SOLVABILITY OF A MULTI-POINT BOUNDARY VALUE PROBLEM OF NEUMANN TYPE

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Received 5 April 1999

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions and $e(t) \in L^1[0,1]$. Let $\xi_i \in (0,1)$, $a_i \in \mathbb{R}$, $i = 1,2,\ldots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ be given. This paper is concerned with the problem of existence of a solution for the m-point boundary value problem $x''(t) = f(t,x(t),x'(t)) + e(t)$, $0 < t < 1$; $x(0) = 0$, $x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$. This paper gives conditions for the existence of a solution for this boundary value problem using some new Poincaré type a priori estimates. This problem was studied earlier by Gupta, Ntouyas, and Tsamatos (1994) when all of the $a_i \in \mathbb{R}$, $i = 1,2,\ldots,m-2$, had the same sign. The results of this paper give considerably better existence conditions even in the case when all of the $a_i \in \mathbb{R}$, $i = 1,2,\ldots,m-2$, have the same sign. Some examples are given to illustrate this point.

1. Introduction

Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions and $e : [0,1] \mapsto \mathbb{R}$ be a function in $L^1[0,1]$, $a_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1,2,\ldots,m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$. We study the problem of existence of solutions for the m-point boundary value problem

$$\begin{gathered} x''(t) = f\big(t, x(t), x'(t)\big) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \sum_{i=1}^{m-2} a_i x'\big(\xi_i\big). \end{gathered} \tag{1.1}$$

This problem was studied earlier by Gupta, Ntouyas, and Tsamatos in [1] when all of the $a_i \in \mathbb{R}$, $i = 1,2,\ldots,m-2$, have the same sign. Gupta, Ntouyas, and Tsamatos have studied problem (1.1) by first studying the three-point boundary value problem, for a given $\alpha \in \mathbb{R}$, $\alpha \neq 1$, $\eta \in (0,1)$,

$$\begin{gathered} x''(t) = f\big(t, x(t), x'(t)\big) + e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta). \end{gathered} \tag{1.2}$$

Abstract and Applied Analysis 4:2 (1999) 71–81
1991 Mathematics Subject Classification: 34B10, 34B15, 34G20
URL: http://aaa.hindawi.com/volume-4/S1085337599000093.html

The purpose of this paper is to obtain conditions for the existence of a solution for the boundary value problem (1.1), using new estimates and inequalities involving a function $x(t)$ and its derivative $x'(t)$. These results are motivated by the so-called *nonlocal* boundary value problem studied by Il'in and Moiseev in [5].

We use the classical spaces $C[0,1]$, $C^k[0,1]$, $L^k[0,1]$, and $L^\infty[0,1]$ of continuous, k-times continuously differentiable, measurable real-valued functions whose kth power of the absolute value is Lebesgue integrable on $[0,1]$, or measurable functions that are essentially bounded on $[0,1]$. We also use the Sobolev spaces $W^{2,k}(0,1)$, $k=1,2$ defined by

$$W^{2,k}(0,1)=\{x:[0,1]\longrightarrow\mathbb{R}\mid x,x' \text{ absolutely continuous on } [0,1] \text{ with } x''\in L^k[0,1]\} \tag{1.3}$$

with its usual norm. We denote the norm in $L^k[0,1]$ by $\|\cdot\|_k$, and the norm in $L^\infty[0,1]$ by $\|\cdot\|_\infty$.

2. A priori estimates

Let $a_i\in\mathbb{R}$, $\xi_i\in(0,1)$, $i=1,2,\dots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, with $\alpha=\sum_{i=1}^{m-2}a_i\neq 1$ be given. Let $x(t)\in W^{2,1}(0,1)$ be such that $x(0)=0$, $x'(1)=\sum_{i=1}^{m-2}a_ix'(\xi_i)$ be given. We are interested in obtaining a priori estimates of the form $\|x'\|_\infty\le C\|x''\|_1$. The following theorem gives such an estimate. We recall that for $a\in\mathbb{R}$, $a_+=\max\{a,0\}$, $a_-=\max\{-a,0\}$ so that $a=a_+-a_-$ and $|a|=a_++a_-$.

