

Research Article

The Numerical Method for Solving Differential Equations of Lane-Emden Type by Padé Approximation

Muhammed Yiğider,¹ Khatereh Tabatabaei,² and Ercan Çelik³

¹ Department of Mathematics, Faculty of Art and Science, Erzincan University, 24100 Erzincan, Turkey

² Department of Mathematics, Young Researchers Club, Azad University, 55518/134 Bonab, Iran

³ Department of Mathematics, Faculty of Science, Atatiürk University, 25240 Erzurum, Turkey

Correspondence should be addressed to Ercan Çelik, ercelik@atauni.edu.tr

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Numerical solution differential equation of Lane-Emden type is considered by Padé approximation. We apply these method to two examples. First differential equation of Lane-Emden type has been converted to power series by one-dimensional differential transformation, then the numerical solution of equation was put into Padé series form. Thus, we have obtained numerical solution differential equation of Lane-Emden type.

1. Introduction

Lane-Emden equations have the following form [1–4]:

$$y'' + \frac{k}{x}y' + f(x, y) = g(x), \quad 0 \leq x \leq 1, \quad k \geq 0, \quad (1.1)$$
$$y(0) = A, \quad y'(0) = B,$$

where A and B are constants, $f(x, y)$ is a continuous real-valued function, and $g(x) \in C[0, 1]$. Lane-Emden equations are singular initial value problems relating to second-order ordinary differential equations (ODEs) which have been used to model several phenomena in mathematical physics and astrophysics such as thermal explosions [5], the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and thermionic currents [5]. Several

methods for the solutions of Lane-Emden equations have been presented. Wazwaz [6] has given a general way to construct exact and series solutions to Lane-Emden equations by employing the Adomian decomposition method. A numerical solution of Lane-Emden equations is given based on the Legendre wavelets methods [4]. The variational iteration method is used to solve differential equations arising in astrophysics including the Lane-Emden equation [7, 8]. The homotopy perturbation method and the integral operator method were suggested in [9–11].

In this paper, we apply these method to two examples. First, differential equation of Lane-Emden type has been converted to power series by one-dimensional differential transformation; differential transformation was introduced first by Zhou [12]. Then the numerical solution of equation was put into Padé series form [13]. The Padé approximation method was used to accelerate the convergence of the power series solution. Thus, we obtain numerical solution differential equation of Lane-Emden type.

2. One-Dimensional Differential Transform

Differential transform of function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0}. \quad (2.1)$$

In (2.1), $y(x)$ is the original function and $Y(k)$ is the transformed function, which is called the T-function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k). \quad (2.2)$$

From (2.1) and (2.2), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \left[\frac{d^k y(x)}{dx^k} \right]_{x=0}. \quad (2.3)$$

Equation (2.3) implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically.

However, relative derivatives are calculated by an iterative way which is described by the transformed equations of the original functions. In this study, we use the lower case letter to represent the original function, and upper case letter represent the transformed function.

From the definitions of (2.1) and (2.2), it is easily proven that the transformed functions comply with the basic mathematics operations shown in Table 1.

Table 1: The fundamental operations of one-dimensional differential transform method.

Original function	Transformed function
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = cw(x)$	$Y(k) = cW(k)$
$y(x) = d\omega/dx$	$Y(k) = (k + 1)W(k + 1)$
$y(x) = d^j\omega/dx^j$	$Y(k) = (k + 1)(k + 2) \cdots (k + j)W(k + j)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{r=0}^k U(r)V(k - r)$
$y(x) = x^j$	$Y(k) = \delta(k - j) = \begin{cases} 1, & k=j \\ 0, & k \neq j \end{cases}$

In actual applications, the function $y(x)$ is expressed by a finite series and (2.2) can be written as

$$y(x) = \sum_{k=0}^m x^k Y(k). \tag{2.4}$$

Equation (2.3) implies that $\sum_{k=m+1}^{\infty} x^k Y(k)$ is negligibly small. In fact, m is decided by the convergence of natural frequency in this study.

