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Research Article

An Extension to Nonlinear Sum-Difference Inequality and Applications

Wu-Sheng Wang[1] and Xiaoliang Zhou[2]

[1] *Department of Mathematics, Hechi University, Yizhou, Guangxi 546300, China*
[2] *Department of Mathematics, Guangdong Ocean University, Zhanjiang 524088, China*

Correspondence should be addressed to Xiaoliang Zhou, zjhdzxl@yahoo.com.cn

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We establish a general form of sum-difference inequality in two variables, which includes both more than two distinct nonlinear sums without an assumption of monotonicity and a nonconstant term outside the sums. We employ a technique of monotonization and use a property of stronger monotonicity to give an estimate for the unknown function. Our result enables us to solve those discrete inequalities considered in the work of W.-S. Cheung (2006). Furthermore, we apply our result to a boundary value problem of a partial difference equation for boundedness, uniqueness, and continuous dependence.

1. Introduction

Gronwall-Bellman inequality [1, 2] is a fundamental tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equation. There are a lot of papers investigating them such as [3–15]. Along with the development of the theory of integral inequalities and the theory of difference equations, more attentions are paid to some discrete versions of Bellman-Gronwall type inequalities (e.g., [16–18]). Starting from the basic form

$$u(n) \le a(n) + \sum_{s=0}^{n-1} f(s)u(s), \tag{1.1}$$

discussed in [19], an interesting direction is to consider the inequality

$$u^2(n) \le P^2u^2(0) + 2\sum_{s=0}^{n-1}\Big[\alpha u^2(s) + Qg(s)u(s)\Big], \tag{1.2}$$

a discrete version of Dafermos' inequality [20], where α, P, Q are nonnegative constants and u, g are nonnegative functions defined on $\{1,2,\dots,T\}$ and $\{1,2,\dots,T-1\}$, respectively. Pang and Agarwal [21] proved for (1.2) that $u(n) \le (1+\alpha)^n[Pu(0)+\sum_{s=0}^{n-1} Qg(s)]$ for all $0 \le n \le T$. Another form of sum-difference inequality

$$u^2(n) \le c^2 + 2\sum_{s=0}^{n-1}\left[f_1(s)u(s)w(u(s)) + f_2(s)u(s)\right] \tag{1.3}$$

was estimated by Pachpatte [22] as $u(n) \le \Omega^{-1}[\Omega(c+\sum_{s=0}^{n-1} f_2(s))+\sum_{s=0}^{n-1} f_1(s)]$, where $\Omega(u) := \int_{u_0}^{u} ds/w(s)$. Recently, Pachpatte [23, 24] discussed the inequalities of two variables

$$\begin{aligned} u(m,n) &\le c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} u(s,t)\left[a(s,t)\log u(s,t) + b(s,t)g(\log u(s,t))\right], \\ u(m,n) &\le c + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} f_1(s,t)g(u(s,t)) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\kappa(s,t,\sigma,\tau)g(u(\sigma,\tau))\right) \\ &\quad + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1}\left(\sum_{\sigma=0}^{s-1}\sum_{\tau=0}^{t-1}\left(\sum_{\xi=0}^{\sigma-1}\sum_{\eta=0}^{\tau-1} h(s,t,\sigma,\tau,\xi,\eta)g(u(\xi,\eta))\right)\right), \end{aligned} \tag{1.4}$$

where g is nondecreasing. In [25] another form of inequality of two variables

$$u^2(m,n) \le c^2 + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u(s,t)w(u(s,t)) \tag{1.5}$$

was discussed. Later, this result was generalized in [26] to the inequality

$$u^p(m,n) \le c + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} d(s,t)u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} e(s,t)u^q(s,t)w(u(s,t)), \tag{1.6}$$

where c, p, and q are all constants, $c \ge 0$, $p > q > 0$, and d, e are both nonnegative real-valued functions defined on a lattice in $\mathbb{Z}_+^2$, and w is a continuous nondecreasing function satisfying $w(u) > 0$ for all $u > 0$.

In this paper we establish a more general form of sum-difference inequality with positive integers m, n,

$$\psi(u(m,n)) \le a(m,n) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} f_i(m,n,s,t)\varphi_i(u(s,t)), \tag{1.7}$$

where $k \ge 2$. In (1.7) we replace the constant c, the functions u^p, $d(s,t)$, $e(s,t)$, u^q and $u^q w(u)$ in (1.6) with a function $a(m,n)$, more general functions $\psi(u)$, $f_1(m,n,s,t)$, $f_2(m,n,s,t)$, $\varphi_1(u)$ and $\varphi_2(u)$, respectively. Moreover, we consider more than two nonlinear terms and do not require the monotonicity of every φ_i $(i = 1,2,\dots,k)$. We employ a technique of monotonization to construct a sequence of functions which possesses stronger monotonicity than the previous one. Unlike the work in [26] for two sum terms, the maximal regions of validity for our estimate of the unknown function u are decided by boundaries of more than two planar regions. Thus we have to consider the inclusion of those regions and find common regions. We demonstrate that inequalities (1.6) and other inequalities considered in [26] can also be solved with our result. Furthermore, we apply our result to boundary value problems of a partial difference equation for boundedness, uniqueness, and continuous dependence.

2. Main Result

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $m_0, n_0 \in \mathbb{N}_0$, $X, Y \in \mathbb{N}_0 \cup \{\infty\}$ are given nonnegative integers. For any integers $s < t$, let $\operatorname{dis}[s,t] = \{j : s \le j \le t,\ j \in \mathbb{N}_0\}$, $I = \operatorname{dis}[m_0, X]$, and $J = \operatorname{dis}[n_0, Y]$. Define $\Lambda = I \times J \subset \mathbb{N}_0^2$, and let $\Lambda_{[s,t]}$ denote the sublattice $\operatorname{dis}[m_0, s] \times \operatorname{dis}[n_0, t]$ in Λ for any $(s,t) \in \Lambda$.

For functions $g(m,n)$, $m,n \in \mathbb{N}_0$, their first-order differences are defined by $\Delta_1 g(m,n) = g(m+1,n) - g(m,n)$ and $\Delta_2 g(m,n) = g(m,n+1) - g(m,n)$. Obviously, the linear difference equation $\Delta x(m) = b(m)$ with the initial condition $x(m_0) = 0$ has the solution $\sum_{s=m_0}^{m-1} b(s)$. In the sequel, for convenience, we complementarily define that $\sum_{s=m_0}^{m_0-1} b(s) = 0$.

