Hindawi Publishing Corporation
Boundary Value Problems
Volume 2009, Article ID 739097, 23 pages
doi:10.1155/2009/739097

Research Article

On Initial Boundary Value Problems with Equivalued Surface for Nonlinear Parabolic Equations

Fengquan Li

Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China

Correspondence should be addressed to Fengquan Li, fqli@dlut.edu.cn

Received 6 January 2009; Revised 12 March 2009; Accepted 22 May 2009

Recommended by Sandro Salsa

We will use the concept of renormalized solution to initial boundary value problems with equivalued surface for nonlinear parabolic equations, discuss the existence and uniqueness of renormalized solution, and give the relation between renormalized solutions and weak solutions.

1. Introduction

Let $\Omega \subset R^N (N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega = \Gamma$. T is a fixed positive constant, $Q = \Omega \times (0,T)$. We consider the following nonlinear parabolic boundary value problems with equivalued surface:

$$\begin{gathered}
\frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x,u)\frac{\partial u}{\partial x_j}\right) = f(x,t) \quad \text{in } Q, \\
u = C(t)(\text{a function of } t \text{ to be determined}) \quad \text{on } \Gamma \times (0,T), \\
\int_\Gamma \frac{\partial u}{\partial n_L} ds = A(t) \quad \forall \text{ a.e. } t \in (0,T), \\
u(x,0) = 0 \quad \text{in } \Omega,
\end{gathered} \tag{P}$$

where $f \in L^2(Q)$ and $A \in L^2(0,T)$, $\mathbf{n} = (n_1, \ldots, n_N)$ denotes the unit outward normal vector on Γ and

$$\frac{\partial u}{\partial n_L} = \sum_{i,j=1}^{N} a_{ij}(x,u)\frac{\partial u}{\partial x_j} n_i. \tag{1.1}$$

There are many concrete physical sources for problem (P), for example, in the petroleum exploitation, u denotes the oil pressure, and $A(t)$ is the rate of total oil flux per unit length of the well at the time t; in the combustion theory, u denotes the temperature, for any fixed time t, the temperature distribution on the boundary is a constant to be determined, while, the total heat $A(t)$ through the boundary is given (cf. [1–7]). For linear equations, the existence, uniqueness of solution to the corresponding problem are well understood (cf. [1–3]), for the purpose, the Galerkin method was used. For semilinear equations, the existence of global smooth solution was obtained in [7] in which a comparison principle was established. If $a_{ij}(x,u)$ is locally Lipschitz continuous with respect to the second variable, the existence and uniqueness of bounded weak solution to problem (P) have been discussed in [8] under the hypotheses of $f \in L^q(Q)$ and $A \in L^r(0,T)$ with $q > N/2+1$, $r > N+2$. However, if $f \in L^2(Q)$ and $A \in L^2(0,T)$, we cannot get a bounded weak solution. In order to deal with this situation, we will introduce the concept of renormalized solution to problem (P) and discuss the existence and uniqueness of renormalized solution.

The paper is organized as follows. In Section 2, we introduce the concept of renormalized solution and prove the existence of renormalized solution to problem (P). In Section 3, uniqueness and a comparison principle of renormalized solution to problem (P) are established. In Section 4, we discuss the relation between renormalized solutions and weak solutions for problem (P).

2. Existence of Renormalized Solution to Problem (P)

In order to prove the existence of renormalized solution to problem (P), we make the following assumptions.

Let $a_{ij} : \Omega \times R \to R$ be Carathéodory functions with $1 \le i,j \le N$. We assume that $a_{ij}(\cdot,0) \in L^\infty(\Omega)$ and for any given $M > 0$ there exist $d_M \in L^\infty(\Omega)$ and a positive constant λ_0 such that for every s, s_1, $s_2 \in R$, $\xi = (\xi_1,\dots,\xi_N) \in R^N$, and a.e. $x \in \Omega$,

$$\left|a_{ij}(x,s_1) - a_{ij}(x,s_2)\right| \le d_M(x)|s_1 - s_2|, \quad |s_k| \le M, k = 1,2, \tag{2.1}$$

$$\sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \ge \lambda_0|\xi|^2. \tag{2.2}$$

Set

$$V = \left\{ v \in H^1\ (\Omega) | v|_\Gamma = \text{constant} \right\}. \tag{2.3}$$

Under hypotheses (2.1)-(2.2) and $f \in L^2(Q)$, $A \in L^2(0,T)$, we cannot obtain an L^∞ estimate on the determined function $C(t)$; thus, we cannot prove the existence of bounded weak solutions to problem (P), hence $a_{ij}(\cdot,u)D_j u$ may not belong to $L^2(Q)$. In order to overcome this difficulty, we will use the concept of renormalized solution introduced by DiPerna and Lions in [9] for Boltzmann equations (see also [10–12]).

As usual, for $k > 0$, T_k denotes the truncation function defined by

$$T_k(v) = \begin{cases} k, & \text{if } v > k, \\ v, & \text{if } |v| \leq k, \\ -k, & \text{if } v < -k. \end{cases} \tag{2.4}$$

Set

$$W = \left\{ \xi \in C^\infty\left(\overline{Q}\right) | \xi(T) = 0, \xi(t)|_\Gamma = C(t)\left(\text{an arbitrary function of } t\right) \right\}. \tag{2.5}$$

Definition 2.1. A renormalized solution to problem (P) is a measurable function $u : Q \to R$, satisfying $u \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega))$ and for all $h \in C_c^1(R)$, $\xi \in W$,

$$\begin{aligned} -\int_Q \xi_t \int_0^u h(r)\,dr\,dx\,dt + \int_Q \sum_{i,j=1}^N a_{ij}(x,u) D_j u D_i(h(u)\xi)\,dx\,dt \\ = \int_Q f h(u)\xi\,dx\,dt + \int_0^T A(t) h(u(t)|_\Gamma)\xi(t)|_\Gamma\,dt, \end{aligned} \tag{2.6}$$

$$\lim_{m\to+\infty} \int_{\{(x,t)\in Q: m\leq |u(x,t)|\leq m+1\}} \sum_{i,j=1}^N a_{ij}(x,u) D_j u D_i u\,dx\,dt = 0. \tag{2.7}$$

Remark 2.2. Each term in (2.6) and (2.7) is well defined. Indeed, the first term on the left side of (2.6) is welldefined as $|\int_0^u h(r)dr| \leq \|h\|_{L^\infty}|u|$ and $u \in L^2(Q)$. The second term on the left side of (2.6) should be understood as

$$\int_{\{(x,t)\in Q: |u|<k\}} \sum_{i,j=1}^N a_{ij}(x, T_k(u)) D_j T_k(u) D_i[h(T_k(u))\xi]\,dx\,dt, \tag{2.8}$$

for $k > 0$ such that $\operatorname{supp} h \subset [-k,k]$. Since $u \in L^2(0,T;V)$, it is the same for $h(u)\xi$ and $h(u(t)|_\Gamma)\xi(t)|_\Gamma$. The integral in (2.7) should be understood as

$$\int_{\{(x,t)\in Q: m\leq|u(x,t)|\leq m+1\}} \sum_{i,j=1}^N a_{ij}(x, T_{m+1}(u)) D_j T_{m+1}(u) D_i T_{m+1}(u)\,dx\,dt. \tag{2.9}$$

Remark 2.3. Note that if u is a renormalized solution of problem (P), we get $B_h(u) = \int_0^u h(r)dr \in L^2(0,T;V)$, $B_h(u)_t \in L^2(0,T;V') + L^1(Q)$; thus, $B_h(u) \in C([0,T];L^1(\Omega))$, hence $B_h(u)(\cdot,0) = 0$ makes sense.

Remark 2.4. By approximation, (2.6) holds for any $h \in W^{1,\infty}(R)$ with compact support and all $\xi \in \{\xi \in L^2(0,T;V) \mid \xi_t \in L^2(Q), \xi(\cdot,T) = 0\}$.