THEOREM 2.1. *Let $a_i\in\mathbb{R}$, $\xi_i\in(0,1)$, $i=1,2,\dots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, with $\alpha=\sum_{i=1}^{m-2}a_i\neq 1$ be given. Then for $x(t)\in W^{2,1}(0,1)$ with $x(0)=0, x'(1)=\sum_{i=1}^{m-2}a_ix'(\xi_i)$ we have*

$$\|x'\|_\infty\le\frac{1}{1-\tau}\|x''\|_1, \tag{2.1}$$

where

$$\tau=\min\left\{\frac{\sum_{i=1}^{m-2}(a_i)_+}{\sum_{i=1}^{m-2}(a_i)_-+1},\frac{\sum_{i=1}^{m-2}(a_i)_-+1}{\sum_{i=1}^{m-2}(a_i)_+}\right\}. \tag{2.2}$$

Proof. We see that the assumption $x'(1)=\sum_{i=1}^{m-2}a_ix'(\xi_i)$ implies

$$x'(1)+\sum_{i=1}^{m-2}(a_i)_-x'(\xi_i)=\sum_{i=1}^{m-2}(a_i)_+x'(\xi_i) \tag{2.3}$$

and thus there exist $\lambda_1,\lambda_2\in[0,1]$ such that

$$\left(1+\sum_{i=1}^{m-2}(a_i)_-\right)x'(\lambda_1)=\sum_{i=1}^{m-2}(a_i)_+x'(\lambda_2). \tag{2.4}$$

If, now, either $x'(\lambda_1)=0$ or $x'(\lambda_2)=0$, then we clearly have

$$\|x'\|_\infty\le\|x''\|_1. \tag{2.5}$$

Suppose, now, that $x'(\lambda_1) \neq 0$ and $x'(\lambda_2) \neq 0$. Then it follows easily from (2.4) that $x'(\lambda_1) \neq x'(\lambda_2)$, in view of the assumption $\alpha = \sum_{i=1}^{m-2} a_i \neq 1$. Then it follows from (2.4), the estimate (2.5), and the equations

$$x'(t) = x'(\lambda_1) + \int_{\lambda_1}^{t} x''(s)\,ds, \qquad x'(t) = x'(\lambda_2) + \int_{\lambda_2}^{t} x''(s)\,ds, \tag{2.6}$$

that

$$\|x'\|_\infty \leq \frac{1}{1-\tau}\|x''\|_1 \tag{2.7}$$

with

$$\tau = \min\left\{ \frac{\sum_{i=1}^{m-2}(a_i)_+}{\sum_{i=1}^{m-2}(a_i)_- + 1}, \frac{\sum_{i=1}^{m-2}(a_i)_- + 1}{\sum_{i=1}^{m-2}(a_i)_+} \right\}. \tag{2.8}$$

This completes the proof of the theorem. □

Remark 2.2. We note that if $a_i \leq 0$ for every $i = 1, 2, \dots, m-2$, then $\tau = 0$ and if $a_i \geq 0$ for every $i = 1, 2, \dots, m-2$ so that $\alpha = \sum_{i=1}^{m-2} a_i = \sum_{i=1}^{m-2}(a_i)_+ \geq 0$, then $\tau = \min\{\alpha, 1/\alpha\} \in [0, 1)$ since $\alpha \neq 1$, by assumption.

The following theorem gives a better estimate for the three-point boundary value in the case of the L^2-norm.

THEOREM 2.3. *Let $\alpha \in \mathbb{R}$, $\alpha \neq 1$, and $\eta \in (0, 1)$ be given. Let $x(t) \in W^{2,2}(0, 1)$ be such that $x'(1) = \alpha x'(\eta)$. Then*

$$\|x'\|_2 \leq C(\alpha, \eta)\|x''\|_2, \tag{2.9}$$

where

$$C(\alpha, \eta) = \begin{cases} \min\left\{\sqrt{F(\alpha, \eta)}, \dfrac{2}{\pi}\right\} & \text{if } \alpha \leq 0, \\ \sqrt{F(\alpha, \eta)} & \text{if } \alpha > 0, \end{cases} \tag{2.10}$$

$$F(\alpha, \eta) = \frac{1}{2(\alpha-1)^2}\left[\alpha^2(1-\eta)^2 + (\alpha^2 - 2\alpha)\eta^2 + 1\right].$$

Proof. If $\alpha \leq 0$, we note from $x'(1) = \alpha x'(\eta)$ that there exists an $\xi \in (\eta, 1)$ such that $x'(\xi) = 0$. It follows from the Wirtinger's inequality (see [4, Theorem 256]) that

$$\|x'\|_2 \leq \frac{2}{\pi}\|x''\|_2. \tag{2.11}$$

Next, we note, again, from $x'(1) = \alpha x'(\eta)$ that

$$x'(t) = \int_0^t x''(s)\,ds + \frac{\alpha}{1-\alpha}\int_0^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_0^1 x''(s)\,ds \quad \text{for } 0 < t < 1. \tag{2.12}$$