We give the following basic assumptions for the inequality (1.7).

(H_1) ψ is a strictly increasing continuous function on $\mathbb{R}_+$ satisfying that $\psi(\infty) = \infty$ and $\psi(u) > 0$ for all $u > 0$.

(H_2) All φ_i $(i = 1, 2, \ldots, k)$ are continuous and positive functions on $\mathbb{R}_+$.

(H_3) $a(m,n) \ge 0$ on Λ.

(H_4) All f_i $(i = 1, 2, \ldots, k)$ are nonnegative functions on $\Lambda \times \Lambda$.

With given functions φ_1, φ_2, and ψ, we technically consider a sequence of functions $w_i(s)$, which can be calculated recursively by

$$
\begin{aligned}
w_1(s) &:= \max_{\tau \in [0,s]} \{\varphi_1(\tau)\}, \\
w_{i+1}(s) &:= \max_{\tau \in [0,s]} \left\{ \frac{\varphi_{i+1}(\tau)}{w_i(\tau)} \right\} w_i(s), \quad i = 1, \ldots, k-1.
\end{aligned}
\tag{2.1}
$$

For given constants $u_i > 0$ and variable $u > 0$, we define

$$
W_i(u, u_i) := \int_{u_i}^{u} \frac{dx}{w_i(\psi^{-1}(x))}, \qquad i = 1, 2, \ldots, k. \tag{2.2}
$$

Obviously, W_i is strictly increasing in $u > 0$ and therefore the inverses W_i^{-1} are well defined, continuous, and increasing. Let

$$
\widetilde{f}_i(m,n,s,t) := \max_{(\tau,\xi) \in [m_0,m] \times [n_0,n]} f_i(\tau, \xi, s, t), \tag{2.3}
$$

which is nondecreasing in m and n for each fixed s and t and satisfies $\widetilde{f}_i(x,y,t,s) \ge f_i(x,y,t,s) \ge 0$ for all $i = 1, \ldots, k$.

Theorem 2.1. *Suppose that* (H_1)–(H_4) *hold and* $u(m,n)$ *is a nonnegative function on* Λ *satisfying* (1.7). *Then, for* $(m,n) \in \Lambda_{[M_1,N_1]}$, *a sublattice in* Λ,

$$u(m,n) \leq \varphi^{-1}\left\{W_k^{-1}\left[W_k(\Upsilon_k(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_k(m,n,s,t)\right]\right\}, \tag{2.4}$$

where $\Upsilon_k(m,n)$ *is determined recursively by*

$$\begin{aligned} \Upsilon_1(m,n) &:= a(m_0,n_0) + \sum_{s=m_0}^{m-1}|a(s+1,n_0) - a(s,n_0)| + \sum_{t=n_0}^{n-1}|a(m,t+1) - a(m,t)|, \\ \Upsilon_{i+1}(m,n) &:= W_i^{-1}\left[W_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(m,n,s,t)\right], \quad i = 1,\ldots,k-1, \end{aligned} \tag{2.5}$$

and $(M_1,N_1) \in \Lambda$ *is arbitrarily given on the boundary of the lattice*

$$U := \left\{(m,n) \in \Lambda : W_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(m,n,s,t) \leq \int_{u_i}^{\infty}\frac{dx}{w_i(\varphi^{-1}(x))},\ i = 1,2,\ldots,k\right\}. \tag{2.6}$$

Remark 2.2. As explained in [3, Remark 2], since different choices of u_i in W_i $(i = 1,2,\ldots,k)$ do not affect our results, we simply let $W_i(u)$ denote $W_i(u,u_i)$ when there is no confusion. For positive constants $v_i \neq u_i$, let $\widetilde{W}_i(u) = \int_{v_i}^{u} dx/w_i(\varphi^{-1}(x))$. Obviously, $\widetilde{W}_i(u) = W_i(u) + \widetilde{W}_i(u_i)$ and $\widetilde{W}_i^{-1}(v) = W_i^{-1}(v - \widetilde{W}_i(u_i))$. It follows that

$$\widetilde{W}_i^{-1}\left[\widetilde{W}_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(m,n,s,t)\right] = W_i^{-1}\left[W_i(\Upsilon_i(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(m,n,s,t)\right], \tag{2.7}$$

that is, we obtain the same expression in (2.4) if we replace W_i with $\widetilde{W}_i$, $i = 1,2,\ldots,k$. Moreover, by replacing W_i with $\widetilde{W}_i$, the condition in the definition of U in Theorem 2.1 reads

$$\widetilde{W}_i(\Upsilon_i(M_1,N_1)) + \sum_{s=m_0}^{M_1-1}\sum_{t=n_0}^{N_1-1}\tilde{f}_i(m,n,s,t) \leq \int_{v_i}^{\infty}\frac{dx}{w_i(\varphi^{-1}(x))}, \quad i = 1,2,\ldots,k, \tag{2.8}$$

the left-hand side of which is equal to

$$\widetilde{W}_i(u_i) + W_i(\Upsilon_i(M_1,N_1)) + \sum_{s=m_0}^{M_1-1}\sum_{t=n_0}^{N_1-1}\tilde{f}_i(m,n,s,t), \tag{2.9}$$

and the right-hand side of which equals

$$\int_{v_i}^{u_i} \frac{dx}{w_i(\psi^{-1}(x))} + \int_{u_i}^{\infty} \frac{dx}{w_i(\psi^{-1}(x))} = \widetilde{W}_i(u_i) + \int_{u_i}^{\infty} \frac{dx}{w_i(\psi^{-1}(x))}. \tag{2.10}$$

The comparison between the both sides implies that (2.8) is equivalent to the condition given in the definition of U in Theorem 2.1 with $(m,n) = (M_1, N_1)$.

Remark 2.3. If we choose $k = 2$, $\psi(u) = u^p$, $\varphi_1(u) = u^q$, $\varphi_2(u) = u^q w(u)$ with $p > q > 0$, $f_1(m,n,s,t) = d(s,t)$ and $f_2(m,n,s,t) = e(s,t)$ and restrict $a(m,n)$ to be a constant c in (1.7), then we can apply Theorem 2.1 to inequality (1.6) discussed in [26].