Now we can state the existence result for prolem (P) as follows.

Theorem 2.5. *Under hypotheses* (2.1)-(2.2) *and* $f \in L^2(Q)$, $A \in L^2(0,T)$, *problem* (P) *admits a renormalized solution* $u \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega))$ *in the sense of Definition 2.1.*

In order to prove Theorem 2.5, we will consider the following problem:

$$\begin{gathered}\frac{\partial u_n}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^{(n)}(x,u_n)\frac{\partial u_n}{\partial x_j}\right) = f \quad \text{in } Q,\\ u_n|_{\Gamma\times(0,T)} = C_n(t) \ \text{(a function of } t \text{ to be determined)} \quad \text{on } \Gamma\times(0,T),\\ \int_\Gamma \frac{\partial u_n}{\partial n_L} ds = A(t) \quad \forall \text{a.e. } t \in (0,T),\\ u_n(x,0) = 0 \quad \text{in } \Omega,\end{gathered} \tag{P_n}$$

where $a_{ij}^{(n)}(x,u) = a_{ij}(x,T_n(u))$, $i,j = 1,2,\dots,N$.

Then problem (P_n) admits a unique weak solution $u_n \in L^2(0,T;V) \cap C([0,T];L^2(\Omega))$ such that $u_n' \in L^2(0,T;V')$ and satisfies

$$\begin{aligned}&\langle u_n'(t), v\rangle_{V',V} + \int_\Omega \sum_{i,j=1}^{N} a_{ij}^{(n)}(x,u_n) D_j u_n D_i v\,dx\\ &\quad= \int_\Omega f(x,t)v(x)dx + A(t)v|_\Gamma, \quad \text{a.e. } t\in(0,T), \quad \forall v \in V,\end{aligned} \tag{2.10}$$

$$u_n(x,0) = 0 \quad \text{a.e. } x \in \Omega. \tag{2.11}$$

In fact, here we can prove the existence of weak solution for problem (P_n) via Galerkin method. Let us consider the operator

$$\begin{gathered}\mathcal{B} : L^2(\Omega) \longrightarrow V,\\ F \longmapsto v,\end{gathered} \tag{2.12}$$

where v is the weak solution of the following problem:

$$\begin{gathered}-\Delta v + v = F \quad \text{in } \Omega,\\ v = C \ \text{(a constant to be determined)} \quad \text{on } \Gamma,\\ \int_\Gamma \frac{\partial v}{\partial n_L} ds = 0.\end{gathered} \tag{E}$$

By Lax-Milgram Theorem, the above problem exists a unique weak solution v which continuously depending on F. Hence $\mathcal{B}$ is a compact self-adjoint operator from $L^2(\Omega)$ to

$L^2(\Omega)$. By Riesz-Schauder's theory, there is a completed orthogonal eigenvalues sequence $\{w^k\}$ of the operator $\mathcal{B}$. Here we may take the special orthogonal system $\{w^k\}$.

Define $\mathcal{A} : V \to V'$,

$$(\mathcal{A}w, v)_{V',V} = \int_\Omega \sum_{i,j=1}^N a_{ij}^n(x,w) D_j w D_i v dx,$$
$$(F(t), v)_{V',V} = \int_\Omega f(x,t)v(x)dx + A(t)v|_\Gamma, \quad \text{a.e. } t \in (0,T), \ \forall w, v \in V. \tag{2.13}$$

Let $u_n^m(x,t) = \sum_{k=1}^m \Phi_n^{km} w^k$, then Galerkin equations can be written as

$$\left((u_n^m)'(t), w^k\right) + \left(\mathcal{A}u_n^m(t), w^k\right) = \left(F(t), w^k\right), \quad \text{a.e. } t \in (0,T),$$
$$u_n^m(x,0) = 0 \quad \text{a.e. } x \in \Omega. \tag{2.14}$$

By using the same arguments as [13, Lemma 30.4], we get a solution $u_n^m \in L^2(0,T;V)$ to the above Galerkin equations such that $(u_n^m)' \in L^2(0,T;V)$. Moreover, we can easily prove the following estimates:

$$\|u_n^m\|_{L^\infty(0,T;L^2(\Omega))} \le C_0,$$
$$\|u_n^m\|_{L^2(0,T;V)} \le C_0,$$
$$\|\mathcal{A}u_n^m\|_{L^2(0,T;V')} \le C_0,$$
$$\left\|(u_n^m)'\right\|_{L^2(0,T;V')} \le C_0, \tag{2.15}$$

where C_0 is a positive constant independent of m.

The above estimates imply that there exists a subsequence of $\{u_n^m\}$ (still be denoted by $\{u_n^m\}$) such that

$$u_n^m \rightharpoonup u_n \quad \text{weak} * \text{ in } L^\infty\left(0,T;L^2(\Omega)\right),$$
$$u_n^m \rightharpoonup u_n \quad \text{weakly in } L^2(0,T;V),$$
$$(u_n^m)' \rightharpoonup u_n' \quad \text{weakly in } L^2\left(0,T;V'\right),$$
$$\mathcal{A}u_n^m \rightharpoonup \mathcal{A}u_n \quad \text{weakly in } L^2\left(0,T;V'\right). \tag{2.16}$$

Thus we can pass to the limit in the above Galerkin equations and obtain the existence of weak solution for problem (P_n). Since it is easy to prove the uniqueness of weak solution for problem (P_n), we omit the details.

To deal with the time derivative of truncation function, we introduce a time regularization of a function $u \in L^2(0,T;V)$. Let

$$u_\nu(x,t) = \int_{-\infty}^{t} \nu \widetilde{u}(x,s) e^{\nu(s-t)} ds, \quad \widetilde{u}(x,s) = u(x,s)\chi_{(0,T)}(s), \tag{2.17}$$

where $\chi_{(0,T)}$ denotes the characteristic function of a set $(0,T)$ and $\nu > 0$. This convolution function has been first used in [14] (see also [10]), and it enjoys the following properties: u_ν belongs to $C([0,T];V)$, $u_\nu(x,0) = 0$, and u_v converges strongly to u in $L^2(0,T;V)$ as ν tends to the infinity. Moreover, we have

$$(u_\nu)_t = \nu(u - u_\nu), \tag{2.18}$$

and finally if $u \in L^\infty(Q)$, then $u_\nu \in L^\infty(Q)$ and

$$\|u_\nu\|_{L^\infty(Q)} \le \|u\|_{L^\infty(Q)}, \quad \forall \nu > 0. \tag{2.19}$$

Taking $v = u_n(t)$ in (2.10), then integrating over $(0,\tau)$ with $\tau \in (0,T)$, we have

$$\begin{aligned} &\int_0^\tau \int_\Omega \frac{d}{dt} |u_n(x,t)|^2 dx dt + \int_0^\tau \int_\Omega \sum_{i,j=1}^{N} a_{ij}^{(n)}(x,u_n) D_j u_n D_i u_n dx dt \\ &\quad = \int_0^\tau \int_\Omega f u_n dx dt + \int_0^\tau A(t) u_n(t)|_\Gamma dt. \end{aligned} \tag{2.20}$$

By (2.2), trace theorem, Hölder's inequality, Young's inequality and Gronwall's inequality, we get

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le C_1, \tag{2.21}$$

$$\|u_n\|_{L^2(0,T;V)} \le C_1, \tag{2.22}$$

where C_1 is a positive constant depending only on $\|f\|_{L^2(Q)}$, $\|A\|_{L^2(0,T)}$, λ_0, but independent of n and u_n.