Accordingly, we have for $t \in [0,\eta]$

$$\begin{aligned} x'(t) &= \int_0^t x''(s)\,ds + \frac{\alpha}{1-\alpha}\int_0^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_0^1 x''(s)\,ds \\ &= \int_0^t\left(1+\frac{\alpha}{1-\alpha}-\frac{1}{1-\alpha}\right)x''(s)ds+\int_t^\eta\left(\frac{\alpha}{1-\alpha}-\frac{1}{1-\alpha}\right)x''(s)\,ds-\frac{1}{1-\alpha}\int_\eta^1 x''(s)\,ds \\ &= -\int_t^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_\eta^1 x''(s)\,ds, \end{aligned} \tag{2.13}$$

and for $t \in [\eta,1]$

$$\begin{aligned} x(t) &= \int_0^t x''(s)\,ds + \frac{\alpha}{1-\alpha}\int_0^\eta x''(s)\,ds - \frac{1}{1-\alpha}\int_0^1 x''(s)\,ds \\ &= \int_0^\eta\left(1+\frac{\alpha}{1-\alpha}-\frac{1}{1-\alpha}\right)x''(s)\,ds+\int_\eta^t\left(1-\frac{1}{1-\alpha}\right)x''(s)\,ds-\frac{1}{1-\alpha}\int_t^1 x''(s)\,ds \\ &= -\int_\eta^t \frac{\alpha}{1-\alpha}x''(s)\,ds - \frac{1}{1-\alpha}\int_t^1 x''(s)\,ds. \end{aligned} \tag{2.14}$$

We now define a function $K : [0,1]\times[0,1] \mapsto \mathbb{R}$ by

$$K(t,s) = \begin{cases} -\chi_{[t,\eta]}(s) - \dfrac{1}{1-\alpha}\chi_{[\eta,1]}(s) & \text{for } t \in [0,\eta],\ s \in [0,1], \\ -\dfrac{\alpha}{1-\alpha}\chi_{[\eta,t]}(s) - \dfrac{1}{1-\alpha}\chi_{[t,1]}(s) & \text{for } t \in [\eta,1],\ s \in [0,1]. \end{cases} \tag{2.15}$$

Now, we see from (2.13) and (2.14) that

$$x'(t) = \int_0^1 K(t,s)x''(s)\,ds \quad \text{for } t \in [0,1], \tag{2.16}$$

$$\|x'\|_2^2 \le \left(\int_0^1\int_0^1 \big(K(t,s)\big)^2\,ds\,dt\right)\|x''\|_2^2. \tag{2.17}$$

Now, it is easy to see that

$$\int_0^1\int_0^1 \big(K(t,s)\big)^2\,ds\,dt = \frac{1}{2(\alpha-1)^2}\big[\alpha^2(1-\eta)^2+\big(\alpha^2-2\alpha\big)\eta^2+1\big]. \tag{2.18}$$

For $\alpha \le 0$ the estimate (2.9) is now immediate from (2.11), (2.17), and (2.18) and for $\alpha > 0$, $\alpha \ne 1$, by (2.17) and (2.18). This completes the proof of the theorem. □

Remark 2.4. It is easy to see that $C(-0.1,\eta) = 2/\pi$, for all $\eta \in (0,1)$, indeed, $\sqrt{F(-0.1,\eta)} \ge 0.648986183$ and $2/\pi \approx 0.6366197724$. Also $C(-2,1/3) = \sqrt{11/54}$ and $C(-2,15/16) = 2/\pi$, since $\sqrt{F(-2,15/16)} = \sqrt{1030}/48 > 2/\pi$.

3. Existence theorems

Definition 3.1. A function $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory's conditions if

(i) for each $(x,y)\in\mathbb{R}^2$, the function $t\in[0,1]\to f(t,x,y)\in\mathbb{R}$ is measurable on $[0,1]$,

(ii) for a.e. $t\in[0,1]$, the function $(x,y)\in\mathbb{R}^2\to f(t,x,y)\in\mathbb{R}$ is continuous on $\mathbb{R}^2$,

(iii) for each $r>0$, there exists $\alpha_r(t)\in L^1[0,1]$ such that $|f(t,x,y)|\le\alpha_r(t)$ for a.e. $t\in[0,1]$ and all $(x,y)\in\mathbb{R}^2$ with $\sqrt{x^2+y^2}\le r$.

