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Research Article

Convergence on Composite Iterative Schemes for Nonexpansive Mappings in Banach Spaces

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Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E, $f : C \to C$ a contractive mapping (or a weakly contractive mapping), and $T : C \to C$ nonexpansive mapping with the fixed point set $F(T) \neq \emptyset$. Let $\{x_n\}$ be generated by a new composite iterative scheme: $y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n$, $x_{n+1} = (1-\beta_n)y_n + \beta_n Ty_n$, $(n \geq 0)$. It is proved that $\{x_n\}$ converges strongly to a point in $F(T)$, which is a solution of certain variational inequality provided that the sequence $\{\lambda_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=1}^{\infty}\lambda_n = \infty$, $\{\beta_n\} \subset [0,a)$ for some $0 < a < 1$ and the sequence $\{x_n\}$ is asymptotically regular.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $x, y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. That is, $\Sigma_C = \{f : C \to C \mid f \text{ is a contraction with constant } k\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in C.

Now let $T : C \to C$ be a nonexpansive mapping (recall that a mapping $T : C \to C$ is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) and $F(T)$ denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

We consider the iterative scheme: for T nonexpansive mapping, $f \in \Sigma_C$ and $\lambda_n \in (0,1)$,

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)Tx_n, \quad n \geq 0. \tag{1.1}$$

As a special case of (1.1), the following iterative scheme:

$$z_{n+1} = \lambda_n u + (1 - \lambda_n)Tz_n, \quad n \geq 0, \tag{1.2}$$

where $u, z_0 \in C$ are arbitrary (but fixed), has been investigated by many authors; see, for example, Cho et al. [1], Halpern [2], Lions [3], Reich [4, 5], Shioji and Takahashi [6], Wittmann [7], and Xu [8]. The authors above showed that the sequence $\{z_n\}$ generated by (1.2) converges strongly to a point in the fixed point set $F(T)$ under appropriate conditions on $\{\lambda_n\}$ in either Hilbert spaces or certain Banach spaces. Recently, many authors also considered the iterative scheme (1.2) for finite or countable families of nonexpansive mappings $\{T_i\}_{i\in\{1,2,\ldots,r \text{ or } \infty\}}$; see, for instance, [9–14].

The viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping in a Hilbert space was proposed by Moudafi [15] (see [16] for finding hierarchically a fixed point). In 2004, Xu [17] extended Theorem 2.2 of Moudafi [15] for the iterative scheme (1.1) to a Banach space setting using the following conditions on $\{\lambda_n\}$:

(H1) $\lim_{n\to\infty}\lambda_n = 0$; $\sum_{n=0}^{\infty}\lambda_n = \infty$ or, equivalently, $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$;

(H2) $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$ or $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$.

We also refer to [18–23] for the iterative scheme (1.1) for finite of countable families of nonexpansive mappings $\{T_i\}_{i\in\{1,2,\ldots,r \text{ or } \infty\}}$. For the iterative scheme (1.1) with generalized contractive mappings instead of contractions, see [22, 24]. We can refer to [25] for the general iteration method for finding a zero of accretive operator.

Recently, Kim and Xu [26] provided a simpler modification of Mann iterative scheme (1.3) in a uniformly smooth Banach space as follows:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)y_n, \end{aligned} \tag{1.3}$$

where $u \in C$ is an arbitrary (but fixed) element, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. They proved that $\{x_n\}$ generated by (1.3) converges to a fixed point of T under the control conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$;

(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$, (or equivalently, $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$), $\sum_{n=0}^{\infty}\beta_n = \infty$;

(iii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n| < \infty$.

In this paper, motivated by the above-mentioned results, as the viscosity approximation method, we consider a new composite iterative scheme for nonexpansive mapping T:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \\ x_{n+1} &= (1-\beta_n)y_n + \beta_n T y_n, \end{aligned} \tag{IS}$$

where $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$. First, we prove the strong convergence of the sequence $\{x_n\}$ generated by (IS) under the suitable conditions on the control parameters $\{\beta_n\}$ and $\{\lambda_n\}$ and the asymptotic regularity on $\{x_n\}$ in reflexive Banach space with a uniformly Gâteaux differentiable norm together with the assumption that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Moreover, we show that the strong limit is a solution of certain variational inequality. Next, we study the viscosity approximation with the weakly contractive mapping to a fixed point of nonexpansive mapping in the same Banach space. The main results improve and complement the corresponding results of [1–8, 15, 17]. In particular, if $\beta_n = 0$, for all $n \geq 0$, then (IS) reduces to (1.1). We point out that the iterative scheme (IS) is a new one for finding a fixed point of T.