By (2.21) and (2.22), there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weak} * \quad \text{in} \quad L^\infty\left(0,T;L^2(\Omega)\right), \\ &u_n \rightharpoonup u \quad \text{weakly} \quad \text{in} \quad L^2(0,T;V) \\ &u_n|_\Gamma \rightharpoonup u|_\Gamma \quad \text{weakly} \quad \text{in} \quad L^2(0,T). \end{aligned} \tag{2.23}$$

Using the same method as [10], we can obtain

$$u_n \longrightarrow u \quad \text{a.e. in } Q \text{ (up to some subsequence).} \tag{2.24}$$

Thus for any given $k > 0$,

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^2(0,T;V), \text{ strongly in } L^2(Q), \text{ a.e. in } Q. \tag{2.25}$$

By [15, Lemma 2 and Lemma 3], we have

$$u_n \longrightarrow u \quad \text{strongly in } L^q(Q), \quad \forall 1 \leq q < 2 + \frac{4}{N}, \tag{2.26}$$

$$u_n \longrightarrow u \quad \text{strongly in } L^r(\Gamma \times (0,T)), \quad \forall 2 \leq r < 2 + \frac{2}{N}. \tag{2.27}$$

impling that

$$u_n|_\Gamma \longrightarrow u|_\Gamma, \text{ a.e. in } (0,T). \tag{2.28}$$

For any given $k > 0$, it follows from (2.27)-(2.28) and Vitali's theorem that

$$T_k(u_n|_\Gamma) \longrightarrow T_k(u|_\Gamma) \text{ strongly in } L^2(0,T). \tag{2.29}$$

Set

$$\eta_\nu(u) = (T_k(u))_\nu. \tag{2.30}$$

Similar to [10], this function has the following properties: $(\eta_\nu(u))_t = \nu(T_k(u) - \eta_\nu(u)), \eta_\nu(u)(0) = 0, |\eta_\nu(u)| \leq k$,

$$\eta_\nu(u) \longrightarrow T_k(u) \quad \text{strongly in } L^2(0,T;V), \text{ as } \nu \text{ tends to the infinity.} \tag{2.31}$$

For any fixed h and k with $h > k > 0$, let

$$w_n = T_{2k}\big(u_n - T_h(u_n) + T_k(u_n) - \eta_\nu(u)\big). \tag{2.32}$$

Then we have the following lemma.

Lemma 2.6. *Under the previous assumptions, we have*

$$\int_0^T < (u_n)_t, w_n > dt \geq \omega(n,\nu,h), \tag{2.33}$$

where $\lim_{h\to+\infty}\lim_{\nu\to+\infty}\lim_{n\to+\infty}\omega(n,\nu,h) = 0$.

Proof. The proof of Lemma 2.6 is the same as [10, Lemma 2.1], and we omit the details. □

Lemma 2.7. *Under the previous assumptions, for any given $k > 0$, we have*

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } L^2(0,T;V). \tag{2.34}$$

Proof. Taking $v = w_n(t)$ in (2.10), then integrating over $(0,T)$, by Lemma 2.6, we have

$$\int_Q \sum_{i,j=1}^N a_{ij}^{(n)}(x,u_n) D_j u_n D_i w_n dx dt \le \int_Q f w_n dx dt + \int_0^T A(t) w_n(t)|_\Gamma dt + \omega(n,\nu,h). \tag{2.35}$$

□

Now note that $Dw_n = 0$ if $|u_n| > h + 4k$; then if we set $M = h + 4k$, splitting the integral on the left side of (2.35) on the sets $\{(x,t) \in Q : |u_n(x,t)| > k\}$ and $\{(x,t) \in Q : |u_n(x,t)| \le k\}$, $\forall n > M$, we get

$$\begin{aligned}
&\int_Q \sum_{i,j=1}^N a_{ij}^{(n)}(x,u_n) D_j u_n D_i w_n dx dt \\
&\quad = \int_Q \sum_{i,j=1}^N a_{ij}(x, T_M(u_n)) D_j T_M(u_n) D_i w_n dx dt \\
&\quad \ge \int_Q \sum_{i,j=1}^N a_{ij}(x, T_k(u_n)) D_j T_k(u_n) D_i \big(T_k(u_n) - \eta_\nu(u)\big) dx dt \\
&\qquad - \int_{\{|u_n|>k\}} \sum_{i,j=1}^N \big|a_{ij}(x, T_M(u_n)) D_j T_M(u_n)\big| \big|D_i \eta_\nu(u)\big| dx dt.
\end{aligned} \tag{2.36}$$

While,

$$\begin{aligned}
&\int_{\{|u_n|>k\}} \sum_{i,j=1}^N \big|a_{ij}(x, T_M(u_n)) D_j T_M(u_n)\big| \big|D_i \eta_\nu(u)\big| dx dt \\
&\quad \le \int_{\{|u_n|>k\}} \sum_{i,j=1}^N \big|a_{ij}(x, T_M(u_n)) D_j T_M(u_n)\big| |D_i T_k(u)| dx dt \\
&\qquad + \int_Q \sum_{i,j=1}^N \big|a_{ij}(x, T_M(u_n)) D_j T_M(u_n)\big| \big|D_i \eta_\nu(u) - D_i T_k(u)\big| dx dt.
\end{aligned} \tag{2.37}$$

For any fixed $h > 0$, (2.1) and (2.22) imply that $a_{ij}(x, T_M(u_n)) D_j T_M(u_n)$ is bounded in $L^2(Q)$ with respect to n, while $|D_i T_k(u)| \chi_{\{|u_n|>k\}}$ strongly converges to zero in $L^2(Q)$. Moreover it follows from (2.31) that

$$\int_{\{|u_n|>k\}} \sum_{i,j=1}^N \big|a_{ij}(x, T_M(u_n)) D_j T_M(u_n)\big| \big|D_i \eta_\nu(u)\big| dx dt \le \omega(n,\nu), \tag{2.38}$$

where $\lim_{\nu\to+\infty}\lim_{n\to+\infty}\omega(n,\nu)=0$. Equations (2.38), (2.36), and (2.35) imply that

$$\begin{aligned}&\int_Q\sum_{i,j=1}^N a_{ij}(x,T_k(u_n))D_jT_k(u_n)D_i\big(T_k(u_n)-\eta_\nu(u)\big)dxdt\\&\qquad\le\int_Q fw_n dxdt+\int_0^T A(t)w_n(t)|_\Gamma dt+\omega(n,\nu)+\omega(n,\nu,h).\end{aligned}\tag{2.39}$$

By (2.25), (2.31), and (2.39), we get

$$\begin{aligned}&\int_Q\sum_{i,j=1}^N a_{ij}(x,T_k(u_n))D_jT_k(u_n)D_i(T_k(u_n)-T_k(u))dxdt\\&\qquad\le\int_Q fw_n dxdt+\int_0^T A(t)w_n(t)|_\Gamma dt+\omega(n,\nu)+\omega(n,\nu,h).\end{aligned}\tag{2.40}$$

By (2.24)-(2.25) and the Lebesgue dominated convergence theorem, we have

$$\int_Q fw_n dxdt=\int_Q fT_{2k}\big(u-T_h(u)+T_k(u)-\eta_\nu(u)\big)dxdt+\omega(n),\tag{2.41}$$

where $\lim_{n\to+\infty}\omega(n)=0$. (2.31) and (2.41) imply that

$$\int_Q fw_n dxdt=\int_Q fT_{2k}(u-T_h(u))dxdt+\omega(n,\nu);\tag{2.42}$$

thus, we get

$$\int_Q fw_n dxdt=\omega(n,\nu,h).\tag{2.43}$$

Similarly to the proof of (2.43), we also have

$$\int_0^T A(t)w_n(t)|_\Gamma dt=\omega(n,\nu,h).\tag{2.44}$$

Therefore we get

$$\begin{aligned}&\int_Q\sum_{i,j=1}^N a_{ij}(x,T_k(u_n))\big(D_jT_k(u_n)-D_jT_k(u)\big)D_i(T_k(u_n)-T_k(u))dxdt\\&\qquad\le\omega(n,\nu,h)-\int_Q\sum_{i,j=1}^N a_{ij}(x,T_k(u_n))D_jT_k(u)D_i(T_k(u_n)-T_k(u))dxdt.\end{aligned}\tag{2.45}$$

Let n, ν, then and h tend to the infinity, respectively, we get

$$\lim_{n\to+\infty}\int_Q \sum_{i,j=1}^{N} a_{ij}(x,T_k(u_n))D_j(T_k(u_n)-T_k(u))D_i(T_k(u_n)-T_k(u))dxdt = 0. \tag{2.46}$$

Using (2.2), (2.25), and (2.46), we obtain (2.34).

