Hindawi Publishing Corporation
Advances in Difference Equations
Volume 2010, Article ID 841643, 11 pages
doi:10.1155/2010/841643

Research Article

Approximation of Solution of Some *m*-Point Boundary Value Problems on Time Scales

Rahmat Ali Khan[1] and Mohammad Rafique[2]

[1] *Centre for Advanced Mathematics and Physics, National University of Sciences and Technology (NUST), H-12, Islamabad 46000, Pakistan*
[2] *Department of Basic Sciences, College of Electrical and Mechanical Engineering, National University of Sciences and Technology (NUST), Peshawar Road, Rawalpindi 46000, Pakistan*

Correspondence should be addressed to Mohammad Rafique, mrdhillon@yahoo.com

Received 24 August 2009; Revised 13 May 2010; Accepted 2 June 2010

Academic Editor: Ondřej Došlý

The method of upper and lower solutions and the generalized quasilinearization technique for second-order nonlinear m-point dynamic equations on time scales of the type $x^{\Delta\Delta}(t) = f(t, x^{\sigma})$, $t \in [0,1]_{\mathbb{T}} = [0,1] \cap \mathbb{T}$, $x(0) = 0$, $x(\sigma^2(1)) = \sum_{i=1}^{m-1} \alpha_i x(\eta_i)$, $\eta_i \in (0,1)_{\mathbb{T}}$, $\sum_{i=1}^{m-1} \alpha_i \leq 1$, are developed. A monotone sequence of solutions of linear problems converging uniformly and quadratically to a solution of the problem is obtained.

1. Introduction

Many dynamical processes contain both continuous and discrete elements simultaneously. Thus, traditional mathematical modeling techniques, such as differential equations or difference equations, provide a limited understanding of these types of models. A simple example of this hybrid continuous-discrete behavior appears in many natural populations: for example, insects that lay their eggs at the end of the season just before the generation dies out, with the eggs laying dormant, hatching at the start of the next season giving rise to a new generation. For more examples of species which follow this type of behavior, we refer the readers to [1].

Hilger [2] introduced the notion of time scales in order to unify the theory of continuous and discrete calculus. The field of dynamical equations on time scales contain, links and extends the classical theory of differential and difference equations, besides many others. There are more time scales than just $\mathbb{R}$ (corresponding to the continuous case) and $\mathbb{N}$ (corresponding to the discrete case) and hence many more classes of dynamic equations. An excellent resource with an extensive bibliography on time scales was produced by Bohner and Peterson [3, 4].

Recently, existence theory for positive solutions of boundary value problems (BVPs) on time scales has attracted the attention of many authors; see, for example, [5–12] and the references therein for the existence theory of some two-point BVPs, and [13–16] for three-point BVPs on time scales. For the existence of solutions of m-point BVPs on time scales, we refer the readers to [17].

However, the method of upper and lower solutions and the quasilinearization technique for BVPs on time scales are still in the developing stage and few papers are devoted to the results on upper and lower solutions technique and the method of quasilinearization on time scales [18–21]. The pioneering paper on multipoint BVPs on time scales has been the one in [21] where lower and upper solutions were combined with degree theory to obtain very wide-ranging existence results. Further, the authors of [21] studied existence results for more general three-point boundary conditions which involve first delta derivatives and they also developed some compatibility conditions. We are very grateful to the reviewer for directing us towards this important work.

Recently, existence results via upper and lower solutions method and approximation of solutions via generalized quasilinearization method for some three-point boundary value problems on time scales have been studied in [16]. Motivated by the work in [16, 17], in this paper, we extend the results studied in [16] to a class of m-point BVPs of the type

$$\begin{gathered} x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) = 0, x\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1} \alpha_i x(\eta_i), \end{gathered} \tag{1.1}$$

where $\eta_i \in (0,1)_{\mathbb{T}}$, $\sum_{i=1}^{m-1} \alpha_i \leq 1$, and t is from a so-called time scale $\mathbb{T}$ (which is an arbitrary closed subset of $\mathbb{R}$). Existence of at least one solution for (1.1) has already been studied in [17] by the Krasnosel'skii and Zabreiko fixed point theorems. We obtain existence and uniqueness results and develop a method to approximate the solutions.