3. Proof of Theorem

First of all, we monotonize some given functions φ_i, f_i in the sums. Obviously, the sequence $w_i(s)$ defined by φ_i $(i = 1,\ldots,k)$ in (2.1) consists of nondecreasing nonnegative functions and satisfies $w_i(s) \geq \varphi_i(s)$, for $i = 1,\ldots,k$. Moreover,

$$w_i \propto w_{i+1}, \quad i = 1,2,\ldots,k-1, \tag{3.1}$$

as defined in [27] for comparison of monotonicity of functions $w_i(s)$ $(i = 1,\ldots,k)$, because every ratio $w_{i+1}(s)/w_i(s)$ is nondecreasing. By the definitions of functions w_i, $\widetilde{f}_i$, ψ, and Υ_1, from (1.7) we get

$$u(m,n) \leq \psi^{-1}\left[\Upsilon_1(m,n) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \widetilde{f}_i(m,n,s,t) w_i(u(s,t))\right], \quad \forall (m,n) \in \Lambda. \tag{3.2}$$

Then, we discuss the case that $a(m,n) > 0$ for all $(m,n) \in \Lambda$. Because Υ_1 satisfies

$$\begin{aligned}\Upsilon_1(m,n) &= a(m_0,n_0) + \sum_{s=m_0}^{m-1} |a(s+1,n_0) - a(s,n_0)| + \sum_{t=n_0}^{n-1} |a(m,t+1) - a(m,t)| \\ &\geq a(m,n),\end{aligned} \tag{3.3}$$

it is positive and nondecreasing on Λ. We consider the auxiliary inequality to (3.2), for all $(m,n) \in \Lambda_{[M,N]}$,

$$u(m,n) \leq \psi^{-1}\left[\Upsilon_1(M,N) + \sum_{i=1}^{k}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} \widetilde{f}_i(M,N,s,t) w_i(u(s,t))\right], \tag{3.4}$$

where $M \in \operatorname{dis}[m_0, M_1]$ and $N \in \operatorname{dis}[n_0, N_1]$ are chosen arbitrarily, and claim that, for $(m,n) \in \Lambda_{[\min\{M_2,M\},\min\{N_2,N\}]}$, a sublattice in $\Lambda_{[M_1,N_1]}$,

$$u(m,n) \leq \varphi^{-1}\left\{W_k^{-1}\left[W_k\left(\widetilde{\Upsilon}_k(M,N,m,n)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_k(M,N,s,t)\right]\right\}, \tag{3.5}$$

where $\widetilde{\Upsilon}_k(M,N,m,n)$ is determined recursively by

$$\begin{gathered}\widetilde{\Upsilon}_1(M,N,m,n) := \Upsilon_1(M,N),\\ \widetilde{\Upsilon}_{i+1}(M,N,m,n) := W_i^{-1}\left[W_i\left(\widetilde{\Upsilon}_i(M,N,m,n)\right) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_i(M,N,s,t)\right],\end{gathered} \tag{3.6}$$

$i = 1,2,\ldots,k-1$, and $(M_2, N_2) \in \Lambda_{[M_1,N_1]}$ is arbitrarily chosen on the boundary of the lattice

$$\begin{aligned}U_1 := \Bigg\{(m,n) \in \Lambda : W_i\left(\widetilde{\Upsilon}_i(M,N,m,n)\right) &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_i(M,N,s,t)\\ &\leq \int_{u_i}^{\infty}\frac{dx}{w_i(\varphi^{-1}(x))},\ i = 1,2,\ldots,k\Bigg\}.\end{aligned} \tag{3.7}$$

We note that M_2, N_2 can be chosen appropriately such that

$$M_2(M,N) = M_1, \quad N_2(M,N) = N_1, \quad \forall (M,N) \in \Lambda_{[M_1,N_1]}. \tag{3.8}$$

In fact, from the fact of (M_1, N_1) being on the boundary of the lattice U, we see that

$$\begin{aligned}W_i\left(\widetilde{\Upsilon}_i(M_1,N_1,M_1,N_1)\right) &+ \sum_{s=m_0}^{M_1-1}\sum_{t=n_0}^{N_1-1}\widetilde{f}_i(M_1,N_1,s,t)\\ &= W_i(\Upsilon_i(M_1,N_1)) + \sum_{s=m_0}^{M_1-1}\sum_{t=n_0}^{N_1-1}\widetilde{f}_i(M_1,N_1,s,t)\\ &\leq \int_{u_i}^{\infty}\frac{dx}{w_i(\varphi^{-1}(x))}, \quad i = 1,2,\ldots,k.\end{aligned} \tag{3.9}$$

Thus, it means that we can take $M_2 = M_1$, $N_2 = N_1$. Moreover, $M = \min\{M_2, M\}$, $N = \min\{N_2, N\}$.

In the following, we will use mathematical induction to prove (3.5).

For $k = 1$, let $z(m,n) = \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,s,t)w_1(u(s,t))$. Then z is nonnegative and nondecreasing in each variable on $\Lambda_{[M,N]}$. From (3.4) we observe that

$$u(m,n) \leq \varphi^{-1}(\Upsilon_1(M,N) + z(m,n)), \quad \forall (m,n) \in \Lambda_{[M\ N]}. \tag{3.10}$$

Moreover, we note that w_1 is nondecreasing and satisfies $w_1(u) > 0$ for $u > 0$ and that $\Upsilon_1(M,N) + z(m,n) > 0$. From (3.10) we have

$$\begin{aligned}\frac{\Delta_1(\Upsilon_1(M,N)+z(m,n))}{w_1\left(\varphi^{-1}(\Upsilon_1(M,N)+z(m,n))\right)} &= \frac{\sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,m,t)w_1(u(m,t))}{w_1\left(\varphi^{-1}(\Upsilon_1(M,N)+z(m,n))\right)} \\ &\le \sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,m,t).\end{aligned} \tag{3.11}$$