2. Preliminaries and lemmas

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$) will denote strong (resp., weak) convergence of the sequence $\{x_n\}$ to x.

The (*normalized*) *duality mapping* J from E into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual E^* is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\} \tag{2.1}$$

for each $x \in E$ [27].

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. The norm is said to be *uniformly Gâteaux differentiable* if for $y \in U$, the limit is attained uniformly for $x \in U$. The space E is said to have a *uniformly Fréchet differentiable norm* (and E is said to be *uniformly smooth*) if the limit in (2.2) is attained uniformly for $(x, y) \in U \times U$. It is known that E is smooth if and only if each duality mapping J is single-valued. It is also well known that if E has a uniformly Gâteaux differentiable norm, J is uniformly norm to weak continuous on each bounded subset of E [27].

Let C be a nonempty closed convex subset of E. C is said to have the *fixed point property* for nonexpansive mappings if every nonexpansive mapping of a bounded closed convex subset D of C has a fixed point in D.

Let D be a subset of C. Then, a mapping $Q : C \to D$ is said to be a retraction from C onto D if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to be *sunny* if $Q(Qx + t(x - Qx)) = Qx$ for all $x \in C$ and $t \geq 0$ with $Qx + t(x - Qx) \in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is well known [28, page 48] that Q is a sunny nonexpansive retraction from C onto D if and only if the following condition holds:

$$\langle x - Qx, J(z - Qx)\rangle \leq 0, \quad x \in C, z \in D. \tag{2.3}$$

We need the following lemmas for the proof of our main results. (Lemma 2.1 was also given by Jung and Morales [29] and Lemma 2.2 is essentially Lemma 2 of Liu [30] (also see [8]).)

Lemma 2.1. *Let X be a real Banach space and let J be the duality mapping. Then, for any given $x, y \in X$, one has*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle \tag{2.4}$$

for all $j(x + y) \in J(x + y)$.

Lemma 2.2. *Let* $\{s_n\}$ *be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\gamma_n + \delta_n, \quad n \geq 0, \tag{2.5}$$

where $\{\alpha_n\}$, $\{\gamma_n\}$, *and* $\{\delta_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$,

(ii) $\limsup_{n\to\infty}\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty}\alpha_n\gamma_n < \infty$,

(iii) $\delta_n \geq 0\ (n \geq 0)$, $\sum_{n=0}^{\infty}\delta_n < \infty$.

Then, $\lim_{n\to\infty}s_n = 0$.

Recall that a mapping $A : C \to C$ is said to be *weakly contractive* if

$$\|Ax - Ay\| \leq \|x-y\| - \psi(\|x-y\|), \quad \forall x, y \in C, \tag{2.6}$$

where $\psi : [0,+\infty) \to [0,+\infty)$ is a continuous and strictly increasing function such that ψ is positive on $(0,\infty)$ and $\psi(0) = 0$. As a special case, if $\psi(t) = (1-k)t$ for $t \in [0,+\infty)$, where $k \in (0,1)$, then the weakly contractive mapping A is a contraction with constant k. Rhoades [31] obtained the following result for weakly contractive mapping.

Lemma 2.3 (see [31, Theorem 2]). *Let* (X,d) *be a complete metric space, and* A *a weakly contractive mapping on* X. *Then,* A *has a unique fixed point* p *in* X. *Moreover, for* $x \in X$, $\{A^n x\}$ *converges strongly to* p.

The following lemma was given in [32, 33].

Lemma 2.4. *Let* $\{s_n\}$ *and* $\{\gamma_n\}$ *be two sequences of nonnegative real numbers and* $\{\lambda_n\}$ *a sequence of positive numbers satisfying the conditions*

(i) $\sum_{n=0}^{\infty}\lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$,

(ii) $\lim_{n\to\infty}(\gamma_n/\lambda_n) = 0$.

Let the recursive inequality

$$s_{n+1} \leq s_n - \lambda_n\psi(s_n) + \gamma_n, \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

be given where $\psi(t)$ *is a continuous and strict increasing function on* $[0,+\infty)$ *with* $\psi(0) = 0$. *Then,* $\lim_{n\to\infty}s_n = 0$.