Assume that $\mathbb{T}$ has a topology that it inherits from the standard topology on $\mathbb{R}$ and define the time scale interval $[0,1]_{\mathbb{T}} = \{t \in \mathbb{T} : 0 \leq t \leq 1\}$. For $t \in \mathbb{T}$, define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\sigma(t) = t$, t is said to be right dense. If $\rho(t) < t$, t is said to be left scattered, and if $\rho(t) = t$, t is said to be left dense.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at all right-dense points of $\mathbb{T}$ and its left-sided limit exists at left-dense points of $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be ld-continuous provided it is continuous at all left-dense points of $\mathbb{T}$ and its right-sided limit exists at right-dense points of $\mathbb{T}$. Define $\mathbb{T}^k = \mathbb{T} - \{m\}$ if $\mathbb{T}$ has a left-scattered maximum at m; otherwise $\mathbb{T}^k = \mathbb{T}$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, the delta derivative $f^{\Delta}(t)$ of f at t (if exists) is defined by the following Given that $\epsilon > 0$, there exists a neighborhood U of t such that

$$\left|[f(\sigma(t) - f(s))] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|, \quad \forall s \in U. \tag{1.2}$$

If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$, F is said to be the delta antiderivative of f and the delta integral is defined by

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a), \quad a, b \in \mathbb{T}. \tag{1.3}$$

Definition 1.1. Define $C^2_{\mathrm{rd}}([0,\sigma^2(1)]_{\mathbb{T}})$ to be the set of all functions $y:\mathbb{T}\to\mathbb{R}$ such that

$$C^2_{\mathrm{rd}}\left(\left[0,\sigma^2(1)\right]_{\mathbb{T}}\right)=\left\{y: y, y^{\Delta}\in C\left(\left[0,\sigma^2(1)\right]_{\mathbb{T}}\right) \text{ and } y^{\Delta\Delta}\in C_{\mathrm{rd}}([0,1]_{\mathbb{T}})\right\}. \tag{1.4}$$

A solution of (1.1) is a function $y\in C^2_{\mathrm{rd}}([0,\sigma^2(1)]_{\mathbb{T}})$ which satisfies (1.1) for each $t\in[0,1]_{\mathbb{T}}$.

Let us denote

$$\begin{aligned}C_{\mathrm{rd}}[[0,1]_{\mathbb{T}}\times\mathbb{R}]=\{&y(t,x): y(\cdot,x) \text{ is } C_{\mathrm{rd}}[0,1]_{\mathbb{T}} \text{ for every } x\in\mathbb{R} \text{ and } y(t,\cdot)\\ &\text{is continuous on } \mathbb{R} \text{ uniformly at each } t\in[0,1]_{\mathbb{T}}\},\\ C^2_{\mathrm{rd}}([0,1]_{\mathbb{T}}\times\mathbb{R})=\{&y(t,x): y(\cdot,x), y_x(\cdot,x), y_x(\cdot,x) \text{ are } C_{\mathrm{rd}}[0,1]_{\mathbb{T}}\\ &\text{for every } x\in\mathbb{R} \text{ and } y(t,\cdot), y_x(t,\cdot), y_{xx}(t,\cdot)\\ &\text{are continuous on } \mathbb{R} \text{ uniformly at each } t\in[0,1]_{\mathbb{T}}\}.\end{aligned} \tag{1.5}$$

The purpose of this paper is to develop the method of upper and lower solutions and the method of quasilinearization [22–26]. Under suitable conditions on f, we obtain a monotone sequence of solutions of linear problems. We show that the sequence of approximants converges uniformly and quadratically to a unique solution of the problem.

2. Upper and Lower Solutions Method

We write the BVP (1.1) as an equivalent Δ-integral equation

$$x(t)=\int_a^{\sigma(b)} G(t,s)f(s,x^{\sigma}(s))\Delta s,\quad t\in[0,\sigma^2(1)]_{\mathbb{T}}, \tag{2.1}$$

where $G(t,s)$ is a Green's function for the problem

$$\begin{gathered}y^{\Delta\Delta}(t)=,\quad t\in[0,1]_{\mathbb{T}},\\ y(0)=0,\qquad y\left(\sigma^2(1)\right)-\sum_{i=1}^{m-1}\alpha_i x(\eta_i)=0,\end{gathered} \tag{2.2}$$

and it is given by [17]

$$G(t,s)=\begin{cases} t\left[1+\dfrac{1}{T}\left[\displaystyle\sum_{i=1}^{k}\alpha_i(\sigma(\eta_i)-\sigma(s))-\sigma(s)\alpha\right]\right], & t\le s,\ \sigma(\eta_k)\le s\le\eta_{k+1},\\[2ex] \sigma(s)+\dfrac{t}{T}\left[\displaystyle\sum_{i=1}^{k}\alpha_i(\sigma(\eta_i)-\sigma(s))-\sigma(s)\alpha\right], & \sigma(s)\le t,\ \sigma(\eta_k)\le s\le\eta_{k+1},\end{cases} \tag{2.3}$$

where $k=0,1,2,\ldots,m-1$, $\eta_0=0$, and $\eta_{k+1}=\sigma^2(1)$.