On the other hand, by the Mean Value Theorem for integral and by the monotonicity of w_1 and φ, for arbitrarily given (m,n), $(m+1,n) \in \Lambda_{[M\ N]}$ there exists ξ in the open interval $(\Upsilon_1(M,N)+z(m,n), \Upsilon_1(M,N)+z(m+1,n))$ such that

$$\begin{aligned}&W_1(\Upsilon_1(M,N)+z(m+1,n)) - W_1(\Upsilon_1(M,N)+z(m,n)) \\ &\quad= \int_{\Upsilon_1(M,N)+z(m,n)}^{\Upsilon_1(M,N)+z(m+1,n)}\frac{du}{w_1\left(\varphi^{-1}(u)\right)} \\ &\quad= \frac{\Delta_1(\Upsilon_1(M,N)+z(m,n))}{w_1\left(\varphi^{-1}(\xi)\right)} \\ &\quad\le \frac{\Delta_1(\Upsilon_1(M,N)+z(m,n))}{w_1\left(\varphi^{-1}(\Upsilon_1(M,N)+z(m,n))\right)}.\end{aligned} \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$W_1(\Upsilon_1(M,N)+z(m+1,n)) - W_1(\Upsilon_1(M,N)+z(m,n)) \le \sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,m,t). \tag{3.13}$$

Substituting m with s and summing both sides of (3.13) from $s = m_0$ to $m-1$, we get, for all $(m,n) \in \Lambda_{[M\ N]}$,

$$W_1(\Upsilon_1(M,N)+z(m,n)) \le W_1(\Upsilon_1(M,N)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,s,t). \tag{3.14}$$

We note from the definition of $z(m,n)$ in (3.2) and the definition of $\sum_{s=m_0}^{m_0-1}$ in Section 2 that $z(m_0,n) = 0$. By the monotonicity of W^{-1} and (3.10) we obtain

$$u(m,n) \le \varphi^{-1}\left\{W_1^{-1}\left(W_1(\Upsilon_1(M,N)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,s,t)\right)\right\}, \quad \forall (m,n) \in \Lambda_{[M\ N]}, \tag{3.15}$$

that is, (3.5) is true for $k = 1$.

Next, we make the inductive assumption that (3.5) is true for $k = l$. Consider

$$u(m,n) \leq \varphi^{-1}\left[\Upsilon_1(M,N) + \sum_{i=1}^{l+1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_i(M,N,s,t)w_i(u(s,t))\right], \tag{3.16}$$

for all $(m,n) \in \Lambda_{[M\ N]}$. Let $y(m,n) = \sum_{i=1}^{l+1}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\widetilde{f}_i(M,N,s,t)w_i(u(s,t))$, which is nonnegative and nondecreasing in each variable on $\Lambda_{[M,N]}$. Then (3.16) is equivalent to

$$u(m,n) \leq \varphi^{-1}\big(\Upsilon_1(M,N) + y(m,n)\big), \quad \forall (m,n) \in \Lambda_{[M\ N]}. \tag{3.17}$$

Since w_i is nondecreasing and satisfies $w_i(u) > 0$ for $u > 0$ $(i = 1,2,\ldots,l+1)$ and $\Upsilon_1(K,L) + y(m,n) > 0$, from (3.17) we obtain, for all $(m,n) \in \Lambda_{[M\ N]}$,

$$\begin{aligned}\frac{\Delta_1\big(\Upsilon_1(M,N) + y(m,n)\big)}{w_1\big(\varphi^{-1}\big(\Upsilon_1(M,N) + y(m,n)\big)\big)} &= \frac{\sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,m,t)w_1(u(m,t))}{w_1\big(\varphi^{-1}\big(\Upsilon_1(M,N) + y(m,n)\big)\big)} \\ &\quad + \frac{\sum_{i=2}^{l+1}\sum_{t=n_0}^{n-1}\widetilde{f}_i(M,N,m,t)w_i(u(m,t))}{w_1\big(\varphi^{-1}\big(\Upsilon_1(M,N) + y(m,n)\big)\big)} \\ &\leq \sum_{t=n_0}^{n-1}\widetilde{f}_1(M,N,m,t) + \sum_{i=1}^{l}\sum_{t=n_0}^{n-1}\widetilde{f}_{i+1}(M,N,m,t)\phi_{i+1}(u(m,t)),\end{aligned} \tag{3.18}$$

where

$$\phi_i(u) := \frac{w_i(u)}{w_1(u)}, \quad i = 2,3,\ldots,l+1. \tag{3.19}$$

On the other hand, by the Mean Value Theorem for integrals and by the monotonicity of w_1 and φ, for arbitrarily given (m,n), $(m+1,n) \in \Lambda_{[M,N]}$ there exists ξ in the open interval $(\Upsilon_1(M,N) + y(m,n),\ \Upsilon_1(M,N) + y(m+1,n))$ such that

$$\begin{aligned}&W_1\big(\Upsilon_1(M,N) + y(m+1,n)\big) - W_1\big(\Upsilon_1(M,N) + y(m,n)\big) \\ &\quad = \int_{\Upsilon_1(M,N)+y(m,n)}^{\Upsilon_1(M,N)+y(m+1,n)}\frac{du}{w_1\big(\varphi^{-1}(u)\big)} \\ &\quad = \frac{\Delta_1\big(\Upsilon_1(M,N) + y(m,n)\big)}{w_1\big(\varphi^{-1}(\xi)\big)} \\ &\quad \leq \frac{\Delta_1\big(\Upsilon_1(M,N) + y(m,n)\big)}{w_1\big(\varphi^{-1}\big(\Upsilon_1(M,N) + y(m,n)\big)\big)}.\end{aligned} \tag{3.20}$$