Proof of Theorem 2.5. For any given $\xi \in W$, $h \in C_c^1(R)$, suppose that $\operatorname{supp} h \subset [-k,k]$, taking $v = h(u_n(t))\xi(t)$ in (2.10) and integrating over $(0,T)$, we have

$$\begin{aligned}\int_0^T \langle (u_n)_t, h(u_n)\xi\rangle dt &+ \int_Q \sum_{i,j=1}^{N} a_{ij}^{(n)}(x,u_n)D_j u_n D_i(h(u_n)\xi)dxdt \\ &= \int_0^T A(t)h(u_n(t)|_\Gamma)\xi(t)|_\Gamma dt + \int_Q f h(u_n)\xi dxdt.\end{aligned} \tag{2.47}$$

By [12, Lemma 1.4], we have

$$\int_0^T \langle (u_n)_t, h(u_n)\xi\rangle dt = -\int_Q \xi_t \int_0^{u_n} h(r)drdxdt. \tag{2.48}$$

However

$$-\int_Q \xi_t \int_0^{u_n} h(r)drdxdt \longrightarrow -\int_Q \xi_t \int_0^{u} h(r)drdxdt. \tag{2.49}$$

In fact, Noting (2.26) we get

$$\begin{aligned}&\left| -\int_Q \xi_t \int_0^{u_n} h(r)drdxdt + \int_Q \xi_t \int_0^{u} h(r)drdxdt \right| \\ &= \left| \int_Q \xi_t \int_u^{u_n} h(r)drdxdt \right| \\ &\le \|\xi_t\|_{L^2(Q)} \|h\|_{L^\infty(R)} \|u_n - u\|_{L^2(Q)} \longrightarrow 0, \quad \text{as } n \longrightarrow +\infty.\end{aligned} \tag{2.50}$$

As $n > k$, we have

$$\int_Q \sum_{i,j=1}^N a_{ij}^{(n)}(x,u_n) D_j u_n D_i(h(u_n)\xi)\,dx\,dt$$
$$= \int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u_n)) D_j T_k(u_n) D_i T_k(u_n) h'(u_n)\xi\,dx\,dt \tag{2.51}$$
$$+ \int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u_n)) D_j T_k(u_n) h(u_n) D_i\xi\,dx\,dt.$$

Equations (2.47)(2.25) and (2.34) imply that

$$\int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u_n)) D_j T_k(u_n) D_i T_k(u_n) h'(u_n)\xi\,dx\,dt$$
$$\longrightarrow \int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u)) D_j T_k(u) D_i T_k(u) h'(u)\xi\,dx\,dt,$$
$$\int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u_n)) D_j T_k(u_n) D_i\xi h(u_n)\,dx\,dt \tag{2.52}$$
$$\longrightarrow \int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u)) D_j T_k(u) D_i\xi h(u)\,dx\,dt.$$

Thus we get

$$\int_Q \sum_{i,j=1}^N a_{ij}^{(n)}(x,u_n) D_j u_n D_i(h(u_n)\xi)\,dx\,dt$$
$$\longrightarrow \int_Q \sum_{i,j=1}^N a_{ij}(x,T_k(u)) D_j T_k(u) D_i(h(u)\xi)\,dx\,dt \tag{2.53}$$
$$= \int_Q \sum_{i,j=1}^N a_{ij}(x,u) D_j u D_i(h(u)\xi)\,dx\,dt.$$

It follows from (2.28) that

$$\int_0^T A(t) h(u_n(t)|_\Gamma)\xi(t)|_\Gamma\,dt \longrightarrow \int_0^T A(t) h(u(t)|_\Gamma)\xi(t)|_\Gamma\,dt. \tag{2.54}$$

Equation (2.24) yields

$$\int_Q fh(u_n)\xi dxdt \longrightarrow \int_Q fh(u)\xi dxdt. \tag{2.55}$$

Taking $n \to +\infty$ in (2.47), by (2.48), (2.49), and (2.53)–(2.55), we obtain

$$\begin{aligned} -\int_Q \xi_t \int_0^u h(r)drdxdt + \int_Q \sum_{i,j=1}^N a_{ij}(x,u)D_juD_i(h(u)\xi)dxdt \\ = \int_0^T A(t)h(u(t)|_\Gamma)\xi(t)|_\Gamma dt + \int_Q fh(u)\xi dxdt. \end{aligned} \tag{2.56}$$

For any given $m > 0$, taking $v = T_1(u_n(t) - T_m(u_n(t)))$ in (2.10), then integrating over $(0,T)$, we get

$$\begin{aligned} \int_0^T < (u_n)_t, T_1(u_n - T_m(u_n)) > dt + \int_Q \sum_{i,j=1}^N a_{ij}^{(n)}(x,u_n)D_ju_nD_iT_1(u_n - T_m(u_n))dxdt \\ = \int_0^T A(t)T_1(u_n(t)|_\Gamma - T_m(u_n(t)|_\Gamma))dt + \int_Q fT_1(u_n - T_m(u_n))dxdt. \end{aligned} \tag{2.57}$$

Setting $S_1(s) = \int_0^s T_1(t - T_m(t))dt$, then $0 \le S_1(s) \le |s|\text{sign}_0^+(|s| - m)$, for all $s \in R$, where $\text{sign}_0^+(s) = 1$ if $s > 0$, $\text{sign}_0^+(s) = 0$ if $s \le 0$. Thus we get

$$\int_{\{m\le|u_n|\le m+1\}} \sum_{i,j=1}^N a_{ij}(x,u_n)D_ju_nD_iu_ndxdt \le \int_{\{t\in(0,T):|u_n(t)|_\Gamma|\ge m\}} |A(t)|dt + \int_{\{|u_n|\ge m\}} |f|dxdt. \tag{2.58}$$

Let n, m tend to the infinity in (2.58), respectively, then one can deduce that u satisfies (2.7). Thus u is a renormalized solution to problem (P) in the sense of Definition 2.1. This finishes the proof of Theorem 2.5. □

Remark 2.8. Using the same approach as before, we can deal with the nonzero initial value $u_0 \neq 0$. In fact, we only replace $\eta_\nu(u) = (T_k(u))_\nu$ by $\eta_\nu(u) = T_k(u)_\nu + e^{-\nu t}T_k(u_0)$ in (2.30).

3. Uniqueness of Renormalized Solution to Problem (P)

In this section, we will present the uniqueness of renormalized solution to problem (P). Here we will modify a method based on Kruzhkov's technique of doubling variables in [12] and prove uniqueness and a comparison principle of renormalized solution for problem (P).

Only simply modifying [12, Lemma 3.1], we can obtain the following result.

Lemma 3.1. *Let u be a renormalized solution to problem (P) for the data (f, A). Then*

$$\int_Q \xi_t \operatorname{sign}_0^+(u)\int_0^u h(r)dr\,dx\,dt + \int_0^T \operatorname{sign}_0^+(u(t)|_\Gamma)A(t)\xi(t)|_\Gamma h(u(t)|_\Gamma)dt + \int_Q \operatorname{sign}_0^+(u) f h(u)\xi\,dx\,dt$$
$$\geq \int_Q \operatorname{sign}_0^+(u)\sum_{i,j=1}^N a_{ij}(x,u)D_j u D_i(h(u)\xi)dx\,dt, \tag{3.1}$$

$$\int_Q \xi_t \operatorname{sign}_0^+(-u)\int_0^u h(r)dr\,dx\,dt + \int_0^T \operatorname{sign}_0^+(-u(t)|_\Gamma)A(t)\xi(t)|_\Gamma h(u(t)|_\Gamma)dt + \int_Q \operatorname{sign}_0^+(-u) f h(u)\xi\,dx\,dt$$
$$\leq \int_Q \operatorname{sign}_0^+(-u)\sum_{i,j=1}^N a_{ij}(x,u)D_j u D_i(h(u)\xi)dx\,dt, \tag{3.2}$$

for any $h \in C_c^1(R), h \geq 0, \xi \in W, \xi \geq 0$.