Notice that $G(t,s) > 0$ on $(0,\sigma^2(1))_{\mathbb{T}} \times (0,\sigma(1))_{\mathbb{T}}$ and is rd-continuous. Define an operator $N : C[0,\sigma^2(1)]_{\mathbb{T}} \to C[0,\sigma^2(1)]_{\mathbb{T}}$ by

$$(Nx)(t) = \int_0^{\sigma(1)} G(t,s) f(s, x^\sigma(s))\Delta s, \quad t \in [0,\sigma^2(1)]_{\mathbb{T}}. \tag{2.4}$$

By a solution of (2.1), we mean a solution of the operator equation

$$(I - N)x = 0, \quad \text{that is, a fixed point of } N, \tag{2.5}$$

where I is the identity. If $f \in C[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded on $[0,1]_{\mathbb{T}} \times \mathbb{R}$, then by Arzela-Ascoli theorem N is compact and Schauder's fixed point theorem yields a fixed point of N. We discuss the case when f is not necessarily bounded on $[0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R}$.

Definition 2.1. We say that $\alpha \in C^2_{\text{rd}}[0,\sigma^2(1)]_{\mathbb{T}}$ is a lower solution of the BVP (1.1), if

$$\begin{gathered} \alpha^{\Delta\Delta}(t) \geq f(t, \alpha^\sigma(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ \alpha(0) \leq 0, \quad \alpha(\sigma^2(1)) \leq \sum_{i=1}^{m-1} \alpha_i \alpha(\eta_i). \end{gathered} \tag{2.6}$$

Similarly, $\beta \in C^2_{\text{rd}}[0,\sigma^2(1)]_{\mathbb{T}}$ is an upper solution of the BVP (1.1) if

$$\begin{gathered} \beta^{\Delta\Delta}(t) \leq f(t, \beta^\sigma(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ \beta(0) \geq 0, \quad \beta(\sigma^2(1)) \geq \sum_{i=1}^{m-1} \alpha_i \beta(\eta_i). \end{gathered} \tag{2.7}$$

Theorem 2.2. *(comparison result) Assume that α, β are lower and upper solutions of the boundary value problem* (1.1)*. If $f(t,x) \in C_{\text{rd}}[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and is strictly increasing in x for each $t \in [0,\sigma^2(1)]_{\mathbb{T}}$, then $\alpha \leq \beta$ on $[0,\sigma^2(1)]_{\mathbb{T}}$.*

Proof. Define $v(t) = \alpha(t) - \beta(t), t \in [0,\sigma^2(1)]_{\mathbb{T}}$. Then $v \in C^2_{\text{rd}}[0,\sigma^2(1)]_{\mathbb{T}}$ and the BCs imply that

$$v(0) \leq 0, v\left(\sigma^2(1)\right) \leq \sum_{i=1}^{m-1} \alpha_i v(\eta_i). \tag{2.8}$$

Assume that the conclusion of the theorem is not true. Then, v has a positive maximum at some $t_0 \in [0,\sigma^2(1)]_{\mathbb{T}}$. Clearly, $t_0 > 0$. If $t_0 \in (0,\sigma^2(1))_{\mathbb{T}}$, then, the point t_0 is not simultaneously left dense and right scattered; see, for example, [12]. Hence by Lemma 1 of [12],

$$v^{\Delta\Delta}(\rho(t_0)) \leq 0. \tag{2.9}$$

On the other hand, using the definitions of lower and upper solutions, we obtain

$$v^{\Delta\Delta}(\rho(t_0)) = \alpha^{\Delta\Delta}(\rho(t_0)) - \beta^{\Delta\Delta}(\rho(t_0)) \geq f(\rho(t_0), \alpha^\sigma(\rho(t_0))) - f(\rho(t_0), \beta^\sigma(\rho(t_0))). \tag{2.10}$$

Since t_0 is not simultaneously left dense and right scattered, it is left scattered and right scattered, left dense and right dense, or left scattered and right dense. In either case $\sigma(\rho(t_0)) = t_0$. Using the increasing property of $f(t,x)$ in x, we obtain

$$v^{\Delta\Delta}(\rho(t_0)) > 0, \tag{2.11}$$

a contradiction. Hence $v(t)$ has no positive local maximum.