Finally, the sequence $\{x_n\}$ in E is said to be *asymptotically regular* if

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.8}$$

3. Main results

First, using the asymptotic regularity, we study a strong convergence theorem for a composite iterative scheme for the nonexpansive mapping with the contractive mapping.

For $T : C \to C$ nonexpansive and so for any $t \in (0,1)$ and $f \in \Sigma_C$, $tf + (1-t)T : C \to C$ defines a strict contraction mapping. Thus, by the Banach contraction mapping principle, there exists a unique fixed point x_t^f satisfying

$$x_t^f = tf(x_t^f) + (1-t)Tx_t^f. \tag{R}$$

For simplicity, we will write x_t for x_t^f provided no confusion occurs.

In 2006, the following result was given by Jung [18] (see also Xu [17] for the result in uniformly smooth Banach spaces).

Theorem J (see Jung [18]). *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and T nonexpansive mapping from C into itself with $F(T) \neq \varnothing$. Then, $\{x_t\}$ defined by* (R) *converges strongly to a point in $F(T)$. If one defines $Q : \Sigma_C \to F(T)$ by*

$$Q(f) := \lim_{t \to 0^+} x_t, \quad f \in \Sigma_C, \tag{3.1}$$

then $Q(f)$ solves a variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p) \rangle \leq 0, \quad f \in \Sigma_C,\ p \in F(T). \tag{3.2}$$

Remark 3.1. In Theorem J, if $f(x) = u \in C$ is a constant, then (3.2) becomes

$$\langle Q(u) - u,\ J(Q(u)-p) \rangle \leq 0, \quad u \in C,\ p \in F(T). \tag{3.3}$$

Hence by (2.3), Q reduces to the sunny nonexpansive retraction from C to $F(T)$. Namely, $F(T)$ is a sunny nonexpansive retraction of C.

Using Theorem J and the asymptotic regularity on the sequence $\{x_n\}$, we have the following result.

Theorem 3.2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and T nonexpansive mappings from C into itself with $F(T) \neq \varnothing$. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences in $(0,1)$ which satisfies the conditions:*

(B1) $\beta_n \in [0, a)$ *for some* $0 < a < 1$ *for all* $n \geq 0$,

(C1) $\lim_{n\to\infty} \lambda_n = 0$; $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Let $f \in \Sigma_C$ and $x_0 \in C$ be chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by

$$\begin{gathered} x_0 = x \in C, \\ y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad n \geq 0. \end{gathered} \tag{IS}$$

If $\{x_n\}$ is asymptotically regular, then $\{x_n\}$ converges strongly to $Q(f) \in F(T)$, where $Q(f)$ is the unique solution of the variational inequality

$$\langle (I-f)(Q(f)),\ J(Q(f)-p) \rangle \leq 0, \quad f \in \Sigma_C,\ p \in F(T). \tag{3.4}$$

Proof. We notice that by Theorem J, there exists a solution $Q(f)$ of a variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C,\ p \in F(T). \tag{3.5}$$

Namely, $Q(f) = \lim_{t\to 0^+} x_t$, where x_t is defined by (R). We will show that $x_n \to Q(f)$.

We proceed with the following steps.

Step 1. We show that $\|x_n - z\| \leq \max\{\|x_0 - z\|, (1/(1-k))\|f(z)-z\|\}$ for all $n \geq 0$ and all $z \in F(T)$ and so $\{x_n\}$, $\{y_n\}$, $\{f(x_n)\}$, $\{Tx_n\}$, and $\{Ty_n\}$ are bounded.

Indeed, let $z \in F(T)$. Then, we have

$$\begin{aligned}
\|y_n - z\| &= \|\lambda_n(f(x_n)-z) + (1-\lambda_n)(Tx_n - z)\| \\
&\leq \lambda_n\|f(x_n)-z\| + (1-\lambda_n)\|x_n - z\| \\
&\leq \lambda_n(\|f(x_n)-f(z)\| + \|f(z)-z\|) + (1-\lambda_n)\|x_n - z\| \\
&\leq \lambda_n k\|x_n - z\| + \lambda_n\|f(z)-z\| + (1-\lambda_n)\|x_n - z\| \\
&= (1-(1-k)\lambda_n)\|x_n - z\| + \lambda_n\|f(z)-z\| \\
&\leq \max\left\{\|x_n - z\|, \frac{1}{1-k}\|f(z)-z\|\right\}, \\
\|x_{n+1} - z\| &= \|(1-\beta_n)(y_n - z) + \beta_n(Ty_n - z)\| \\
&\leq (1-\beta_n)\|y_n - z\| + \beta_n\|y_n - z\| \\
&= \|y_n - z\| \leq \max\left\{\|x_n - z\|, \frac{1}{1-k}\|f(z)-z\|\right\}.
\end{aligned} \tag{3.6}$$