Let sign^+ denote the multivalued function defined by $\operatorname{sign}^+(r) = 0$ if $r < 0$, and $\operatorname{sign}^+(0) \subset [0,1]$, $\operatorname{sign}^+(r) = 1$ if $r > 0$.

Lemma 3.2. *For $i = 1, 2$, let $f_i \in L^2(Q)$, $A_i \in L^2(0,T)$, u_i be a renormalized solution to problem (P) for the data (f_i, A_i). Then there exist $\mathcal{K}_1 \in \operatorname{sign}^+(u_1 - u_2)$ and $\mathcal{K}_2 \in \operatorname{sign}^+(u_1|_\Gamma - u_2|_\Gamma)$ such that*

$$-\int_Q \xi_t \operatorname{sign}_0^+(u_1-u_2)\int_{u_2}^{u_1} h(r)dr\,dx\,dt$$
$$+\int_Q \operatorname{sign}_0^+(u_1-u_2)\sum_{i,j=1}^N \left[h(u_1)a_{ij}(x,u_1)D_j u_1 - h(u_2)a_{ij}(x,u_2)D_j u_2\right]D_i\xi\,dx\,dt$$
$$+\int_Q \sum_{i,j=1}^N \operatorname{sign}_0^+(u_1-u_2)\left[h'(u_1)a_{ij}(x,u_1)D_j u_1 D_i u_1 - h'(u_2)a_{ij}(x,u_2)D_j u_2 D_i u_2\right]\xi\,dx\,dt$$
$$\leq \int_Q \mathcal{K}_1\left[h(u_1)f_1 - h(u_2)f_2\right]\xi\,dx\,dt + \int_0^T \mathcal{K}_2\left[h(u_1(t)|_\Gamma)A_1(t) - h(u_2(t)|_\Gamma)A_2(t)\right]\xi|_\Gamma dt,$$
$$\forall h \in C_c^1(R), h \geq 0, \quad \forall \xi \in W, \xi \geq 0. \tag{3.3}$$

Proof. Let $\xi \in W$, $\xi \geq 0$, ρ_l be a sequence of mollifiers in R with supp $\rho_l \subset (-2/l, 0)$ and $\rho_l \geq 0$. Define

$$\xi_l(x,t,s) = \xi(x,t)\rho_l(t-s). \tag{3.4}$$

Note that for l sufficiently large,

$$(x,s) \longmapsto \xi_l(x,t,s) \in W, \quad \forall t \in [0,T],$$
$$(x,t) \longmapsto \xi_l(x,t,s) \in W, \quad \forall s \in [0,T]. \tag{3.5}$$

Let $h \in C_c^1(R)$, $h \geq 0$, $H_\varepsilon \in W^{1,\infty}(R)$ be defined by $H_\varepsilon(r) = H(r/\varepsilon)$, where $H \in W^{1,\infty}(R)$, $H(r) = 0$ for $r \leq 0$, $H(r) = r$ for $0 < r < 1$ and $H(r) = 1$ if $r \geq 1$. As u_1, u_2 are renormalized solutions , according to (2.6), for a.e. $t \in (0,T)$, we have

$$\begin{aligned}
&\int_Q (\xi_l)_s \int_0^{u_1} h(r) H_\varepsilon(r - u_2(t,x))\,dr\,dx\,ds + \int_Q f_1 h(u_1) H_\varepsilon(u_1(s,x) - u_2(t,x))\xi_l\,dx\,ds \\
&\quad + \int_0^T A_1(s) h(u_1(s)|_\Gamma) H_\varepsilon(u_1(s)|_\Gamma - u_2(t)|_\Gamma)\xi_l(s)|_\Gamma\,ds \\
&\quad = \int_Q \sum_{i,j=1}^N a_{ij}(x,u_1) D_j u_1 D_i[h(u_1) H_\varepsilon(u_1(s,x) - u_2(t,x))\xi_l]\,dx\,ds
\end{aligned} \tag{3.6}$$

and for a.e. $s \in (0,T)$, we have

$$\begin{aligned}
&\int_Q (\xi_l)_t \int_0^{u_2} h(r) H_\varepsilon(u_1(s,x) - r)\,dr\,dx\,dt + \int_Q f_2 h(u_2) H_\varepsilon(u_1(s,x) - u_2(t,x))\xi_l\,dx\,dt \\
&\quad + \int_0^T A_2(t) h(u_2(t)|_\Gamma) H_\varepsilon(u_1(s)|_\Gamma - u_2(t)|_\Gamma)\xi_l(t)|_\Gamma\,dt \\
&\quad = \int_Q \sum_{i,j=1}^N a_{ij}(x,u_2) D_j u_2 D_i[h(u_2) H_\varepsilon(u_1(s,x) - u_2(t,x))\xi_l]\,dx\,dt.
\end{aligned} \tag{3.7}$$

Integrating the above two equalities in t, respectively, s over $(0,T)$ and taking their difference, we get

$$\begin{aligned}
&\int_0^T \int_Q \left[(\xi_l)_s \int_0^{u_1} h(r) H_\varepsilon(r - u_2(t,x))\,dr - (\xi_l)_t \int_0^{u_2} h(r) H_\varepsilon(u_1(s,x) - r)\,dr\right] dx\,ds\,dt \\
&\quad + \int_0^T \int_Q \left[f_1 h(u_1) H_\varepsilon(u_1(s,x) - u_2(t,x)) - f_2 h(u_2) H_\varepsilon(u_1(s,x) - u_2(t,x))\right]\xi_l\,dx\,ds\,dt \\
&\quad + \int_0^T \int_0^T [A_1(s) h(u_1(s)|_\Gamma) H_\varepsilon(u_1(s)|_\Gamma - u_2(t)|_\Gamma)\xi_l(s)|_\Gamma \\
&\qquad - A_2(t) h(u_2(t)|_\Gamma) H_\varepsilon(u_1(s)|_\Gamma - u_2(t)|_\Gamma)\xi_l(t)|_\Gamma]\,ds\,dt
\end{aligned}$$

$$= \int_0^T \int_Q \sum_{i,j=1}^N \left[a_{ij}(x,u_1) D_j u_1 h(u_1) D_i (H_\varepsilon(u_1(s,x) - u_2(t,x))\xi_l) \right.$$
$$\left. - a_{ij}(x,u_2) D_j u_2 h(u_2) D_i (H_\varepsilon(u_1(s,x) - u_2(t,x))\xi_l) \right] dxdsdt$$
$$+ \int_0^T \int_Q \sum_{i,j=1}^N \left[h'(u_1) a_{ij}(x,u_1) D_j u_1 D_i u_1 H_\varepsilon(u_1(s,x) - u_2(t,x)) \right.$$
$$\left. - h'(u_2) a_{ij}(x,u_2) D_j u_2 D_i u_2 H_\varepsilon(u_1(s,x) - u_2(t,x)) \right] \xi_l dxdsdt. \tag{3.8}$$

Denote the three integrals on the left-hand side by I_1, I_2, I_3, the two integrals on the right-hand side by I_4, I_5.