If $t_0 = \sigma^2(1)$, then $v(\sigma^2(1)) > 0$. If any one of the η_i is such that $v(\eta_i) = v(\sigma^2(1))$, then v has a positive local maximum, a contradiction. Hence

$$v(\eta_i) < v\left(\sigma^2(1)\right), \quad \text{for each } i = 1,2,3,\ldots,m-1. \tag{2.12}$$

Moreover, if $\alpha_i = 0$ for each $i = 1,2,3,\ldots,m-1$, then, from the BCs

$$v\left(\sigma^2(1)\right) \le \sum_{i=1}^{m-1} \alpha_i v(\eta_i), \tag{2.13}$$

we have $v(\sigma^2(1)) \le 0$, a contradiction. Hence, $\alpha_i \ne 0$ for some $i = 1,2,3,\ldots,m-1$, and consequently, in view of (2.12) and the BCs, it follows that

$$v\left(\sigma^2(1)\right) \le \sum_{i=1}^{m-1} \alpha_i v(\eta_i) < \sum_{i=1}^{m-1} \alpha_i v\left(\sigma^2(1)\right). \tag{2.14}$$

Hence, $[1 - \sum_{i=1}^{m-1} \alpha_i] v(\sigma^2(1)) < 0$, which leads to $\sum_{i=1}^{m-1} \alpha_i > 1$, a contradiction.

Hence $t_0 \ne \sigma^2(1)$. Thus, $v(t) \le 0$ on $[0,\sigma^2(1)]_{\mathbb{T}}$. □

Corollary 2.3. *Under the hypotheses of Theorem 2.2, the solutions of the BVP* (1.1)*, if they exist, are unique.*

The following theorem establishes existence of solutions to the BVP (1.1) in the presence of well-ordered lower and upper solutions.

Theorem 2.4. *Assume that α, β are lower and upper solutions of the BVP* (1.1) *such that $\alpha \le \beta$ on $[0,\sigma^2(1)]_{\mathbb{T}}$. If $f(t,x) \in C_{\text{rd}}[[0,1]_{\mathbb{T}} \times \mathbb{R}]$, then the BVP* (1.1) *has a solution x such that*

$$\alpha \le x \le \beta, \quad \text{on } [0,\sigma^2(1)]_{\mathbb{T}}. \tag{2.15}$$

The proof essentially is a minor modification of the ideas in [21] and so is omitted.

3. Generalized Approximations Technique

We develop the approximation technique and show that, under suitable conditions on f, there exists a bounded monotone sequence of solutions of linear problems that converges uniformly and quadratically to a solution of the nonlinear original problem. If $(\partial^2/\partial x^2)f(t,x) \in C[[0,1]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded on $[0,\sigma^2(1)]_{\mathbb{T}} \times [\overline{\alpha},\overline{\beta}]$, where

$$\overline{\alpha} = \min\left\{\alpha(t), t \in [0,\sigma^2(1)]_{\mathbb{T}}\right\}, \qquad \overline{\beta} = \max\left\{\beta(t), t \in [0,\sigma^2(1)]_{\mathbb{T}}\right\}, \tag{3.1}$$

there always exists a function Φ such that

$$\frac{\partial^2}{\partial x^2}\left[f(t,x) + \Phi(t,x)\right] \le 0, \quad \text{on } [0,1]_{\mathbb{T}} \times \left[\overline{\alpha},\overline{\beta}\right], \tag{3.2}$$

where $\Phi \in C^2_{\mathrm{rd}}([0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R})$, and it is such that $(\partial^2/\partial x^2)\Phi(t,x) \le 0$ on $[0,\sigma^2(1)]_{\mathbb{T}} \times [\overline{\alpha},\overline{\beta}]$. For example, let $M = \max\{|f_{xx}(t,x)| : (t,x) \in [0,\sigma^2(1)]_{\mathbb{T}} \times [\overline{\alpha},\overline{\beta}]\}$, then we choose $\Phi = -t - (M/2)x^2$. Clearly,

$$\frac{\partial^2}{\partial x^2}\left[f(t,x) + \Phi(t,x)\right] \le 0, \quad \text{on } [0,\sigma^2(1)]_{\mathbb{T}} \times \left[\overline{\alpha},\overline{\beta}\right]. \tag{3.3}$$

Define $F : [0,1]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ by $F(t,x) = f(t,x) + \Phi(t,x)$. Note that $F \in C^2_{\mathrm{rd}}([0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R})$ and

$$\frac{\partial^2}{\partial x^2}F(t,x) \le 0, \quad \text{on } [0,1]_{\mathbb{T}} \times \left[\overline{\alpha},\overline{\beta}\right]. \tag{3.4}$$

Theorem 3.1. *Assume that*

- (A_1) *α,β are lower and upper solutions of the BVP* (1.1) *such that $\alpha \le \beta$ on $[0,\sigma^2(1)]_{\mathbb{T}}$,*
- (A_2) *$f \in C^2_{rd}([0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R})$ and f is increasing in x for each $t \in [0,\sigma^2(1)]_{\mathbb{T}}$.*

Then, there exists a monotone sequence $\{w_n\}$ of solutions of linear problems converging uniformly and quadratically to a unique solution of the BVP (1.1).