Using an induction, we obtain

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{1}{1-k}\|f(z)-z\|\right\} \tag{3.7}$$

for all $n \geq 0$. Hence, $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{Tx_n\}$, $\{Ty_n\}$, and $\{f(x_n)\}$. Moreover, it follows from condition (C1) that

$$\|y_n - Tx_n\| = \lambda_n\|f(x_n) - Tx_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.8}$$

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Indeed, by the condition (B1)

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \beta_n\|Ty_n - y_n\| \\
&\leq \beta_n(\|Ty_n - Tx_n\| + \|Tx_n - y_n\|) \\
&\leq a(\|y_n - x_n\| + \|Tx_n - y_n\|) \\
&\leq a(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|Tx_n - y_n\|)
\end{aligned} \tag{3.9}$$

which implies that

$$\|x_{n+1} - y_n\| \leq \frac{a}{1-a}(\|x_{n+1} - x_n\| + \|Tx_n - y_n\|). \tag{3.10}$$

So, by asymptotic regularity of $\{x_n\}$ and (3.8), we have $\|x_{n+1} - y_n\| \to 0$, and also

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{3.11}$$

Step 3. We show that $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$. By (3.8) and Step 2, we have

$$\begin{aligned}\|y_n - Ty_n\| &\le \|y_n - Tx_n\| + \|Tx_n - Ty_n\| \\ &\le \|y_n - Tx_n\| + \|x_n - y_n\| \longrightarrow 0.\end{aligned} \tag{3.12}$$

Step 4. We show that $\limsup_{n\to\infty}\langle Q(f) - f(Q(f)), J(Q(f) - y_n)\rangle \le 0$. To prove this, let a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ be such that

$$\limsup_{n\to\infty}\langle Q(f) - f(Q(f)), J(Q(f) - y_n)\rangle = \lim_{j\to\infty}\langle Q(f) - f(Q(f)), J(Q(f) - y_{n_j})\rangle \tag{3.13}$$

and $y_{n_j} \rightharpoonup p$ for some $p \in E$. From Step 3, it follows that $\lim_{j\to\infty}\|y_{n_j} - Ty_{n_j}\| = 0$.

Now let $Q(f) = \lim_{t\to 0^+} x_t$, where $x_t = tf(x_t) + (1-t)Tx_t$. Then, we can write

$$x_t - y_{n_j} = t(f(x_t) - y_{n_j}) + (1-t)(Tx_t - y_{n_j}). \tag{3.14}$$

Putting

$$a_j(t) = (1-t)^2\|Ty_{n_j} - y_{n_j}\|(2\|x_t - y_{n_j}\| + \|Ty_{n_j} - y_{n_j}\|) \longrightarrow 0 \quad (j \longrightarrow \infty) \tag{3.15}$$

by Step 3 and using Lemma 2.1, we obtain

$$\begin{aligned}\|x_t - y_{n_j}\|^2 &\le (1-t)^2\|Tx_t - y_{n_j}\|^2 + 2t\langle f(x_t) - y_{n_j}, J(x_t - y_{n_j})\rangle \\ &\le (1-t)^2(\|Tx_t - Ty_{n_j}\| + \|Ty_{n_j} - y_{n_j}\|)^2 + 2t\langle f(x_t) - x_t, J(x_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|^2 \\ &\le (1-t)^2\|x_t - y_{n_j}\|^2 + a_j(t) + 2t\langle f(x_t) - x_t, J(x_t - y_{n_j})\rangle + 2t\|x_t - y_{n_j}\|^2.\end{aligned} \tag{3.16}$$

The last inequality implies

$$\langle x_t - f(x_t), J(x_t - y_{n_j})\rangle \le \frac{t}{2}\|x_t - y_{n_j}\|^2 + \frac{1}{2t}a_j(t). \tag{3.17}$$