Therefore, it follows from (3.18) and (3.20) that

$$\begin{aligned} &W_1\big(\Upsilon_1(M,N)+y(m+1,n)\big)-W_1\big(\Upsilon_1(M,N)+y(m,n)\big)\\ &\quad\le\sum_{t=n_0}^{n-1}\tilde{f}_1(M,N,m,t)+\sum_{i=1}^{l}\sum_{t=n_0}^{n-1}\tilde{f}_{i+1}(M,N,m,t)\phi_{i+1}(u(m,t)). \end{aligned}\tag{3.21}$$

substituting m with s in (3.21) and summing both sides of (3.21) from $s=m_0$ to $m-1$, we get, for all $(m,n)\in\Lambda_{[M,N]}$,

$$\begin{aligned} &W_1\big(\Upsilon_1(M,N)+y(m,n)\big)-W_1(\Upsilon_1(M,N))\\ &\quad\le\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_1(M,N,s,t)+\sum_{i=1}^{l}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{i+1}(M,N,s,t)\phi_{i+1}(u(s,t)), \end{aligned}\tag{3.22}$$

where we note that $y(m_0,n)=0$. For convenience, let

$$\begin{aligned} \psi(\Xi(m,n))&:=W_1\big(\Upsilon_1(M,N)+y(m,n)\big),\\ \theta(M,N,m,n)&:=W_1(\Upsilon_1(M,N))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_1(M,N,s,t). \end{aligned}\tag{3.23}$$

From (3.17) and (3.22) we can get

$$\Xi(m,n)\le\psi^{-1}\left\{\theta(M,N,M,N)+\sum_{i=1}^{l}\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{i+1}(M,N,s,t)\phi_{i+1}\Big[\psi^{-1}\Big(W_1^{-1}(\psi(\Xi(m,n)))\Big)\Big]\right\},\tag{3.24}$$

the same form as (3.4) for $k=l$, for all $(m,n)\in\Lambda_{[M,N]}$, where we note that $\theta(M,N,M,N)\ge\theta(M,N,m,n)$ for all $(m,n)\in\Lambda_{[M,N]}$. We are ready to use the inductive assumption for (3.24). In order to demonstrate the basic condition of monotonicity, let $h(s)=\psi^{-1}(W_1^{-1}(\psi(s)))$, obviously which is a continuous and nondecreasing function on $\mathbb{R}_+$. Thus each $\phi_i(h(s))$ is continuous and nondecreasing on $\mathbb{R}_+$ and satisfies $\phi_i(h(s))>0$ for $s>0$. Moreover,

$$\frac{\phi_{i+1}(h(s))}{\phi_i(h(s))}=\frac{w_{i+1}(h(s))}{w_i(h(s))}=\max_{\tau\in[0,h(s)]}\left\{\frac{\varphi_{i+1}(\tau)}{w_i(\tau)}\right\},\tag{3.25}$$

which is also continuous nondecreasing on $\mathbb{R}_+$ and positive on $\mathbb{R}_+$. This implies that $\phi_i(h(s))\propto\phi_{i+1}(h(s))$, for $i=2,\ldots,l$. Therefore, the inductive assumption for (3.5) can be used to (3.24) and we obtain, for all $(m,n)\in\Lambda_{[\min\{M,M_3\},\min\{N,N_3\}]}$,

$$\Xi(m,n)\le\psi^{-1}\left\{\Phi_{l+1}^{-1}\left[\Phi_{l+1}(\theta_{l+1}(M,N,m,n))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{l+1}(M,N,s,t)\right]\right\},\tag{3.26}$$

where $\Phi_i(u) := \int_{\varpi(u_i)}^{u}(ds/\phi_i(h(s)))$, $u > 0$, $\varpi(u) = \psi^{-1}(W_1(u))$, Φ_i^{-1} is the inverse of Φ_i (for $i = 2,3,\ldots,l+1$), $\theta_{l+1}(M,N,m,n)$ is determined recursively by

$$\theta_1(M,N,m,n) := \theta(M,N,M,N),$$
$$\theta_{i+1}(M,N,m,n) := \Phi_i^{-1}\left[\Phi_i(\theta_i(M,N,m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(M,N,s,t)\right], \quad i = 1,2,\ldots,l, \tag{3.27}$$

and M_3, N_3 are functions of (M,N) such that $M_3(M,N)$, $N_3(M,N) \in \Lambda_{[M_1,N_1]}$ lie on the boundary of the lattice

$$U_2 := \left\{(m,n) \in \Lambda : \Phi_i(\theta_i(M,N,m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_i(M,N,s,t) \le \int_{\varpi(u_i)}^{\varpi(\infty)}\frac{ds}{\phi_i(h(s))},\ i = 2,3,\ldots,l+1\right\}, \tag{3.28}$$

where $\varpi(\infty)$ denotes either $\lim_{u\to\infty}\varpi(u)$ if it converges or ∞. Note that

$$\begin{aligned}\Phi_i(u) &= \int_{\varpi(u_i)}^{u}\frac{ds}{\theta\left(\psi^{-1}\left(W_1^{-1}(\psi(s))\right)\right)}\\ &= \int_{\varpi(u_i)}^{u}\frac{w_1\left(\psi^{-1}\left(W_1^{-1}(\psi(s))\right)\right)ds}{w_i\left(\psi^{-1}\left(W_1^{-1}(\psi(s))\right)\right)}\\ &= \int_{u_i}^{W_1^{-1}(\psi(u))}\frac{dx}{w_i(\psi^{-1}(x))}\\ &= W_i\Big(W_1^{-1}(\psi(u))\Big), \quad i = 2,3,\ldots,l+1.\end{aligned} \tag{3.29}$$

Thus, from (3.17), (3.23), and (3.27), (3.26) can be equivalently written as

$$\begin{aligned}u(m,n) &\le \psi^{-1}\Big(W_1^{-1}(\psi(\Xi(m,n)))\Big)\\ &\le \psi^{-1}\left\{W_{l+1}^{-1}\left[W_{l+1}\Big(W_1^{-1}(\psi(\theta_{l+1}(M,N,m,n)))\Big) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\tilde{f}_{l+1}(M,N,s,t)\right]\right\}, \quad \forall (m,n) \in \Lambda_{[\min\{M,M_3\},\min\{N,N_3\}]}.\end{aligned} \tag{3.30}$$

We further claim that the term $W_1^{-1}(\psi(\theta_i(M,N,m,n)))$ is the same as $\tilde{\Upsilon}_i(M,N,m,n)$, defined in (3.6), $i = 1,2,\ldots,l+1$. For convenience, let $\tilde{\theta}_i(M,N,m,n) = W_1^{-1}(\psi(\theta_i(M,N,m,n)))$. Obviously, it is that $\tilde{\theta}_1(M,N,m,n) = \tilde{\Upsilon}_1(M,N,m,n)$.