Proof. Conditions (A_1) and (A_2) ensure the existence of a unique solution x of the BVP (1.1) such that

$$\alpha(t) \le x(t) \le \beta(t), \quad t \in [0,\sigma^2(1)]_{\mathbb{T}}. \tag{3.5}$$

In view of (3.4), we have

$$f(t,x) \le f(t,y) + F_x(t,y)(x-y) - [\Phi(t,x) - \Phi(t,y)], \quad \text{on } [0,1]_{\mathbb{T}} \times \left[\overline{\alpha},\overline{\beta}\right]. \tag{3.6}$$

The mean value theorem and the fact that Φ_x is nonincreasing in x on $[\overline{\alpha},\overline{\beta}]$ for each $t \in [0,\sigma^2(1)]_{\mathbb{T}}$ yield

$$\Phi(t,x) - \Phi(t,y) = \Phi_x(t,c)(x-y) \ge \Phi_x\left(t,\overline{\beta}\right)(x-y), \quad \text{for } x \ge y, \tag{3.7}$$

where $x, y \in [\overline{\alpha}, \overline{\beta}]$ such that $y \le c \le x$. Substituting in (3.6), we have

$$f(t,x) \le f(t,y) + \left[F_x(t,y) - \Phi_x\left(t,\overline{\beta}\right)\right](x-y), \quad \text{for } x \ge y, \tag{3.8}$$

on $[0,\sigma^2(1)]_{\mathbb{T}} \times [\overline{\alpha}, \overline{\beta}]$. Define $g : [0,\sigma^2(1)]_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ by

$$g(t,x,y) = f(t,y) + \left[F_x(t,y) - \Phi_x\left(t,\overline{\beta}\right)\right](x-y). \tag{3.9}$$

We note that $g(t,.,.)$ is continuous for each $t \in [0,1]_{\mathbb{T}}$ and $g(.,x,y)$ is rd-continuous for each $(x,y) \in \mathbb{R}^2$. Moreover, g satisfies the following relations on $[0,1]_{\mathbb{T}} \times [\overline{\alpha}, \overline{\beta}]$:

$$g_x(t,x,y) = F_x(t,y) - \Phi_x\left(t,\overline{\beta}\right) \ge F_x(t,y) - \Phi_x(t,y) = f_x(t,y) \ge 0, \tag{3.10}$$

$$\begin{gathered} f(t,x) \le g(t,x,y), \quad \text{for } x \ge y, \\ f(t,x) = g(t,x,x). \end{gathered} \tag{3.11}$$

Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear problem

$$\begin{gathered} x^{\Delta\Delta}(t) = g\left(t, x^{\sigma}(t), w_0^{\sigma}(t)\right), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) = 0, \qquad x\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1} \alpha_i x(\eta_i). \end{gathered} \tag{3.12}$$

Using (3.11) and the definition of lower and upper solutions, we get

$$\begin{gathered} g\left(t, w_0^{\sigma}(t), w_0^{\sigma}(t)\right) = f\left(t, w_0^{\sigma}(t)\right) \le w_0^{\Delta\Delta}(t), \quad t \in [0,1]_{\mathbb{T}}, \\ g\left(t, \beta^{\sigma}(\mathrm{t}), w_0^{\sigma}(t)\right) \ge f\left(t, \beta^{\sigma}(t)\right) \ge \beta^{\Delta\Delta}(t), \quad t \in [0,1]_{\mathbb{T}}, \end{gathered} \tag{3.13}$$

which imply that w_0 and β are lower and upper solutions of (3.12), respectively. Hence by Theorem 2.4 and Corollary 2.3, there exists a unique solution $w_1 \in C_{\mathrm{rd}}^2[0,\sigma^2(1)]_{\mathbb{T}}$ of (3.12) such that

$$w_0(t) \le w_1(t) \le \beta(t), \quad \text{on } \left[0,\sigma^2(1)\right]_{\mathbb{T}}. \tag{3.14}$$

Using (3.11) and the fact that w_1 is a solution of (3.12), we obtain

$$\begin{gathered} w_1^{\Delta\Delta}(t) = g\left(t, w_1^{\sigma}(t), w_0^{\sigma}(t)\right) \ge f\left(t, w_1^{\sigma}(t)\right), \quad t \in [0,1]_{\mathbb{T}}, \\ w_1(0) = 0, \qquad w_1\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1} \alpha_i w_1(\eta_i), \end{gathered} \tag{3.15}$$

which implies that w_1 is a lower solution of the problem (1.1). Similarly, in view of (A_1), (3.11), and (3.15), we can show that w_1 and β are lower and upper solutions of the problem

$$\begin{gathered} x^{\Delta\Delta}(t) = g(t, x^{\sigma}(t), w_1^{\sigma}(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) = 0, \qquad x\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1} \alpha_i x(\eta_i). \end{gathered} \tag{3.16}$$