It follows that

$$\limsup_{j\to\infty}\langle x_t - f(x_t), J(x_t - y_{n_j})\rangle \le \frac{t}{2}M, \tag{3.18}$$

where $M > 0$ is a constant such that $M \ge \|x_t - y_n\|^2$ for all $n \ge 0$ and $t \in (0,1)$. Taking the lim sup as $t \to 0$ in (3.18) and noticing the fact that the two limits are interchangeable due to the fact that J is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*, we have

$$\limsup_{j\to\infty}\langle Q(f) - f(Q(f)), J(Q(f) - y_{n_j})\rangle \le 0. \tag{3.19}$$

Indeed, letting $t \to 0$, from (3.18) we have

$$\limsup_{t\to 0}\limsup_{j\to\infty}\langle x_t - f(x_t),\, J(x_t - y_{n_j})\rangle \le 0. \tag{3.20}$$

So, for any $\varepsilon > 0$, there exists a positive number δ_1 such that for any $t \in (0, \delta_1)$,

$$\limsup_{j\to\infty}\langle x_t - f(x_t),\, J(x_t - y_{n_j})\rangle \le \frac{\varepsilon}{2}. \tag{3.21}$$

Moreover, since $x_t \to Q(f)$ as $t \to 0$, the set $\{x_t - y_{n_j}\}$ is bounded and the duality mapping J is norm-to-weak* uniformly continuous on bounded subset of E, there exists $\delta_2 > 0$ such that, for any $t \in (0, \delta_2)$,

$$\begin{aligned}
&\left|\langle Q(f) - f(Q(f)),\, J(Q(f) - y_{n_j})\rangle - \langle x_t - f(x_t),\, J(x_t - y_{n_j})\rangle\right| \\
&\quad= \left|\langle Q(f) - f(Q(f)),\, J(Q(f) - y_{n_j}) - J(x_t - y_{n_j})\rangle \right.\\
&\qquad\left. + \langle Q(f) - f(Q(f)) - (x_t - f(x_t)),\, J(x_t - y_{n_j})\rangle\right| \\
&\quad\le \left|\langle Q(f) - f(Q(f)),\, J(x_t - y_{n_j}) - J(Q(f) - y_{n_j})\rangle\right| \\
&\qquad + \|Q(f) - f(Q(f)) - (x_t - f(x_t))\|\,\|x_t - y_{n_j}\| < \frac{\varepsilon}{2}.
\end{aligned} \tag{3.22}$$

Choose $\delta = \min\{\delta_1, \delta_2\}$, we have for all $t \in (0, \delta)$ and $j \in \mathbb{N}$,

$$\langle Q(f) - f(Q(f)),\, J(Q(f) - y_{n_j})\rangle < \langle x_t - f(x_t),\, J(x_t - y_{n_j})\rangle + \frac{\varepsilon}{2}, \tag{3.23}$$

which implies that

$$\limsup_{j\to\infty}\langle Q(f) - f(Q(f)),\, J(Q(f) - y_{n_j})\rangle \le \limsup_{j\to\infty}\langle x_t - f(x_t),\, J(x_t - y_{n_j})\rangle + \frac{\varepsilon}{2}. \tag{3.24}$$

Since $\limsup_{j\to\infty}\langle x_t - f(x_t),\, J(x_t - y_{n_j})\rangle \le \varepsilon/2$, we have

$$\limsup_{j\to\infty}\langle Q(f) - f(Q(f)),\, J(Q(f) - y_{n_j})\rangle \le \varepsilon. \tag{3.25}$$

Since ε is arbitrary, we obtain that

$$\limsup_{j\to\infty}\langle Q(f) - f(Q(f)),\, J(Q(f) - y_{n_j})\rangle \le 0. \tag{3.26}$$

Step 5. We show that $\lim_{n\to\infty}\|x_n - Q(f)\| = 0$. By using (IS), we have

$$\|x_{n+1} - Q(f)\| \le \|y_n - Q(f)\| = \|\lambda_n(f(x_n) - Q(f)) + (1 - \lambda_n)(Tx_n - Q(f))\|. \tag{3.27}$$