The remainder case is that $a(m,n)=0$ for some $(m,n)\in\Lambda$. Let

$$\Upsilon_{1,\varepsilon}(m,n)=\Upsilon_1(m,n)+\varepsilon, \tag{3.31}$$

where $\varepsilon>0$ is an arbitrary small number. Obviously, $\Upsilon_{1,\varepsilon}(m,n)>0$ for all $(m,n)\in\Lambda$. Using the same arguments as above and replacing $\Upsilon_1(m,n)$ with $\Upsilon_{1,\varepsilon}(m,n)$, we get

$$u(m,n)\le\varphi^{-1}\left\{W_2^{-1}\left[W_2\left(W_1^{-1}\left(W_1(\Upsilon_{1,\varepsilon}(m,n))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_1(s,t)\right)\right)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}f_2(s,t)\right]\right\} \tag{3.32}$$

for all $(m,n)\in\Lambda_{(m_1,n_1)}$.

Considering continuities of W_i and W_i^{-1} for $i=1,2$ as well as of $\Upsilon_{i,\varepsilon}$ in ε and letting $\varepsilon\to 0_+$, we obtain (2.4). This completes the proof.

We remark that m_1, n_1 lie on the boundary of the lattice U. In particular, (2.4) is true for all $(m,n)\in\Lambda$ when every w_i $(i=1,2)$ satisfies $\int_{u_i}^{\infty}dx/w_i(\varphi^{-1}(x))=\infty$. Therefore, we may take $m_1=M$, $n_1=N$.

4. Applications to a Difference Equation

In this section we apply our result to the following boundary value problem (simply called BVP) for the partial difference equation:

$$\begin{aligned}&\Delta_1\Delta_2\varphi(z(m,n))=F(m,n,z(m,n)),\quad (m,n)\in\Lambda,\\ &z(m,n_0)=f(m),\quad z(m_0,n)=g(n),\quad (m,n)\in\Lambda,\end{aligned} \tag{4.1}$$

where $\Lambda:=I\times J$ is defined as in the beginning of Section 2, $\varphi\in C^0(\mathbb{R},\mathbb{R})$ is strictly increasing odd function satisfying $\varphi(u)>0$ for $u>0$, $F:\Lambda\times\mathbb{R}\to\mathbb{R}$ satisfies

$$|F(m,n,u)|\le h_1(m,n)\varphi_1(|u|)+h_2(m,n)\varphi_2(|u|), \tag{4.2}$$

for given functions $h_1,h_2:\Lambda\to\mathbb{R}_+$ and $\varphi_i\in C^0(\mathbb{R}_+,\mathbb{R}_+)$ $(i=1,2)$ satisfying $\varphi_i(u)>0$ for $u>0$, and functions $f:I\to\mathbb{R}$ and $g:J\to\mathbb{R}$ satisfy that $f(m_0)=g(n_0)=0$. Obviously, (4.1) is a generalization of the BVP problem considered by [26, Section 3], and the theorems of [26] are not able to solve it. In the following we first apply our main result to the discussion of boundedness of (4.1).

Corollary 4.1. *All solutions $z(m,n)$ of BVP* (4.1) *have the following estimation for all $(m,n) \in \Lambda_{(m_1,n_1)}$*

$$|z(m,n)| \leq \psi^{-1}\left\{W_2^{-1}\left[W_2(\Upsilon_2(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\right]\right\}, \tag{4.3}$$

where m_1, n_1 are given as in Theorem 2.1 and

$$\begin{aligned}
W_2(u) &= \int_1^u \frac{dx}{\{\max_{\tau\in[0,x]}\{\varphi_2(\psi^{-1}(\tau))/\max_{\tau_1\in[0,\tau]}\{\varphi_1(\psi^{-1}(\tau_1))\}\}\max_{\tau\in[0,x]}\{\varphi_1(\psi^{-1}(\tau))\}\}},\\
W_1(u) &= \int_1^u \frac{dx}{\max_{\tau\in[0,x]}\{\varphi_1(\psi^{-1}(\tau))\}},\\
\Upsilon_2(m,n) &= W_1^{-1}\left[W_1(\Upsilon_1(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(t,s)\right],\\
\Upsilon_1(m,n) &\leq \sum_{s=m_0}^{m-1}\left|\psi(f(s+1)) - \psi(f(s))\right| + \sum_{t=n_0}^{n-1}\left|\psi(g(t+1)) - \psi(g(t))\right|.
\end{aligned} \tag{4.4}$$

Proof. Clearly, the difference equation of BVP (4.1) is equivalent to

$$\psi(z(m,n)) = \psi(f(m)) + \psi(g(n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} F(s,t,z(s,t)). \tag{4.5}$$

It follows, by (4.2), that

$$\begin{aligned}
\left|\psi(z(m,n))\right| \leq \left|\psi(f(m)) + \psi(g(n))\right| &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t)\varphi_1(|z(s,t)|)\\
&+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\varphi_2(|z(s,t)|).
\end{aligned} \tag{4.6}$$

Let $a(m,n) = |\psi(f(m)) + \psi(g(n))|$. Since $|\psi(z(m,n))| = \psi(|z(m,n)|)$, (4.6) is of the form (1.6). Applying our Theorem 2.1 to inequality (4.6), we obtain the estimate of $z(m,n)$ as given in this corollary. □

Corollary 4.1 gives a condition of boundedness for solutions. Concretely, if

$$\Upsilon_1(m,n) < \infty, \qquad \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t) < \infty, \qquad \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t) < \infty \tag{4.7}$$

for all $(m,n) \in \Lambda_{(m_1,n_1)}$, then every solution $z(m,n)$ of BVP (4.1) is bounded on $\Lambda_{(m_1,n_1)}$.

Next, we discuss the uniqueness of solutions for BVP (4.1).

