx)$ is lower semicontinuous.

We give a simple proof of each of these theorems.

Proof of Theorem 5.1. Define a lower semi-continuous function φ by $\varphi(x) = 1/(b-c)d(x,Tx)$. For any $x \in X$, we can find some $f(x) \in Tx$ satisfying

$$bd(x,f(x)) \le d(x,Tx), \qquad d(f(x),Tf(x)) \le cd(x,f(x)). \tag{5.4}$$

We show that $d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),\ x \in X$. Let $x \in X$. Clearly,

$$\begin{aligned} d(x, f(x)) &= \frac{1}{b-c}\big(bd(x, f(x)) - cd(x, f(x))\big) \leq \frac{1}{b-c}\big(bd(x, f(x)) - d(f(x), Tf(x))\big) \\ &\leq \frac{1}{b-c}\big(d(x, Tx) - d(f(x), Tf(x))\big) = \varphi(x) - \varphi(f(x)). \end{aligned} \tag{5.5}$$

Hence T has a fixed point by Theorem 2.1. □

Proof of Theorem 5.2. Let $k = \inf (\varphi(r) - 1)^2/(2 - \varphi(r)) > 0$ and $\psi(x) = (1/k)d(x, Tx)$. For each $x \in X$, there exists $f(x) \in Tx$ such that

$$d(x, f(x)) \leq (2 - \varphi(d(x, f(x))))d(x, Tx), \qquad d(f(x), Tf(x)) \leq \varphi(d(x, f(x)))d(x, f(x)). \tag{5.6}$$

Furthermore, $d(x, f(x)) \leq \psi(x) - \psi(f(x))$, $x \in X$. Indeed,

$$\begin{aligned} d(x, f(x)) &= \frac{1}{k}\big((\varphi(d(x, f(x))) + k)d(x, f(x)) - \varphi(d(x, f(x)))d(x, f(x))\big) \\ &\leq \frac{1}{k}\left(\left(\varphi(d(x, f(x))) + \frac{(\varphi(d(x, f(x))) - 1)^2}{2 - \varphi(d(x, f(x)))}\right)d(x, f(x)) \right. \\ &\quad \left. -\varphi(d(x, f(x)))d(x, f(x))\right) \\ &\leq \frac{1}{k}\left(\left(\frac{1}{2 - \varphi(d(x, f(x)))}\right)d(x, f(x)) - \varphi(d(x, f(x)))d(x, f(x))\right) \\ &\leq \frac{1}{k}\big(d(x, Tx) - d(f(x), Tf(x))\big) = \psi(x) - \psi(f(x)). \end{aligned} \tag{5.7}$$

Thus T has a fixed point by Theorem 2.1. □

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References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.
[2] E. H. Connell, "Properties of fixed point spaces," *Proceedings of the American Mathematical Society*, vol. 10, no. 6, pp. 974–979, 1959.
[3] R. Kannan, "Some results on fixed points—II," *The American Mathematical Monthly*, vol. 76, no. 4, pp. 405–408, 1969.
[4] P. V. Subrahmanyam, "Completeness and fixed-points," *Monatshefte für Mathematik*, vol. 80, no. 4, pp. 325–330, 1975.

[5] T. Suzuki, “A generalized Banach contraction principle that characterizes metric completeness,” *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.
[6] M. Kikkawa and T. Suzuki, “Three fixed point theorems for generalized contractions with constants in complete metric spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2942–2949, 2008.
[7] G. Moţ and A. Petruşel, “Fixed point theory for a new type of contractive multivalued operators,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 9, pp. 3371–3377, 2009.
[8] A. Petruşel, “Caristi type operators and applications,” *Studia Universitatis Babeş-Bolyai. Mathematica*, vol. 48, no. 3, pp. 115–123, 2003.
[9] M. Kikkawa and T. Suzuki, “Some similarity between contractions and Kannan mappings,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 649749, 8 pages, 2008.
[10] L. Ćirić, “Fixed point theorems for multi-valued contractions in complete metric spaces,” *Journal of Mathematical Analysis and Applications*, vol. 348, no. 1, pp. 499–507, 2008.