Hence by Theorem 2.4 and Corollary 2.3, there exists a unique solution $w_2 \in C_{\mathrm{rd}}^2[0,\sigma^2(1)]_{\mathbb{T}}$ of the problem (3.16) such that

$$w_1(t) \le w_2(t) \le \beta(t), \quad \text{on } \left[0,\sigma^2(1)\right]_{\mathbb{T}}. \tag{3.17}$$

Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$w_0(t) \le w_1(t) \le w_2(t) \le w_3(t) \le \cdots \le w_n(t) \le \beta(t), \quad \text{on } [0,\sigma^2(1)]_{\mathbb{T}}, \tag{3.18}$$

where the element w_n of the sequence is a solution of the linear problem

$$\begin{gathered} x^{\Delta\Delta}(t) = g(t, x^{\sigma}(t), w_{n-1}^{\sigma}(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ x(0) = 0, \qquad x\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1} \alpha_i x(\eta_i), \end{gathered} \tag{3.19}$$

and is given by

$$w_n(t) = \int_0^{\sigma(1)} G(t,s) g(s, w_n^{\sigma}(s), w_{n-1}^{\sigma}(s)) \Delta s, \quad t \in \left[0,\sigma^2(1)\right]_{\mathbb{T}}. \tag{3.20}$$

By standard arguments as in [19], the sequence converges to a solution of (1.1).

Now, we show that the convergence is quadratic. Set $v_{n+1}(t) = x(t) - w_{n+1}(t), t \in [0,\sigma^2(1)]_{\mathbb{T}}$, where x is a solution of (1.1). Then, $v_{n+1}(t) \ge 0$ on $[0,\sigma^2(1)]_{\mathbb{T}}$ and the boundary conditions imply that

$$v_{n+1}(0) = 0, v_{n+1}\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1} \alpha_i v_{n+1}(\eta_i). \tag{3.21}$$

Now, in view of the definitions of F and g, we obtain

$$\begin{aligned}
v_n^{\Delta\Delta}(t) &= f(t, x^\sigma(t)) - g\left(s, w_n^\sigma(t), w_{n-1}^\sigma(t)\right) \\
&= [F(t, x^\sigma(t)) - \Phi(t, x^\sigma(t))] \\
&\quad - \left[f\left(t, w_{n-1}^\sigma(t)\right) + \left(F_x\left(t, w_{n-1}^\sigma(t)\right) - \Phi_x\left(t, \overline{\beta}\right)\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right)\right] \\
&= \left[F(t, x^\sigma(t)) - F\left(t, w_{n-1}^\sigma(t)\right) - F_x\left(t, w_{n-1}^\sigma(t)\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right)\right] \\
&\quad - \left[\Phi(t, x^\sigma(t)) - \Phi\left(t, w_{n-1}^\sigma(t)\right) - \Phi_x\left(t, \overline{\beta}\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right)\right], \quad t \in [0,1]_{\mathbb{T}}.
\end{aligned} \tag{3.22}$$

Using the mean value theorem repeatedly and the fact that $\Phi_{xx} \leq 0$ on $[0,1]_{\mathbb{T}} \times [\overline{\alpha}, \overline{\beta}]$, we obtain

$$\begin{aligned}
&\Phi(t, x^\sigma(t)) - \Phi\left(t, w_{n-1}^\sigma(t)\right) \leq \Phi_x\left(t, w_{n-1}^\sigma(t)\right)\left(x^\sigma(t) - w_{n-1}^\sigma(t)\right), \\
&F(t, x^\sigma(t)) - F\left(t, w_{n-1}^\sigma(t)\right) - F_x\left(t, w_{n-1}^\sigma(t)\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right) \\
&\qquad = F_x\left(t, w_{n-1}^\sigma(t)\right)\left(x^\sigma(t) - w_{n-1}^\sigma(t)\right) + \frac{F_{xx}(t, \xi)}{2}\left(x^\sigma(t) - w_{n-1}^\sigma(t)\right)^2 \\
&\qquad\quad - F_x\left(t, w_{n-1}^\sigma(t)\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right) \\
&\qquad = F_x\left(t, w_{n-1}^\sigma(t)\right)(x^\sigma(t) - w_n^\sigma(t)) + \frac{F_{xx}(t, \xi)}{2}\left(x^\sigma(t) - w_{n-1}^\sigma(t)\right)^2 \\
&\qquad \geq F_x\left(t, w_{n-1}^\sigma(t)\right)(x^\sigma(t) - w_n^\sigma(t)) - d\|v_{n-1}\|^2,
\end{aligned} \tag{3.23}$$