It is easy to prove that

$$\lim_{l\to\infty}\lim_{\varepsilon\to 0} I_5 = \int_Q \sum_{i,j=1}^N \operatorname{sign}_0^+(u_1 - u_2)\xi \left[h'(u_1) a_{ij}(x,u_1) D_j u_1 D_i u_1 - h'(u_2) a_{ij}(x,u_2) D_j u_2 D_i u_2 \right] dxdt. \tag{3.9}$$

Similarly to the estimates for I_2 in [12], (c.f. page 102), we can obtain

$$\begin{aligned} \overline{\lim_{l\to\infty}}\lim_{\varepsilon\to 0} I_2 &\le \int_Q \mathcal{K}_1 \left[f_1 h(u_1) - f_2 h(u_2) \right] \xi dxdt, \\ \overline{\lim_{l\to\infty}}\lim_{\varepsilon\to 0} I_3 &\le \int_0^T \xi(t)|_\Gamma \mathcal{K}_2 [h(u_1(t)|_\Gamma) A_1(t) - h(u_2(t)|_\Gamma) A_2(t)] dt, \end{aligned} \tag{3.10}$$

where $\mathcal{K}_1 = \chi_{\{u_1>u_2\}} + \chi_{\{u_1=u_2\}}\operatorname{sign}_0^+(f_1 - f_2)$, $\mathcal{K}_2 = \chi_{\{t\in(0,T):u_1(t)|_\Gamma > u_2(t)|_\Gamma\}} + \operatorname{sign}_0^+(A_1 - A_2)\chi_{\{t\in(0,T):u_1(t)|_\Gamma = u_2(t)|_\Gamma\}}$.

As for I_1, recall that $\operatorname{supp}\rho_l \subset (-2/l, 0)$, hence

$$\lim_{\varepsilon\to 0} I_1 = \int_0^T \int_Q \operatorname{sign}_0^+(u_1(x,s) - u_2(x,t)) \left[(\xi_l)_s \int_{u_2(x,t)}^{u_1(x,s)} h(r)dr + (\xi_l)_t \int_{u_2(x,t)}^{u_1(x,s)} h(r)dr \right] dxdsdt$$
$$+ \int_Q \xi_l(x,t,0) \operatorname{sign}_0^+(-u_2(x,t)) \int_{u_2(x,t)}^0 h(r)drdxdt$$
$$+ \int_Q \xi_l(x,0,s) \operatorname{sign}_0^+(u_1(x,s)) \int_0^{u_1(x,s)} h(r)drdxds$$
$$= \int_0^T \int_Q \operatorname{sign}_0^+(u_1(x,s) - u_2(x,t)) \xi_t(x,t) \rho_l(t-s) \int_{u_2(x,t)}^{u_1(x,s)} h(r)drdsdt$$
$$+ \int_Q \xi(x,0)\rho_l(-s) \operatorname{sign}_0^+(u_1(x,s)) \int_0^{u_1(x,s)} h(r)drdxds = I_{11} + I_{12}. \tag{3.11}$$

We have

$$\lim_{l\to\infty} I_{11} = \int_Q \operatorname{sign}_0^+(u_1 - u_2)\xi_t \int_{u_2}^{u_1} h(r)\,dr\,dx\,dt. \tag{3.12}$$

Consider the function

$$\phi_l(x,s) = \int_s^T \rho_l(-r)\,dr\,\xi(x,0) = \int_{\inf\{s,2/l\}}^{2/l} \rho_l(-r)\,dr\,\xi(x,0). \tag{3.13}$$

Note that $\xi \in W$, thus for l sufficiently large, $\phi_l \in W$. Applying (3.1) with $u = u_1$, $\xi = \phi_l$, $f = f_1$, $A(t) = A_1(s)$, and $t = s$, we have

$$\begin{aligned} I_{12} &= -\int_Q (\phi_l)_s \operatorname{sign}_0^+(u_1) \int_0^{u_1} h(r)\,dr\,dx\,ds \\ &\le \int_0^T \operatorname{sign}_0^+(u_1(s)|_\Gamma) A_1(s)\phi_l(s)|_\Gamma h(u_1(s)|_\Gamma)\,ds \\ &\quad + \int_Q \operatorname{sign}_0^+(u_1)\phi_l f h(u_1)\,dx\,ds - \int_Q \operatorname{sign}_0^+(u_1)\sum_{i,j=1}^N a_{ij}(x,u_1) D_j u_1 D_i(\phi_l h(u_1))\,dx\,ds. \end{aligned} \tag{3.14}$$

It is easy to prove that the integrals on the right-hand side of (3.14) converge to 0 as $l \to +\infty$. Thus we get

$$\overline{\lim_{l\to\infty}} I_{12} \le 0. \tag{3.15}$$

It remains to consider I_4. We have

$$\begin{aligned} I_4 &= \int_0^T \int_Q \sum_{i,j=1}^N \big[h(u_1)a_{ij}(x,u_1)D_j u_1 H_\varepsilon(u_1 - u_2(t,\cdot)) \\ &\qquad\qquad -h(u_2)a_{ij}(x,u_2)D_j u_2 H_\varepsilon(u_1(s,\cdot) - u_2)\big] D_i\xi_l\,dx\,ds\,dt \\ &\quad + \frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \sum_{i,j=1}^N \big[h(u_1)a_{ij}(x,u_1)D_j u_1 D_i(u_1 - u_2(t,\cdot)) \\ &\qquad\qquad -h(u_2)a_{ij}(x,u_2)D_j u_2 D_i(u_1(s,\cdot) - u_2)\big]\xi_l\,dx\,ds\,dt \\ &= I_{41} + I_{42}. \end{aligned} \tag{3.16}$$

It is easy to see that

$$\lim_{l\to\infty}\lim_{\varepsilon\to 0} I_{41} = \int_Q \operatorname{sign}_0^+(u_1-u_2)\sum_{i,j=1}^{N}\left[h(u_1)a_{ij}(x,u_1)D_ju_1 - h(u_2)a_{ij}(x,u_2)D_ju_2\right]D_i\xi\, dx\, dt. \tag{3.17}$$

Since

$$\begin{aligned} I_{42} &= \frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \xi_l \sum_{i,j=1}^{N}(h(u_1(x,s)) - h(u_2(x,t))) \\ &\qquad \times a_{ij}(x,u_1)D_ju_1D_i(u_1-u_2(t,\cdot))\,dx\,ds\,dt \\ &\quad + \frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \xi_l \sum_{i,j=1}^{N} h(u_2(x,t)) \\ &\qquad \times \left[a_{ij}(x,u_1)D_ju_1 - a_{ij}(x,u_2)D_ju_2\right]D_i(u_1-u_2)\,dx\,ds\,dt \\ &= I_{42}^{(1)} + I_{42}^{(2)}, \end{aligned} \tag{3.18}$$

let $k>0$ such that $\operatorname{supp} h \subset (-k,k)$, for $\varepsilon<1$, we have

$$\begin{aligned} \left|I_{42}^{(1)}\right| &\le \int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \sum_{i,j=1}^{N}|\xi_l|\left|h'\right||d_{k+1}(x)|\left|k+1+a_{ij}(x,0)\right| \\ &\qquad \times \left(2|DT_{k+1}(u_1)|^2 + |DT_{k+1}(u_2)|^2\right)dx\,ds\,dt. \end{aligned} \tag{3.19}$$

Noting that the right side integral of (3.19) belongs to $L^1((0,T)\times Q)$, we get

$$\lim_{\varepsilon\to 0} I_{42}^{(1)} = 0. \tag{3.20}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} I_{42}^{(2)} &= \frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \xi_l \\ &\quad \times \sum_{i,j=1}^{N} h(u_2(x,t))\left[a_{ij}(x,u_1)D_ju_1 - a_{ij}(x,u_1)D_ju_2\right]D_i(u_1-u_2)\,dx\,ds\,dt \\ &\quad + \frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \xi_l \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{i,j=1}^{N} h(u_2(x,t))\left[a_{ij}(x,u_1) - a_{ij}(x,u_2)\right] D_j u_2 D_i (u_1 - u_2)\,dx\,ds\,dt \\
&\geq -\frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \xi_l \\
&\quad \times \sum_{i,j=1}^{N} h(u_2)|d_{k+1}(x)|\left(|DT_{k+1}(u_1)|^2 + 2|DT_{k+1}(u_2)|^2\right)|u_1 - u_2|\,dx\,ds\,dt.
\end{aligned}
\tag{3.21}
$$