THEOREM 3.2. *Let $f : [0,1]\times\mathbb{R}^2\to\mathbb{R}$ be a function satisfying Carathéodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, and $r(t)$ in $L^1(0,1)$ such that*

$$\left|f\big(t,x_1,x_2\big)\right|\le p(t)|x_1|+q(t)|x_2|+r(t) \tag{3.1}$$

for a.e. $t\in[0,1]$ and all $(x_1,x_2)\in\mathbb{R}^2$. Also let $a_i\in\mathbb{R}$, $\xi_i\in(0,1)$, $i=1,2,\dots,m-2$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, with $\alpha=\sum_{i=1}^{m-2}a_i\ne 1$ be given. Then the boundary value problem (1.1) has at least one solution in $C^1[0,1]$ provided

$$\|tp(t)\|_1+\|q(t)\|_1+\tau<1, \tag{3.2}$$

where τ is as defined in Theorem 2.1.

Proof. Let X denote the Banach space $C^1[0,1]$ and Y denote the Banach space $L^1(0,1)$ with their usual norms. We define a linear mapping $L : D(L)\subset X\to Y$ by setting

$$D(L)=\left\{x\in W^{2,1}(0,1)\,\middle|\,x(0)=0,\ x'(1)=\sum_{i=1}^{m-2}a_i x'\big(\xi_i\big)\right\}, \tag{3.3}$$

and for $x\in D(L)$,

$$Lx=x''. \tag{3.4}$$

We also define a nonlinear mapping $N : X\to Y$ by setting

$$(Nx)(t)=f\big(t,x(t),x'(t)\big),\quad t\in[0,1]. \tag{3.5}$$

We note that N is a bounded mapping from X into Y. Next, it is easy to see that the linear mapping $L : D(L)\subset X\to Y$, is a one-to-one mapping. Next, the linear mapping $K : Y\to X$, defined for $y\in Y$ by

$$(Ky)(t)=\int_0^t (t-s)y(s)\,ds+At, \tag{3.6}$$

where A is given by,

$$A\left(1-\sum_{i=1}^{m-2}a_i\right)=\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}y(s)\,ds-\int_0^1 y(s)\,ds, \tag{3.7}$$

is such that for $y \in Y$, $Ky \in D(L)$, and $LKy = y$; and for $u \in D(L)$, $KLu = u$. Furthermore, it follows easily using the Arzela-Ascoli theorem that KN maps a bounded subset of X into a relatively compact subset of X. Hence $KN : X \to X$ is a compact mapping.

We, next, note that $x \in C^1[0,1]$ is a solution of the boundary value problem (1.2) if and only if x is a solution to the operator equation

$$Lx = Nx + e. \tag{3.8}$$

Now, the operator equation $Lx = Nx + e$ is equivalent to the equation

$$x = KNx + Ke. \tag{3.9}$$

We apply the Leray-Schauder continuation theorem (cf. [6, Corollary IV.7]) to obtain the existence of a solution for $x = KNx + Ke$ or equivalently to the boundary value problem (1.2).

To do this, it suffices to verify that the set of all possible solutions of the family of equations

$$\begin{gathered} x''(t) = \lambda f\big(t, x(t), x'(t)\big) + \lambda e(t), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \sum_{i=1}^{m-2} a_i x'\big(\xi_i\big), \end{gathered} \tag{3.10}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda \in [0,1]$.

We observe that if $x \in W^{2,1}(0,1)$, with $x(0) = 0$, $x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$, then $x(t) = \int_0^t x'(s)ds$. Hence, $|x(t)| \le t\|x'\|_\infty$ for $t \in [0,1]$ and $\|x'\|_\infty \le (1/(1-\tau))\|x''\|_1$, where τ is as defined in Theorem 2.1.