where $w_{n-1}^\sigma(t) \leq \xi \leq x^\sigma(t)$, $d = \max\{|F_{xx}(t,x)|/2 : (t,x) \in [0,\sigma^2(1)]_{\mathbb{T}} \times [\overline{\alpha}, \overline{\beta}]\}$, and $\|v\| = \max\{v(t) : t \in [0,\sigma^2(1)]_{\mathbb{T}}\}$. Hence (3.22) can be rewritten as

$$\begin{aligned}
v_n^{\Delta\Delta}(t) &\geq F_x\left(t, w_{n-1}^\sigma(t)\right)(x^\sigma(t) - w_n^\sigma(t)) - d\|v_{n-1}\|^2 \\
&\quad - \Phi_x\left(t, w_{n-1}^\sigma(t)\right)\left(x^\sigma(t) - w_{n-1}^\sigma(t)\right) + \Phi_x\left(t, \overline{\beta}\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right) \\
&= f_x\left(t, w_{n-1}^\sigma(t)\right)(x^\sigma(t) - w_n^\sigma(t)) - d\|v_{n-1}\|^2 \\
&\quad + \left[\Phi_x\left(t, \overline{\beta}\right) - \Phi_x\left(t, w_{n-1}^\sigma(t)\right)\right]\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right) \\
&= f_x\left(t, w_{n-1}^\sigma\right)(x^\sigma(t) - w_n^\sigma(t)) - d\|v_{n-1}\|^2 + \Phi_{xx}(t, \xi_1)\left(\overline{\beta} - w_{n-1}^\sigma(t)\right)\left(w_n^\sigma(t) - w_{n-1}^\sigma(t)\right) \\
&\geq f_x\left(t, w_{n-1}^\sigma\right)(x^\sigma(t) - w_n^\sigma(t)) - d\|v_{n-1}\|^2 + \Phi_{xx}(t, \xi_1)\left(\overline{\beta} - w_{n-1}^\sigma(t)\right)\left(x_n^\sigma(t) - w_{n-1}^\sigma(t)\right) \\
&\geq -d\|v_{n-1}\|^2 - d_1\left|\overline{\beta} - w_{n-1}^\sigma(t)\right|\left|x^\sigma(t) - w_{n-1}^\sigma(t)\right|, \quad t \in [0,1]_{\mathbb{T}},
\end{aligned} \tag{3.24}$$

where $w_{n-1}^{\sigma}(t) \le \xi_1 \le w_n^{\sigma}(t)$, $d_1 = \max\{|\Phi_{xx}| : (t,x) \in [0,1]_{\mathbb{T}} \times [\overline{\alpha}, \overline{\beta}]\}$, and we used the fact that $f_x \ge 0$ on $[0,1]_{\mathbb{T}} \times [\overline{\alpha}, \overline{\beta}]$. Choose $r > 1$ such that

$$\left|\beta^{\sigma}(t) - w_{n-1}^{\sigma}(t)\right| \le r\left|x^{\sigma}(t) - w_{n-1}^{\sigma}(t)\right|, \quad \text{on } [0,1]_{\mathbb{T}}. \tag{3.25}$$

Therefore, we obtain

$$v_n^{\Delta\Delta}(t) \ge -d_2\|v_{n-1}\|^2, \quad t \in [0,1]_{\mathbb{T}}, \tag{3.26}$$

where $d_2 = d + rd_1$.

By comparison result, $v_n(t) \le z(t), t \in [0,1]_{\mathbb{T}}$, where $z(t)$ is the unique solution of the linear BVP

$$\begin{gathered} z^{\Delta\Delta}(t) = d_2\|v_{n-1}\|^2, \quad t \in [a,b]_{\mathbb{T}}, \\ z(0) = 0, z\left(\sigma^2(1)\right) = \sum_{i=1}^{m-1}\alpha_i z(\eta_i). \end{gathered} \tag{3.27}$$

Hence,

$$v_n(t) \le z(t) = d_2\int_0^{\sigma(1)} G(t,s)\|v_{n-1}\|^2\Delta s \le d_3\|v_{n-1}\|^2, \tag{3.28}$$

where $d_3 = d_2 \max\{\int_0^{\sigma(1)} |G(t,s)|\Delta s : t \in [0,\sigma^2(1)]_{\mathbb{T}}\}$. Taking the maximum over $[0,1]_{\mathbb{T}}$, we obtain

$$\|v_n\| \le d_3\|v_{n-1}\|^2, \tag{3.29}$$

which shows the quadratic convergence. □

Acknowledgement

The authors are thankful to reviewers for their valuable comments and suggestions.