Using the same approach as in (3.20), we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{(0,T)\times Q\cap\{0<|u_1(x,s)-u_2(x,t)|<\varepsilon\}} \xi_l \\
&\quad \times \sum_{i,j=1}^{N} h(u_2)|d_{k+1}(x)|\left(|DT_{k+1}(u_1)|^2 + 2|DT_{k+1}(u_2)|^2\right)|u_1 - u_2|\,dx\,ds\,dt = 0.
\end{aligned}
\tag{3.22}
$$

By (3.20)–(3.22) and (3.18) we have

$$
\underline{\lim}_{\varepsilon\to 0} I_{42} \geq 0. \tag{3.23}
$$

Equations (3.8)–(3.12), (3.15)–(3.18), (3.20), and (3.23) imply that (3.3) holds. Thus Lemma 3.2 is proved. □

Remark 3.3. In fact, by the density result, (3.3) is satisfied by any given $\xi \in W_1 = \{\xi \in W^{1,\infty}(Q) \mid \xi(T) = 0,\ \xi(t)|_\Gamma = C(t)$ (an arbitary function of t)$\}$ and $\xi \geq 0$.

Now we state the uniqueness and comparison principle of renormalized solution to problem (P) as follows.

Theorem 3.4. *Under hypotheses* (2.1) *and* (2.2), *for* $i = 1,2$, *let* $f_i \in L^2(Q)$, $A_i \in L^2(0,T)$, u_i *be a renormalized solution to problem* (P) *for the data* (f_i, A_i). *Then there exist* $\mathcal{K}_1 \in \operatorname{sign}^+(u_1 - u_2)$ *and* $\mathcal{K}_2 \in \operatorname{sign}^+(u_1|_\Gamma - u_2|_\Gamma)$ *such that for a.e.* $0 < \tau < T$,

$$
\int_\Omega (u_1(\tau) - u_2(\tau))^+ dx \leq \int_0^\tau\int_\Omega \mathcal{K}_1(f_1 - f_2)\,dx\,dt + \int_0^\tau \mathcal{K}_2(A_1 - A_2)\,dt. \tag{3.24}
$$

In particular, for any given $A \in L^2(0,T)$ and $f \in L^2(Q)$, the renormalized solution u to problem (P) is unique.

Proof. For any given $\tau \in (0, T)$ and any given $\varepsilon > 0$ sufficiently small, let $\alpha_\varepsilon(s)$ be defined by

$$\alpha_\varepsilon(t) = \begin{cases} 1, & \text{if } 0 \le t \le \tau - \varepsilon, \\ \dfrac{\tau - t}{\varepsilon}, & \text{if } \tau - \varepsilon < t < \tau, \\ 0, & \text{if } T \ge t \ge \tau. \end{cases} \tag{3.25}$$

Defining $\xi(x,t) = \alpha_\varepsilon(t)$, $\forall (x,t) \in \overline{Q}$, it is easy to see that $\xi \in W_1$ and $\xi \ge 0$.

Taking $\xi = \alpha_\varepsilon$ in (3.3), we get

$$\begin{aligned} &- \int_Q (\alpha_\varepsilon)_t \operatorname{sign}_0^+(u_1 - u_2) \int_{u_2}^{u_1} h(r)\,dr\,dx\,dt \\ &\quad + \int_Q \sum_{i,j=1}^N \operatorname{sign}_0^+(u_1 - u_2)\alpha_\varepsilon \left[h'(u_1)a_{ij}(x,u_1)D_j u_1 D_i u_1 - h'(u_2)a_{ij}(x,u_2)D_j u_2 D_i u_2\right] dx\,dt \\ &\le \int_Q \mathcal{K}_1 \left[h(u_1)f_1 - h(u_2)f_2\right]\alpha_\varepsilon(t)\,dx\,dt \\ &\quad + \int_0^T \mathcal{K}_2 [h(u_1(t)|_\Gamma)A_1(t) - h(u_2(t)|_\Gamma)A_2(t)]\alpha_\varepsilon(t)\,dt. \end{aligned} \tag{3.26}$$

Defining $h_m(r) = \inf((m + 1 - |r|)^+, 1)$ and replacing h with h_m in (3.26), then letting $m \to +\infty$, we obtain

$$-\int_Q (\alpha_\varepsilon)_t (u_1 - u_2)^+ dx\,dt \le \int_Q \alpha_\varepsilon(t)\mathcal{K}_1(f_1 - f_2)\,dx\,dt + \int_0^T \mathcal{K}_2(A_1(t) - A_2(t))\alpha_\varepsilon(t)\,dt. \tag{3.27}$$

In fact, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} &- \int_Q (\alpha_\varepsilon)_t \operatorname{sign}_0^+(u_1 - u_2) \int_{u_2}^{u_1} h_m(r)\,dr\,dx\,dt \longrightarrow -\int_Q (\alpha_\varepsilon)_t (u_1 - u_2)^+ dx\,dt, \\ &\int_Q \mathcal{K}_1 \left[h_m(u_1)f_1 - h_m(u_2)f_2\right]\alpha_\varepsilon(t)\,dx\,dt \longrightarrow \int_Q \alpha_\varepsilon(t)\mathcal{K}_1(f_1 - f_2)\,dx\,dt, \\ &\int_0^T \mathcal{K}_2 [h_m(u_1(t)|_\Gamma)A_1(t) - h_m(u_2(t)|_\Gamma)A_2(t)]\alpha_\varepsilon(t)\,dt, \longrightarrow \int_0^T \mathcal{K}_2(A_1(t) - A_2(t))\alpha_\varepsilon(t)\,dt. \end{aligned} \tag{3.28}$$

As for the second term in (3.26), we have

$$\begin{aligned}
&\int_Q \sum_{i,j=1}^{N} \operatorname{sign}_0^+(u_1-u_2)\alpha_\varepsilon\left[h_m'(u_1)a_{ij}(x,u_1)D_ju_1D_iu_1 - h_m'(u_2)a_{ij}(x,u_2)D_ju_2D_iu_2\right]dxdt \\
&\quad= \int_Q \sum_{i,j=1}^{N} \operatorname{sign}_0^+(u_1-u_2)\alpha_\varepsilon h_m'(u_1)a_{ij}(x,u_1)D_ju_1D_iu_1\,dxdt \\
&\qquad- \int_Q \sum_{i,j=1}^{N} \operatorname{sign}_0^+(u_1-u_2)\alpha_\varepsilon h_m'(u_2)a_{ij}(x,u_2)D_ju_2D_iu_2\,dxdt = J_1 + J_2.
\end{aligned} \tag{3.29}$$

Moreover,

$$|J_k| \le \int_{\{m\le|u_k|\le m+1\}} \sum_{i,j=1}^{N} a_{ij}(x,u_k)D_ju_kD_iu_k\,dxdt, \quad k=1,2. \tag{3.30}$$

As u_1, u_2 are renormalized solutions, noting (2.7), we prove

$$\lim_{m\to+\infty} J_1 = \lim_{m\to+\infty} J_2 = 0. \tag{3.31}$$

By (3.29) and (3.31), the second term in (3.26) tends to zero. □

Letting ε tend to zero in (3.27), (3.24) follows from (3.25) and (3.27).

4. The Relation between Weak Solutions and Renormalized Solutions for Problem (P)

In this section, we will see that the concept of renormalized solution is an extension of the concept of weak solution. The main result in this section is the following theorem.