Definition 2.1. If $y(x) = u(x)v(x)$, $u(x) = D^{-1}[U(k)]$, $v(x) = D^{-1}[V(k)]$, and \otimes denote convolution. Then $[y(x)] = D[u(x)v(x)] = U(k) \otimes V(k) = \sum_{r=0}^k U(r)V(k - r)$.

3. Padé Approximation

Suppose that we are given a power series $\sum_{i=0}^{\infty} a_i x^i$, representing a function $f(x)$, so that

$$f(x) = \sum_{i=0}^{\infty} a_i x^i. \tag{3.1}$$

A Padé approximation is a rational fraction

$$\left[\begin{matrix} L \\ M \end{matrix} \right] = \frac{p_0 + p_1 x + \cdots + p_L x^L}{q_0 + q_1 x + \cdots + q_M x^M}, \tag{3.2}$$

which has a Maclaurin expansion which agrees with (3.1) as much as possible. Notice that in (3.2), there are $L+1$ numerator coefficients and $M+1$ denominator coefficients. There is a more or less irrelevant common factor between them, and for definiteness, we take $q_0 = 1$. This choice turns out to be an essential part of the precise definition, and (3.2) is our conventional notation with this choice for q_0 . So, there are $L + 1$ independent numerator coefficients and M independent denominator coefficients, making $L + M + 1$ unknown coefficients in all. This number suggests that normally the $[L/M]$ ought to fit the power series (3.1) through the

orders $1, x, x^2, \dots, x^{L+M}$ in the notation of formal power series

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1 x + \dots + p_L x^L}{q_0 + q_1 x + \dots + q_M x^M} + O(x^{L+M+1}). \quad (3.3)$$

Multiply both sides of (3.3) by the denominator of right side in (3.3), and compare the coefficients of both sides of (3.3), then we have

$$a_l + \sum_{k=1}^m a_{l-k} q_k = p_l, \quad (l = 0, \dots, M), \quad (3.4)$$

$$a_l + \sum_{k=1}^L a_{l-k} q_k = 0, \quad (l = M+1, \dots, M+L). \quad (3.5)$$

Solve the linear equation in (3.5), then we have $q_k (k = 1, \dots, L)$. And substitute q_k into (3.4), then we have $p_l (l = 0, \dots, M)$. Therefore, we have constructed a $[L/M]$ Padé approximation, which agrees with $\sum_{i=0}^{\infty} a_i x^i$ through order x^{L+M} . If $M \leq L \leq M+2$, where M and L are the degree of numerator and denominator in Padé series, respectively, then Padé series gives an A-stable formula for an ordinary differential equation.

4. Numerical Examples

Example 4.1 (see Table 2 and Figure 1).

$$\begin{aligned} u''(x) + \frac{2}{x} u'(x) &= 2(2x^2 + 3)u(x), \quad 0 \leq x \leq 1, \\ u(0) &= 1, \quad u'(0) = 0, \end{aligned} \quad (4.1)$$

where the exact solution is $u(x) = e^{x^2}$. Multiplying both sides of (4.1) by x ,

$$x u''(x) + 2 u'(x) = 4x^3 u(x) + 6x u(x), \quad 0 \leq x \leq 1. \quad (4.2)$$

By using the fundamental operations of one-dimensional differential transform method in Table 1, we obtained following recurrence relation:

$$\delta(k-1) \otimes (k+1)(k+2)U(k+2) + 2(k+1)U(k+1) = 4\delta(k-3) \otimes U(k) + 6\delta(k-1) \otimes U(k). \quad (4.3)$$

Table 2: Numerical solution of $u(x)$.

x	$u(x)$	$P[5/4]$	$ u(x) - P[5/4] $
0.1	1.010050167	1.010050167	0
0.2	1.040810774	1.040810774	0
0.3	1.094174234	1.094174275	4.1×10^{-8}
0.4	1.173509974	1.173510700	7.26×10^{-7}
0.5	1.284016927	1.284023668	0.000006741
0.6	1.433275840	1.43317285	0.000041445
0.7	1.632060167	1.632251266	0.000191099
0.8	1.895481174	1.896191187	0.000710013
0.9	2.244559634	2.246777235	0.002217601
1.0	2.708333334	2.714285714	0.005952380