Applying Lemma 2.1, we obtain

$$
\begin{aligned}
\|x_{n+1} - Q(f)\|^2 &\leq \|y_n - Q(f)\|^2 \\
&\leq (1-\lambda_n)^2 \|Tx_n - Q(f)\|^2 + 2\lambda_n \langle f(x_n) - Q(f), J(y_n - Q(f))\rangle \\
&\leq (1-\lambda_n)^2 \|x_n - Q(f)\|^2 + 2\lambda_n \langle f(x_n) - f(Q(f)), J(y_n - Q(f))\rangle \\
&\quad + 2\lambda_n \langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle \\
&\leq (1-\lambda_n)^2 \|x_n - Q(f)\|^2 + 2k\lambda_n \|x_n - Q(f)\| \|y_n - Q(f)\| \\
&\quad + 2\lambda_n \langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle \\
&\leq (1-\lambda_n)^2 \|x_n - Q(f)\|^2 + 2k\lambda_n \|x_n - Q(f)\|^2 \\
&\quad + 2\lambda_n \langle f(Q(f)) - Q(f), J(y_n - Q(f))\rangle.
\end{aligned}
\tag{3.28}
$$

It then follows that

$$
\begin{aligned}
\|x_{n+1} - Q(f)\|^2 &\leq (1 - 2(1-k)\lambda_n + \lambda_n^2) \|x_n - Q(f)\|^2 + 2\lambda_n \langle Q(f) - f(Q(f)), J(Q(f) - y_n)\rangle \\
&\leq (1 - (2-k)\lambda_n) \|x_n - Q(f)\|^2 + \lambda_n^2 M^2 + 2\lambda_n \langle Q(f) - f(Q(f)), J(Q(f) - y_n)\rangle,
\end{aligned}
\tag{3.29}
$$

where $M = \sup_{n \geq 0} \|x_n - Q(f)\|$. Put

$$
\begin{aligned}
\alpha_n &= 2(1-k)\lambda_n, \\
\gamma_n &= \frac{\lambda_n}{2(1-k)} M^2 + \frac{1}{1-k} \langle Q(f) - f(Q(f)), J(Q(f) - y_n)\rangle.
\end{aligned}
\tag{3.30}
$$

From the condition (C1) and Step 4, it follows that $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} \gamma_n \leq 0$. Since (3.29) reduces to

$$
\|x_{n+1} - Q(f)\|^2 \leq (1-\alpha_n) \|x_n - Q(f)\|^2 + \alpha_n \gamma_n,
\tag{3.31}
$$

from Lemma 2.2 with $\delta_n = 0$, we conclude that $\lim_{n\to\infty} \|x_n - Q(f)\| = 0$. This completes the proof. □

Corollary 3.3. *Let E be a uniformly smooth Banach space. Let C, T, f, $\{\beta_n\}$, $\{\lambda_n\}$, f, x_0, and $\{x_n\}$ be the same as in Theorem 3.2. Then, the conclusion of Theorem 3.2 still holds.*

Proof. Since E is a uniformly smooth Banach space, E is reflexive and the norm is uniformly Gâteaux differentiable norm and its every nonempty weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Thus, the conclusion of Corollary 3.3 follows from Theorem 3.2 immediately. □

Remark 3.4. (1) If $\{\beta_n\}$ and $\{\lambda_n\}$ in Theorem 3.2 satisfy the conditions

(B2) $\sum_{n=0}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(C1) $\lim_{n\to\infty}\lambda_n = 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$,

(C2) $\sum_{n=0}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$, or

(C3) $\lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$, or

(C4) $|\lambda_{n+1} - \lambda_n| \leq \circ(\lambda_{n+1}) + \sigma_n$, $\sum_{n=0}^{\infty}\sigma_n < \infty$ (the perturbed control condition),

then the sequence $\{x_n\}$ generated by (IS) is asymptotically regular. Now, we only give the proof in case when $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions (B2), (C1), and (C4). Indeed, from (IS), we have for every $n \geq 1$,

$$\begin{aligned} y_n &= \lambda_n f(x_n) + (1 - \lambda_n)Tx_n, \\ y_{n-1} &= \lambda_{n-1} f(x_{n-1}) + (1 - \lambda_{n-1})Tx_{n-1}, \end{aligned} \tag{3.32}$$

and so, for every $n \geq 1$, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|(1 - \lambda_n)(Tx_n - Tx_{n-1}) + \lambda_n(f(x_n) - f(x_{n-1})) + (\lambda_n - \lambda_{n-1})(f(x_{n-1}) - Tx_{n-1})\| \\ &\leq (1 - \lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}| + k\lambda_n\|x_n - x_{n-1}\| \\ &= (1 - (1 - k)\lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}|, \end{aligned} \tag{3.33}$$

where $L = \sup\{\|f(x_n) - Tx_n\| : n \geq 0\}$.