Corollary 4.2. *Suppose additionally that*

$$|F(m,n,u_1) - F(m,n,u_2)| \le h_1(m,n)\varphi_1\left(\left|\psi(u_1) - \psi(u_2)\right|\right) + h_2(m,n)\varphi_2\left(\left|\psi(u_1) - \psi(u_2)\right|\right) \tag{4.8}$$

for $u_1, u_2 \in \mathbb{R}$ *and* $(m,n) \in \Lambda := I \times J$, *where* $I = [m_0, M) \cap \mathbb{N}_0$, $J = [n_0, N) \cap \mathbb{N}_0$ *as assumed in the beginning of Section 2 with natural numbers* M *and* N, h_1, h_2 *are both nonnegative functions defined on the lattice* Λ, $\varphi_1, \varphi_2 \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ *are both nondecreasing with the nondecreasing ratio* φ_2/φ_1 *such that* $\varphi_i(0) = 0$, $\varphi_i(u) > 0$ *for all* $u > 0$ *and* $\int_0^1 ds/\varphi_i(s) = +\infty$ *for* $i = 1,2$ *and* $\psi \in C^0(\mathbb{R}, \mathbb{R})$ *is strictly increasing odd function satisfying* $\psi(u) > 0$ *for* $u > 0$. *Then BVP* (4.1) *has at most one solution on* Λ.

Proof. Assume that both $z(m,n)$ and $\tilde{z}(m,n)$ are solutions of BVP (4.1). From the equivalent form (4.5) of (4.1) we have

$$\begin{aligned}\left|\psi(z(m,n)) - \psi(\tilde{z}(m,n))\right| \le &\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t)\varphi_1\left(\left|\psi(z(s,t)) - \psi(\tilde{z}(s,t))\right|\right) \\ &+ \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\varphi_2\left(\left|\psi(z(s,t)) - \psi(\tilde{z}(s,t))\right|\right)\end{aligned} \tag{4.9}$$

for all $(m,n) \in \Lambda$, which is an inequality of the form (1.7), where $a(m,n) \equiv 0$. Applying our Theorem 2.1 with the choice that $u_1 = u_2 = 1$, we obtain an estimate of the difference $|\psi(z(m,n)) - \psi(\tilde{z}(m,n))|$ in the form (2.4), where $\Upsilon_1(m,n) \equiv 0$ because $a(m,n) \equiv 0$. Furthermore, by the definition of W_i we see that

$$\lim_{u\to 0} W_i(u) = -\infty, \qquad \lim_{u\to -\infty} W_i^{-1}(u) = 0, \quad i = 1,2. \tag{4.10}$$

It follows that

$$W_1(\Upsilon_1(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t) = -\infty, \tag{4.11}$$

since $m < M$, $n < N$. Thus, by (4.10),

$$\Upsilon_2(m,n) = W_1^{-1}\left[W_1(\Upsilon_1(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_1(s,t)\right] = 0. \tag{4.12}$$

Similarly, we get $W_2(\Upsilon_2(m,n)) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} h_2(s,t) = -\infty$ and therefore

$$W_2^{-1}\left[W_2(\Upsilon_2(m,n)) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} h_2(s,t)\right] = 0. \tag{4.13}$$

Thus we conclude from (2.4) that $|\varphi(z(m,n)) - \varphi(\tilde{z}(m,n))| \le 0$, implying that $z(m,n) = \tilde{z}(m,n)$ for all $(m,n) \in \Lambda$ since φ is strictly increasing. It proves the uniqueness. □

Remark 4.3. If $h_1 \equiv 0$ or $h_2 \equiv 0$ in (4.8), the conclusion of the Corollary 4.2 also can be obtained.

Finally, we discuss the continuous dependence of solutions of BVP (4.1) on the given functions F, f, and g. Consider a variation of BVP (4.1)

$$\begin{aligned} &\Delta_1\Delta_2\varphi(z(m,n)) = \tilde{F}(m,n,z(m,n)), \quad (m,n) \in \Lambda, \\ &z(m,n_0) = \tilde{f}(m), \quad z(m_0,n) = \tilde{g}(n), \quad (m,n) \in \Lambda, \end{aligned} \tag{4.14}$$

where $\varphi \in C^0(\mathbb{R},\mathbb{R})$ is strictly increasing odd function satisfying $\varphi(u) > 0$ for $u > 0$, $\tilde{F} \in C^0(\Lambda \times \mathbb{R},\mathbb{R})$, and $\tilde{f} : I \to \mathbb{R}$, $\tilde{g} : J \to \mathbb{R}$ are functions satisfying $\tilde{f}(m_0) = \tilde{g}(n_0) = 0$.

Corollary 4.4. *Let F be a function as assumed in the beginning of Section 4 and satisfy* (4.2) *and* (4.8) *on the same lattice Λ as assumed in Corollary 4.2. Suppose that the three differences*

$$\max_{m\in I}\left|\tilde{f} - f\right|, \qquad \max_{n\in J}|\tilde{g} - g|, \qquad \max_{(s,t,u)\in\Lambda\times\mathbb{R}}\left|\tilde{F}(s,t,u) - F(s,t,u)\right| \tag{4.15}$$

are all sufficiently small. Then solution $\tilde{z}(m,n)$ of BVP (4.14) *is sufficiently close to the solution $z(m,n)$ of BVP* (4.1).

Proof. By Corollary 4.2, the solution $z(m,n)$ is unique. By the continuity and the strict monotonicity of φ, we suppose that

$$\begin{aligned} &\max_{m\in I}\left|\varphi\left(\tilde{f}(m)\right) - \varphi(f(m))\right| < \epsilon, \qquad \max_{n\in J}|\varphi(\tilde{g}(n)) - \varphi(g(n))| < \epsilon, \\ &\max_{(s,t,u)\in I\times J\times\mathbb{R}}\left|\tilde{F}(s,t,u) - F(s,t,u)\right| < \epsilon, \end{aligned} \tag{4.16}$$

where $\epsilon > 0$ is a small number. By the equivalent difference equation (4.5) and the inequality (4.8) we get

$$\begin{aligned}
\left|\varphi(\tilde{z}(m,n)-\varphi(z(m,n)))\right| &\leq \left|\varphi\left(\tilde{f}(m)\right)-\varphi(f(m))+\varphi(\tilde{g}(n))-\varphi(g(n))\right| \\
&\quad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|\tilde{F}(s,t,\tilde{z}(s,t))-F(s,t,z(s,t))\right| \\
&\leq 2\epsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|\tilde{F}(s,t,\tilde{z}(s,t))-F(s,t,\tilde{z}(s,t))\right| \\
&\quad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F(s,t,\tilde{z}(s,t))-F(s,t,z(s,t))\right| \\
&\leq \{2+(m_1-m_0)(n_1-n_0)\}\epsilon \\
&\quad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_1(s,t)\varphi_1(\left|\varphi(\tilde{z}(s,t))-\varphi(z(s,t))\right|) \\
&\quad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_2(s,t)\varphi_2(\left|\varphi(\tilde{z}(s,t))-\varphi(z(s,t))\right|),
\end{aligned} \tag{4.17}$$