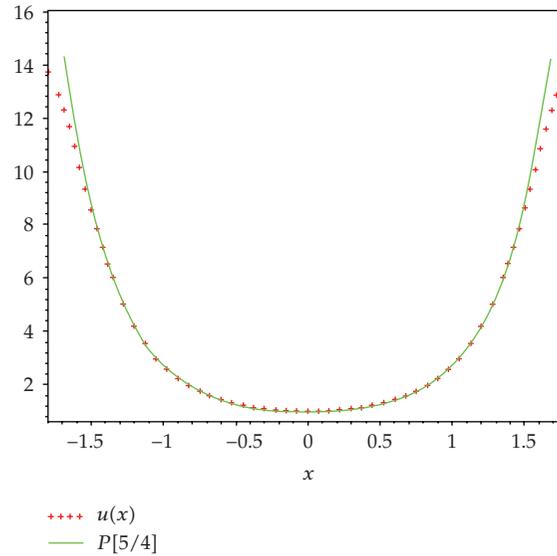


Figure 1: Values of $u(x)$ and its $P[5/4]$ Padé approximant.

Then,

$$\begin{aligned} & \sum_{l=0}^k \delta(l-1)(k-l+1)(k-l+2)U(k-l+2) + 2(k+1)U(k+1) \\ &= 4 \sum_{l=0}^k \delta(l-3)U(k-l) + \sum_{l=0}^k 6\delta(l-1)U(k-l), \end{aligned}$$

$$U(0) = 1, \quad U(1) = 0, \quad U(2) = 1, \quad U(3) = 0, \quad U(4) = \frac{1}{2}, \quad U(5) = 0,$$

$$U(6) = \frac{1}{6}, \quad U(7) = 0, \quad U(8) = \frac{1}{24}, \quad U(9) = 0, \dots,$$

$$u^*(x) = U(0) + xU(1) + x^2U(2) + x^3U(3) + \dots,$$

$$u^*(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \dots = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \dots + \frac{1}{n!}x^{2n}. \quad (4.4)$$

Power series $u^*(x)$ can be transformed into Padé series

$$P\left[\frac{5}{4}\right] = \frac{(1 + (1/2)x^2 + (1/12)x^4)}{(1 - (1/2)x^2 + (1/12)x^4)} = \frac{(1 + 0.5x^2 + 0.8333333333x^4)}{(1 - 0.5x^2 + 0.8333333333x^4)}. \quad (4.5)$$

Example 4.2 (see Table 3 and Figure 2). Consider the Lane-Emden-type equation

$$y'' + \frac{2}{x}y' + (8e^y + 4e^{y/2}) = 0, \quad (4.6)$$

with initial values

$$y(0) = 0, \quad y'(0) = 0, \quad (4.7)$$

where the exact solution is $y(x) = -2 \ln(1 + x^2)$.

Multiplying both sides of equation by x ,

$$xy'' + 2y' + 8xe^y + 4xe^{y/2} = 0. \quad (4.8)$$

Also, e^y and $e^{y/2}$ are defined as follows:

$$e^y = 1 + y, \quad e^{y/2} = 1 + \frac{y}{2}. \quad (4.9)$$

Then,

$$xy'' + 2y' + 12x + 10xy = 0. \quad (4.10)$$

By using the fundamental operations of one-dimensional differential transform method in Table 1, we obtain the following recurrence relation:

$$\delta(k-1) \otimes (k+1)(k+2)Y(k+2) + 2(k+1)Y(k+1) + 12\delta(k-1) + 10\delta(k-1) \otimes Y(k) = 0. \quad (4.11)$$

Table 3: Numerical solution of $y(x)$.