On the other hand, by (IS), we also have for every $n \geq 1$,

$$\begin{aligned} x_{n+1} &= (1 - \beta_n)y_n + \beta_n Ty_n, \\ x_n &= (1 - \beta_{n-1})y_{n-1} + \beta_{n-1}Ty_{n-1}. \end{aligned} \tag{3.34}$$

Simple calculations show that

$$x_{n+1} - x_n = (1 - \beta_n)(y_n - y_{n-1}) + \beta_n(Ty_n - Ty_{n-1}) + (\beta_n - \beta_{n-1})(Ty_{n-1} - y_{n-1}), \tag{3.35}$$

then it follows that

$$\|x_{n+1} - x_n\| \leq (1 - \beta_n)\|y_n - y_{n-1}\| + \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|Ty_{n-1} - y_{n-1}\|. \tag{3.36}$$

Substituting (3.33) into (3.36) and using the condition (C4), we derive

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1 - (1 - k)\lambda_n)\|x_n - x_{n-1}\| + L|\lambda_n - \lambda_{n-1}| + M|\beta_n - \beta_{n-1}| \\ &\leq (1 - (1 - k)\lambda_n)\|x_n - x_{n-1}\| + L(\circ(\lambda_n) + \sigma_{n-1}) + M|\beta_n - \beta_{n-1}|, \end{aligned} \tag{3.37}$$

where $M = \sup\{\|Ty_n - y_n\| : n \geq 0\}$. By taking $s_{n+1} = \|x_{n+1} - x_n\|$, $\alpha_n = (1-k)\lambda_n$, $\alpha_n\gamma_n = L\circ(\lambda_n)$, and $\delta_n = L\sigma_{n-1} + M|\beta_n - \beta_{n-1}|$, we have

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \delta_n. \tag{3.38}$$

Hence, by the conditions (B2), (C1), and (C4) and Lemma 2.2, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Moreover, from (3.33) and the condition (C4), it follows that $\lim_{n\to\infty}\|y_n - y_{n-1}\| = 0$.

(2) The control conditions (C2) and (C3) are not comparable (coupled with condition (C1)), that is, neither of them implies the others. For this fact, see [13, 34]. We also refer to [13] for the examples which satisfy condition (C1) and the perturbed control condition (C4) but fail to satisfy both conditions (C2) and (C3). See also [1].

From these facts in Remark 3.4, we have the following.

Corollary 3.5. *Let E, C, and T be the same as in Theorem 3.2. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences in $(0,1)$ which satisfy the conditions (B1), (B2), (C1), and (C4) (or the conditions (B1), (B2), (C1), and (C2), or the conditions (B1), (B2), (C1), and (C3)), with $f \in \Sigma_C$ and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_0 = x \in C, \\ y_n = \lambda_n f(x_n) + (1-\lambda_n)Tx_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad n \geq 0. \end{gathered} \tag{3.39}$$

Then $\{x_n\}$ converges strongly to $Q(f) \in F(T)$, where $Q(f)$ is the unique solution of the variational inequality

$$\langle (I-f)(Q(f)), J(Q(f)-p)\rangle \leq 0, \quad f \in \Sigma_C,\ p \in F(T). \tag{3.40}$$

Remark 3.6. (1) Theorem 3.2 and Corollary 3.5 extend and improve the corresponding results by Moudafi [15] and Xu [17]. In particular, if $\beta_n = 0$ in (IS), then Corollary 3.5 with the conditions (C1) and (C2) (or the conditions (C1) and (C3)) reduces Theorem 4.2 in the paper of Xu [17].

(2) Even $\beta_n = 0$ in (IS), Corollary 3.5 generalizes the corresponding results by Halpern [2], Lions [3], Reich [4, 5], Shioji and Takahashi [6], Wittmann [7], and Xu [8] to the viscosity methods along with the perturb control condition (C4).

Next, we consider the viscosity approximation method with the weakly contractive mapping for the nonexpansive mapping.