Let $x(t)$ be a solution of (3.10) for some $\lambda \in [0,1]$, so that $x \in W^{2,1}(0,1)$ with $x(0) = 0$, $x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$. We then get from the equation in (3.10) and Theorem 2.1 that

$$\begin{aligned} \|x'\|_\infty &\le \frac{\lambda}{1-\tau} \big\| f\big(t, x(t), x'(t)\big) + e(t) \big\|_1 \\ &\le \frac{1}{1-\tau} \big(\big\| p(t)|x(t)| + q(t)\big|x'(t)\big| + r(t) \big\|_1 + \|e(t)\|_1 \big) \\ &\le \frac{1}{1-\tau} \big(\big\| tp(t) \big\| x' \big\|_\infty + q(t)\big|x'(t)\big| + r(t) \big\|_1 + \|e(t)\|_1 \big) \\ &\le \frac{1}{1-\tau} \big(\|tp(t)\|_1 + \|q(t)\|_1 \big) \big\| x' \big\|_\infty + \frac{1}{1-\tau} \big(\|r(t)\|_1 + \|e(t)\|_1 \big). \end{aligned} \tag{3.11}$$

It follows from assumption (3.2) that there is a constant c, independent of $\lambda \in [0,1]$, such that

$$\|x\|_\infty \le \big\|x'\big\|_\infty \le c. \tag{3.12}$$

It is now immediate that the set of solutions of the family of equations (3.10) is, a priori, bounded in $C^1[0,1]$ by a constant, independent of $\lambda \in [0,1]$.

This completes the proof of the theorem. □

THEOREM 3.3. *Let $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be a function satisfying Carathéodory's conditions. Assume that there exist functions $p(t)$, $q(t)$, and $r(t)$ in $L^2(0,1)$ such that*

$$\left|f\left(t,x_1,x_2\right)\right| \le p(t)|x_1| + q(t)|x_2| + r(t) \tag{3.13}$$

for a.e. $t\in[0,1]$ and all $(x_1,x_2)\in\mathbb{R}^2$. Also let $\alpha\neq 1$, and $\eta\in(0,1)$ be given. Then for any given $e(t)$ in $L^2(0,1)$ the boundary value problem (1.2) has at least one solution in $C^1[0,1]$ provided

$$C(\alpha,\eta)\left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right) < 1, \tag{3.14}$$

where $C(\alpha,\eta)$ is as in Theorem 2.3.

Proof. As in the proof of Theorem 3.2 it suffices to prove that the set of all possible solutions of the family of equations

$$\begin{gathered} x''(t) = \lambda f\left(t,x(t),x'(t)\right) + \lambda e(t), \quad 0<t<1, \\ x(0)=0, \qquad x'(1)=\alpha x'(\eta), \end{gathered} \tag{3.15}$$

is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda\in[0,1]$. For $x\in W^{2,2}(0,1)$, with $x(0)=0$, and $x'(1)=\alpha x'(\eta)$ we use the Wirtinger's inequality (see [4, Theorem 256]) and Theorem 2.3, above, to note that

$$\|x\|_2 \le \frac{2}{\pi}\left\|x'\right\|_2 \qquad \text{and} \qquad \left\|x'\right\|_2 \le C(\alpha,\eta)\left\|x''\right\|_2. \tag{3.16}$$

Now, for a solution x of the family of equations (3.15) for some $\lambda\in[0,1]$ we have

$$\begin{aligned} \left\|x''\right\|_2 &\le \lambda\left\| f\left(t,x(t),x'(t)\right) + e(t)\right\|_2 \\ &\le \left\|p(t)|x(t)| + q(t)\left|x'(t)\right| + r(t)\right\|_2 + \|e\|_2 \\ &\le \|p\|_2\|x\|_2 + \|q\|_2\left\|x'\right\|_2 + \|r\|_2 + \|e\|_2 \\ &\le \left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right)\left\|x'\right\|_2 + \|r(t)\|_2 + \|e\|_2 \\ &\le C(\alpha,\eta)\left(\frac{2}{\pi}\|p\|_2 + \|q\|_2\right)\left\|x''\right\|_2 + \|r(t)\|_2 + \|e\|_2, \end{aligned} \tag{3.17}$$

in view of estimate (3.16), for a solution x of the family of equations (3.15) for some $\lambda\in[0,1]$. It then follows from (3.14) that there is a constant c independent of $\lambda\in[0,1]$ such that

$$\left\|x''\right\|_2 \le c, \tag{3.18}$$

for a solution x of the family of equations (3.15) for some $\lambda\in[0,1]$. Finally, we see, using Theorem 2.1 that $\|x\|_\infty \le \|x'\|_\infty \le (1/(1-\tau))\|x''\|_1 \le (1/(1-\tau))\|x''\|_2$ and accordingly, the set of solutions of the family of equations (3.15) is, a priori, bounded in $C^1[0,1]$ by a constant independent of $\lambda\in[0,1]$. This completes the proof of Theorem 3.3. □

We next give an existence condition independent of α and η for the three-point boundary value problem (1.2).