x	$y(x)$	$P[5/4]$	$ y(x) - P[5/4] $
0.1	-0.01990023712	-0.01990023713	1×10^{-11}
0.2	-0.07841498810	-0.07841499080	2.70×10^{-9}
0.3	-0.1720671642	-0.1720673162	1.520×10^{-7}
0.4	-0.2953112381	-0.2953138699	0.0000026318
0.5	-0.4408387684	-0.4408625091	0.0000237407
0.6	-0.5998683214	-0.6000097952	0.0001414738
0.7	-0.7622819782	-0.7629141475	0.0006321693
0.8	-0.9164312381	-0.9187160286	0.0022847905
0.9	-1.048395758	-1.055408536	0.007012778
1.0	-1.140438988	-1.159333345	0.018894357

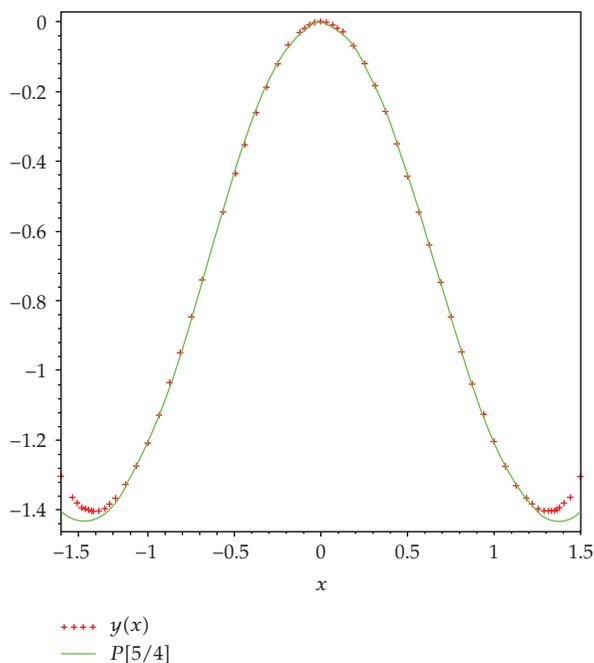


Figure 2: Values of $y(x)$ and its $P[5/4]$ Padé approximant.

Then,

$$\sum_{l=0}^k \delta(l-1)(k-l+1)(k-l+2)Y(k-l+2) + 2(k+1)Y(k+1) + 12\delta(k-1) + 10 \sum_{l=0}^k \delta(l-1)Y(k-l) = 0,$$

$$y(0) = 0, \quad y(1) = 0, \quad y(2) = -2, \quad y(3) = 0, \quad y(4) = 1, \quad y(5) = 0,$$

$$\begin{aligned}
y(6) &= -\frac{5}{21}, & y(7) &= 0, & y(8) &= \frac{25}{756}, \dots, \\
y^*(x) &= y(0) + xy(1) + x^2y(2) + x^3y(3) + \dots, \\
y^*(x) &= -2x^2 + x^4 - \frac{5}{21}x^6 + \frac{25}{756}x^8 + \dots
\end{aligned}
\tag{4.12}$$

Power series $y^*(x)$ can be transformed into Padé series

$$\begin{aligned}
P\left[\frac{5}{4}\right] &= \frac{((589/704)x^4 - 2x^2)}{(1 + (115/1408)x^2 - (4625/59136)x^4)} \\
&= \frac{(0.8366477273x^4 - 2x^2)}{(1 + 0.08167613636x^2 - 0.07820955087x^4)}.
\end{aligned}
\tag{4.13}$$

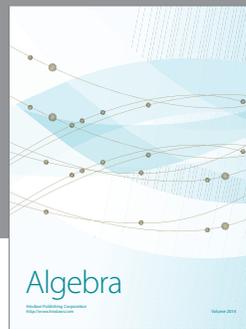
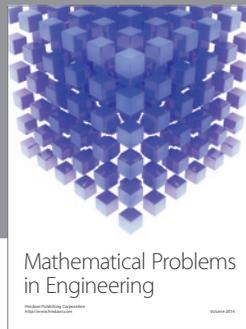
5. Conclusion

A Padé approximation method has proposed for solving differential equation of Lane-Emden type in this study. This method is very simple and effective for most of Lane-Emden equations. Numerical results explicitly reveal the complete reliability of the proposed Padé approximation.

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