Theorem 3.7. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm. Suppose that every weakly compact convex subset of E has the fixed point property for nonexpansive mappings. Let C be a nonempty closed convex subset of E and T nonexpansive mappings from C into itself with $F(T) \neq \emptyset$. Let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences in $(0,1)$ which satisfy the conditions (B1), (B2), (C1), and (C4) (or the conditions (B1), (B2), (C1), and (C2), or the conditions (B1), (B2), (C1), and (C3)). Let $A : C \to C$ be a weakly contractive mapping and $x_0 \in C$ chosen arbitrarily. Let $\{x_n\}$ be the sequence generated by*

$$\begin{gathered} x_0 = x \in C, \\ y_n = \lambda_n A x_n + (1-\lambda_n)Tx_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T y_n, \quad n \geq 0. \end{gathered} \tag{3.41}$$

Then, $\{x_n\}$ converges strongly to $Q(Ax^*) = x^* \in F(T)$, where Q is a sunny nonexpansive retraction from C onto $F(T)$.

Proof. It follows from Remark 3.1 that $F(T)$ is the sunny nonexpansive retract of C. Denote by Q the sunny nonexpansive retraction of C onto F. Then, QA is a weakly contractive mapping of C into itself. Indeed,

$$\|Q(Ax) - Q(Ay)\| \leq \|Ax - Ay\| \leq \|x - y\| - \psi(\|x - y\|), \quad \forall x, y \in C. \tag{3.42}$$

Lemma 2.3 assures that there exists a unique element $x^* \in C$ such that $x^* = Q(Ax^*)$. Such an $x^* \in C$ is an element of $F(T)$.

Now we define an iterative scheme as follows:

$$\begin{aligned} z_n &= \lambda_n Ax^* + (1 - \lambda_n) Tw_n, \\ w_{n+1} &= (1 - \beta_n) z_n + \beta_n T z_n, \quad n \geq 0. \end{aligned} \tag{3.43}$$

Let $\{w_n\}$ be the sequence generated by (3.43). Then, Corollary 3.5 with $f = Ax^*$ a constant assures that $\{w_n\}$ converges strongly to $Q(Ax^*) = x^*$ as $n \to \infty$. For any n, we have

$$\begin{aligned} \|x_{n+1} - w_{n+1}\| &\leq (1 - \beta_n)\|y_n - z_n\| + \beta_n \|Ty_n - Tz_n\| \\ &\leq \|y_n - z_n\| \\ &\leq \lambda_n \|Ax_n - Ax^*\| + (1 - \lambda_n)\|Tx_n - Tw_n\| \\ &\leq \lambda_n (\|Ax_n - Aw_n\| + \|Aw_n - Ax^*\|) + (1 - \lambda_n)\|x_n - w_n\| \\ &\leq \|x_n - y_n\| - \lambda_n \psi(\|x_n - w_n\|) + \lambda_n (\|w_n - x^*\| - \psi(\|w_n - x^*\|)) \\ &\leq \|x_n - w_n\| - \lambda_n \psi(\|x_n - w_n\|) + \lambda_n \|w_n - x^*\|. \end{aligned} \tag{3.44}$$

Thus, we obtain for $s_n = \|x_n - w_n\|$ the following recursive inequality:

$$s_{n+1} \leq s_n - \lambda_n \psi(s_n) + \lambda_n \|w_n - x^*\|. \tag{3.45}$$

Since $\|w_n - x^*\| \to 0$, it follows from Lemma 2.4 that $\lim_{n\to\infty}\|x_n - w_n\| = 0$. Hence,

$$\lim_{n\to\infty} \|x_n - x^*\| \leq \lim_{n\to\infty} (\|x_n - w_n\| + \|w_n - x^*\|) = 0. \tag{3.46}$$

This completes the proof. □

Corollary 3.8. *Let E be a uniformly smooth Banach space. Let C, T, A, x_0, $\{\beta_n\}$, $\{\lambda_n\}$, and $\{x_n\}$ be the same as in Theorem 3.7. Then, the conclusion of Theorem 3.7 still holds.*

Remark 3.9. (1) Theorem 3.7 (as well as Corollary 3.8) develops and complements the corresponding results by Cho et al. [1], Halpern [2], Lions [3], Moudafi [15], Reich [4, 5], Shioji and Takahashi [6], Wittmann [7], and Xu [8, 17].

(2) Even $\beta_n = 0$ in Theorem 3.7, Theorem 3.7 appears to be independent of Theorem 5.6 of Wong et al. [24] in which the control conditions (C1) and (C3) were utilized. In fact, it appears to be unknown whether a reflexive and strictly convex space satisfies the fixed point property for nonexpansive mappings.

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