References

[1] F. B. Christiansen and T. M. Fenchel, "Theories of Populations in Biological Communities, Ecological Studies," vol. 20, Springer, Berlin, Germany, 1977.
[2] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.
[3] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[4] M. Bohner and A. Peterson, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.
[5] R. P. Agarwal and D. O'Regan, "Nonlinear boundary value problems on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 44, no. 4, pp. 527–535, 2001.

[6] D. Anderson, R. Avery, and J. Henderson, "Existence of solutions for a one dimensional p-Laplacian on time-scales," *Journal of Difference Equations and Applications*, vol. 10, no. 10, pp. 889–896, 2004.
[7] F. M. Atici and G. Sh. Guseinov, "On Green's functions and positive solutions for boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.
[8] R. I. Avery and D. R. Anderson, "Existence of three positive solutions to a second-order boundary value problem on a measure chain," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 65–73, 2002.
[9] L. Erbe and A. Peterson, "Positive solutions for a nonlinear differential equation on a measure chain," *Mathematical and Computer Modelling*, vol. 32, no. 5-6, pp. 571–585, 2000.
[10] J. Henderson, "Double solutions of impulsive dynamic boundary value problems on a time scale," *Journal of Difference Equations and Applications*, vol. 8, no. 4, pp. 345–356, 2002.
[11] J. Henderson, A. Peterson, and C. C. Tisdell, "On the existence and uniqueness of solutions to boundary value problems on time scales," *Advances in Difference Equations*, vol. 2004, no. 2, pp. 93–109, 2004.
[12] C. C. Tisdell and H. B. Thompson, "On the existence of solutions to boundary value problems on time scales," *Dynamics of Continuous, Discrete & Impulsive Systems. Series A*, vol. 12, no. 5, pp. 595–606, 2005.
[13] D. R. Anderson and R. I. Avery, "An even-order three-point boundary value problem on time scales," *Journal of Mathematical Analysis and Applications*, vol. 291, no. 2, pp. 514–525, 2004.
[14] T. G. Bhaskar, "Comparison theorem for a nonlinear boundary value problem on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 117–122, 2002.
[15] E. R. Kaufmann, "Positive solutions of a three-point boundary-value problem on a time scale," *Electronic Journal of Differential Equations*, vol. 2003, no. 82, pp. 1–11, 2003.
[16] R. A. Khan, J. J. Nieto, and V. Otero-Espinar, "Existence and approximation of solution of three-point boundary value problems on time scales," *Journal of Difference Equations and Applications*, vol. 14, no. 7, pp. 723–736, 2008.
[17] B. Karna and B. A. Lawrence, "An existence result for a multipoint boundary value problem on a time scale," *Advances in Difference Equations*, vol. 2006, Article ID 63208, 8 pages, 2006.
[18] E. Akin-Bohner and F. M. Atici, "A quasilinearization approach for two point nonlinear boundary value problems on time scales," *The Rocky Mountain Journal of Mathematics*, vol. 35, no. 1, pp. 19–45, 2005.
[19] F. M. Atici, P. W. Eloe, and B. Kaymakçalan, "The quasilinearization method for boundary value problems on time scales," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 1, pp. 357–372, 2002.
[20] F. M. Atici and S. G. Topal, "The generalized quasilinearization method and three point boundary value problems on time scales," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 577–585, 2005.
[21] A. C. Peterson, Y. N. Raffoul, and C. C. Tisdell, "Three point boundary value problems on time scales," *Journal of Difference Equations and Applications*, vol. 10, no. 9, pp. 843–849, 2004.
[22] R. E. Bellman and R. E. Kalaba, *Quasilinearization and Nonlinear Boundary-Value Problems*, vol. 3 of *Modern Analytic and Computional Methods in Science and Mathematics*, American Elsevier, New York, NY, USA, 1965.
[23] R. A. Khan, "Approximations and rapid convergence of solutions of nonlinear three point boundary value problems," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 957–968, 2007.
[24] V. Lakshmikantham and A. S. Vatsala, *Generalized Quasilinearization for Nonlinear Problems*, vol. 440 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1998.
[25] V. Lakshmikantham and A. S. Vatsala, "Generalized quasilinearization versus Newton's method," *Applied Mathematics and Computation*, vol. 164, no. 2, pp. 523–530, 2005.
[26] J. J. Nieto, "Generalized quasilinearization method for a second order ordinary differential equation with Dirichlet boundary conditions," *Proceedings of the American Mathematical Society*, vol. 125, no. 9, pp. 2599–2604, 1997.