Theorem 4.1. (i) *Assume that* $u \in L^2(0,T;V)\cap L^\infty(0,T;L^2(\Omega))$ *and* $a_{ij}(\cdot,u)D_ju \in L^2(Q)$, $i,j = 1,2,\dots N$. *Then* u *is a weak solution to problem* (P) *if and only if* u *is a renormalized solution to problem* (P).

(ii) *If* $u \in L^2(0,T;V)\cap L^\infty(Q)$, *then* u *is a weak solution to problem* (P) *if and only if* u *is a renormalized solution to problem* (P).

Proof. (i) If u is a weak solution to problem (P), we have

$$\begin{aligned}
&\int_0^T \langle u_t, v\rangle_{V',V}dt + \int_Q \sum_{i,j=1}^{N} a_{ij}(x,u)D_juD_iv\,dxdt \\
&\quad= \int_Q fv\,dxdt + \int_0^T A(t)v(t)|_\Gamma dt, \quad \forall v \in L^2(0,T;V),
\end{aligned} \tag{4.1}$$

$$u(x,0) = 0. \tag{4.2}$$

Noting that $u \in L^2(0,T;V)$, $a_{ij}(x,u)D_ju \in L^2(Q), i,j = 1,2\dots N$, we have $u \in L^2(0,T;V')$, hence $u \in C([0,T];L^2(\Omega))$, for any given $h \in C_c^1(R)$ and $\xi \in W$. Taking $v = \xi h(u)$ in (4.1), we have

$$\begin{aligned}\int_0^T \langle u_t, \xi h(u)\rangle dt = -\int_Q \sum_{i,j=1}^N a_{ij}(x,u)D_juD_i(\xi h(u))dxdt + \int_Q f\xi h(u)dxdt \\ + \int_0^T A(t)h(u(t)|_\Gamma)\xi(t)|_\Gamma dt.\end{aligned} \tag{4.3}$$

By [12, Lemma 1.4], we have

$$\int_0^T \langle u_t, \xi h(u)\rangle dt = -\int_Q \xi_t \int_0^u h(r)drdxdt. \tag{4.4}$$

Equations (4.3) and (4.4) imply that

$$\begin{aligned}-\int_Q \xi_t \int_0^u h(r)dr + \int_Q \sum_{i,j=1}^N a_{ij}(x,u)D_juD_i(\xi h(u))dxdt \\ = \int_Q f\xi h(u)dxdt + \int_0^T A(t)h(u(t)|_\Gamma)\xi(t)|_\Gamma dt. \quad \forall \xi \in W,\ h \in C_c^1(R).\end{aligned} \tag{4.5}$$

For any given m, let $S_m(r) = \int_0^r [T_{m+1}(s) - T_m(s)]ds$, it is easy to see that $0 \le S_m(r) \le |r|$. Taking $v = T_{m+1}(u) - T_m(u)$ in (4.1), we get

$$\begin{aligned}\int_\Omega S_m(u(x,T))dx + \int_Q \sum_{i,j=1}^N a_{ij}(x,u)D_juD_i[T_{m+1}(u) - T_m(u)]dxdt \\ = \int_Q f[T_{m+1}(u) - T_m(u)]dxdt + \int_0^T A(t)[T_{m+1}(u(t)|_\Gamma) - T_m(u(t)|_\Gamma)]dt.\end{aligned} \tag{4.6}$$

By Lebesgue's dominated convergence theorem, we obtain the first term on the left side, and all terms on the right side of (4.6) converge to zero as $m \to +\infty$. Then it follows from (4.6) that

$$\begin{aligned}\lim_{m\to+\infty}\int_{\{m\le|u|\le m+1\}} \sum_{i,j=1}^N a_{ij}(x,u)D_juD_iudxdt \\ = \lim_{m\to+\infty}\int_Q \sum_{i,j=1}^N a_{ij}(x,u)D_juD_i[T_{m+1}(u) - T_m(u)]dxdt = 0.\end{aligned} \tag{4.7}$$

Conversely, assume that u is a renormalized solution. Applying (2.6) with $h(u) = H(n+1-|u|)$, where $H \in C^\infty(R)$, $H' \geq 0$, $H = 0$ on $(-\infty, 0]$, and $H = 1$ on $[1, +\infty)$, as $n \to +\infty$, we get

$$-\int_Q \xi_t u\,dx\,dt + \int_Q \sum_{i,j=1}^{N} a_{ij}(x,u) D_j u D_i \xi\,dx\,dt = \int_Q f\xi\,dx\,dt + \int_0^T A(t)\xi(t)|_\Gamma dt. \tag{4.8}$$

Hence u is a weak solution to problem (P).

(ii) Due to $u \in L^\infty(Q)$, assumption (3.1) and the definition of a renormalized solution to problem (P), (ii) is an immediate consequence of (i). □

Remark 4.2. Theorems 2.5 and 3.4 improve those results of [1, 3, 8].

Acknowledgments

This work is supported by NCET of China(no: 060275) and Foundation of Maths +X of DLUT.

References

[1] W. X. Shen, "On mixed initial-boundary value problems of second order parabolic equations with equivalued surface boundary conditions," *Journal of Fudan University. Natural Science*, vol. 4, pp. 15–24, 1978 (Chinese).

[2] Z. X. Chen and L. S. Jiang, "Exact solution for the system of flow equations through a medium with double-porosity," *Scientia Sinica*, vol. 23, no. 7, pp. 880–896, 1980.

[3] A. Damlamian and T. Li, "Comportement limite des solutions de certains problèmes mixtes pour les équations paraboliques," *Journal de Mathématiques Pures et Appliquées*, vol. 61, no. 2, pp. 113–130, 1982.

[4] T. Li, "A class of nonlocal boundary value problems for partial differential equations and its applications in numerical analysis," *Journal of Computational and Applied Mathematics*, vol. 28, pp. 49–62, 1989.

[5] T. Li, S. Zheng, Y. Tan, and W. Shen, *Boundary Value Problems with Equivalued Surface and Resistivity Well-Logging*, vol. 382 of *Pitman Research Notes in Mathematics Series*, Longman, Harlow, UK, 1998.

[6] G. Chavent and J. Jaffre, *Mathematical Models and Finite Elements for Reservoir Simulation*, North-Holland, Amsterdam, The Netherlands, 1986.

[7] W. X. Shen and S. M. Zheng, "Nonlocal initial-boundary value problems for nonlinear parabolic equations," *Journal of Fudan University. Natural Science*, vol. 24, no. 1, pp. 47–57, 1985 (Chinese).

[8] F. Li, "Existence and uniqueness of bounded weak solution for non-linear parabolic boundary value problem with equivalued surface," *Mathematical Methods in the Applied Sciences*, vol. 27, no. 10, pp. 1115–1124, 2004.

[9] R. J. DiPerna and P.-L. Lions, "On the Cauchy problem for Boltzmann equations: global existence and weak stability," *Annals of Mathematics*, vol. 130, no. 2, pp. 321–366, 1989.

[10] A. Porretta, "Existence results for nonlinear parabolic equations via strong convergence of truncations," *Annali di Matematica Pura ed Applicata*, vol. 177, pp. 143–172, 1999.

[11] D. Blanchard and F. Murat, "Renormalised solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 127, no. 6, pp. 1137–1152, 1997.

[12] J. Carrillo and P. Wittbold, "Uniqueness of renormalized solutions of degenerate elliptic-parabolic problems," *Journal of Differential Equations*, vol. 156, no. 1, pp. 93–121, 1999.

[13] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. Part 2B: Nonlinear Monotone Operators*, Springer, New York, NY, USA, 1990.

[14] R. Landes, “On the existence of weak solutions for quasilinear parabolic initial-boundary value problems,” *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 89, no. 3-4, pp. 217–237, 1981.

[15] J. Filo and J. Kačur, “Local existence of general nonlinear parabolic systems,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 24, no. 11, pp. 1597–1618, 1995.