that is an inequality of the form (1.7). Applying Theorem 2.1 to (4.17), we obtain, for all $(m,n)\in\Lambda_{(m_1,n_1)}$, that

$$\left|\varphi(\tilde{z}(m,n)-\varphi(z(m,n)))\right| \leq W_2^{-1}\left[W_2(\Upsilon_2(m,n))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_2(s,t)\right], \tag{4.18}$$

where m_1, n_1 are given as in Theorem 2.1,

$$\begin{aligned}
\Upsilon_2(m,n) &= W_1^{-1}\left[W_1(\Upsilon_1(m,n))+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}h_1(t,s)\right], \\
\Upsilon_1(m,n) &= \{2+(m_1-m_0)(n_1-n_0)\}\epsilon.
\end{aligned} \tag{4.19}$$

By (4.10) we see that $\Upsilon_i(m,n)\to 0\,(i=1,2)$ as $\epsilon\to 0$. It follows from (4.18) that $\lim_{\epsilon\to 0}|\varphi(\tilde{z}(m,n)-\varphi(z(m,n))|=0$ and hence $z(m,n)$ depends continuously on F, f, and g. □

Remark 4.5. Our requirement of the small difference $\tilde{F}-F$ in Corollary 4.4 is stronger than the condition (iii) in [26, Theorem 3.3], but it is easier to check than the condition of them.

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References

[1] T. H. Gronwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Annals of Mathematics*, vol. 20, no. 4, pp. 292–296, 1919.

[2] R. Bellman, "The stability of solutions of linear differential equations," *Duke Mathematical Journal*, vol. 10, pp. 643–647, 1943.

[3] R. P. Agarwal, S. Deng, and W. Zhang, "Generalization of a retarded Gronwall-like inequality and its applications," *Applied Mathematics and Computation*, vol. 165, no. 3, pp. 599–612, 2005.

[4] D. Baĭnov and P. Simeonov, *Integral Inequalities and Applications*, vol. 57 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[5] W.-S. Cheung and Q.-H. Ma, "On certain new Gronwall-Ou-Iang type integral inequalities in two variables and their applications," *Journal of Inequalities and Applications*, vol. 10, no. 4, pp. 347–361, 2005.

[6] O. Lipovan, "Integral inequalities for retarded Volterra equations," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 349–358, 2006.

[7] Q.-H. Ma and E.-H. Yang, "On some new nonlinear delay integral inequalities," *Journal of Mathematical Analysis and Applications*, vol. 252, no. 2, pp. 864–878, 2000.

[8] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, vol. 53 of *Mathematics and Its Applications (East European Series)*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.

[9] B. G. Pachpatte, *Inequalities for Differential and Integral Equations*, vol. 197 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1998.

[10] W.-S. Wang, "A generalized retarded Gronwall-like inequality in two variables and applications to BVP," *Applied Mathematics and Computation*, vol. 191, no. 1, pp. 144–154, 2007.

[11] W.-S. Wang, "A generalized sum-difference inequality and applications to partial difference equations," *Advances in Difference Equations*, vol. 2008, Article ID 695495, 12 pages, 2008.

[12] W.-S. Wang and C.-X. Shen, "On a generalized retarded integral inequality with two variables," *Journal of Inequalities and Applications*, vol. 2008, Article ID 518646, 9 pages, 2008.

[13] W.-S. Wang, "Estimation on certain nonlinear discrete inequality and applications to boundary value problem," *Advances in Difference Equations*, vol. 2009, Article ID 708587, 8 pages, 2009.

[14] W. Zhang and S. Deng, "Projected Gronwall-Bellman's inequality for integrable functions," *Mathematical and Computer Modelling*, vol. 34, no. 3-4, pp. 393–402, 2001.

[15] K. Zheng, Y. Wu, and S. Deng, "Nonlinear integral inequalities in two independent variables and their applications," *Journal of Inequalities and Applications*, vol. 2007, Article ID 32949, 11 pages, 2007.

[16] T. E. Hull and W. A. J. Luxemburg, "Numerical methods and existence theorems for ordinary differential equations," *Numerische Mathematik*, vol. 2, pp. 30–41, 1960.

[17] B. G. Pachpatte and S. G. Deo, "Stability of discrete-time systems with retarded argument," *Utilitas Mathematica*, vol. 4, pp. 15–33, 1973.

[18] D. Willett and J. S. W. Wong, "On the discrete analogues of some generalizations of Gronwall's inequality," *Monatshefte für Mathematik*, vol. 69, pp. 362–367, 1965.

[19] B. G. Pachpatte, "On some fundamental integral inequalities and their discrete analogues," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 2, no. 2, article 15, pp. 1–13, 2001.

[20] C. M. Dafermos, "The second law of thermodynamics and stability," *Archive for Rational Mechanics and Analysis*, vol. 70, no. 2, pp. 167–179, 1979.

[21] P. Y. H. Pang and R. P. Agarwal, "On an integral inequality and its discrete analogue," *Journal of Mathematical Analysis and Applications*, vol. 194, no. 2, pp. 569–577, 1995.

[22] B. G. Pachpatte, "On some new inequalities related to certain inequalities in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 1, pp. 128–144, 1995.

[23] B. G. Pachpatte, *Inequalities for Finite Difference Equations*, vol. 247 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2002.

[24] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, vol. 205 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2006.

[25] W.-S. Cheung, "Some discrete nonlinear inequalities and applications to boundary value problems for difference equations," *Journal of Difference Equations and Applications*, vol. 10, no. 2, pp. 213–223, 2004.

[26] W.-S. Cheung and J. Ren, "Discrete non-linear inequalities and applications to boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 708–724, 2006.

[27] M. Pinto, "Integral inequalities of Bihari-type and applications," *Funkcialaj Ekvacioj*, vol. 33, pp. 387–430, 1990.