Let $p(t), q(t)$ be given functions in $L^1(0,1)$. For, a given measurable function $x(t)$ on $[0,1]$, we define for $t \in [0,1]$,

$$\begin{gathered} P(t) = \int_t^1 p(u)\,du, \qquad (Vx)(t) = \int_t^1 q(s)x(s)\,ds, \\ (Sx)(t) = P(t)\int_0^t x(u)\,du + \int_t^1 P(u)x(u)\,du; \end{gathered} \tag{3.19}$$

provided that the integrals in (3.19) exist. We, further, suppose that the operator $M : L^2(0,1) \mapsto L^2(0,1)$ defined for $x(t) \in L^2(0,1)$ by

$$(Mx)(t) = (Sx)(t) + (Vx)(t), \quad 0 < t < 1; \tag{3.20}$$

maps $L^2(0,1)$ into itself and is continuous.

THEOREM 3.4. *Let $p(t), q(t)$, and M be as above. Let $f : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}$ be a given function satisfying Carathéodory conditions. Suppose that $p(t), q(t) \in L^1(0,1)$ and $r(t) \in L^2(0,1)$ be such that*

$$|f(t,x,y)| \le p(t)|x| + q(t)|y| + r(t) \quad \text{for } t \in [0,1],\ x, y \in \mathbb{R}. \tag{3.21}$$

Then, given $\alpha \in \mathbb{R}$, $\alpha \le 0$, and $\eta \in (0,1)$, the three-point boundary value problem

$$\begin{gathered} x''(t) = f\big(t, x(t), x'(t)\big), \quad 0 < t < 1, \\ x(0) = 0, \qquad x'(1) = \alpha x'(\eta), \end{gathered} \tag{3.22}$$

has at least one solution if the spectral radius, $r(M)$ of the operator M is less than one.

Proof. Let $x(t)$ be a solution of the boundary value problem (3.22), so that $x(0) = 0$, $x'(1) = \alpha x'(\eta)$. It is then easy to see that there exists a $\mu \in (0,1)$ such that $x'(\mu) = 0$. The rest of the proof is identical to the proof of Theorem 5 of [2] and is omitted. □

COROLLARY 3.5. *Let $p(t), q(t)$ in Theorem 3.4 be such that $p(t), q^2(t) \in L^1(\sigma,1)$ for every $\sigma > 0$, and $\sqrt{t}\int_t^1 q^2(s)ds \in L^2(0,1)$. Suppose, further, that*

$$\big\|\sqrt{2t}\,P(t)\big\|_2 + \left\|\sqrt{2t}\int_t^1 q^2(s)\,ds\right\|_2^{1/2} < 1. \tag{3.23}$$

Then, given $\alpha \in \mathbb{R}$, $\alpha \le 0$, and $\eta \in (0,1)$, the boundary value problem (3.22) has at least one solution.

The proof of the corollary is identical to the proof of Theorem 3 of [3] and is omitted.

Example 3.6. Let $\alpha \le 0$ and $\eta \in (0,1)$ be given and $A \in \mathbb{R}$. For $e(t) \in L^1(0,1)$, we consider the three-point boundary value problem

$$\begin{aligned} x''(t) &= t^{-1/2}|x(t)| + At\left|x'(t)\right| + e(t), \quad 0 < t < 1, \\ x(0) &= 0, \qquad x'(1) = \alpha x'(\eta). \end{aligned} \tag{3.24}$$

We apply Theorem 3.2 to obtain a condition for the existence of a solution for the three-point boundary value problem (3.24). Here $p(t) = t^{-1/2}$, $q(t) = At$, and $\tau = 0$. Now, $\|tp(t)\|_1 = 2/3$ and $\|q(t)\|_1 = (1/2)|A|$. Now, if

$$\frac{2}{3} + \frac{1}{2}|A| < 1, \tag{3.25}$$

or, equivalently

$$|A| < \frac{2}{3}, \tag{3.26}$$

then Theorem 3.2 implies the existence of a solution for the three-point boundary value problem (3.24).

Example 3.7. Let $\alpha = -2$, $\eta = 1/3$, and $A \in \mathbb{R}$. For $e(t) \in L^2(0,1)$, we, next, consider the three-point boundary value problem

$$\begin{aligned} x''(t) &= t^{-1/4}|x(t)| + At^{-1/4}\left|x'(t)\right| + e(t), \quad 0 < t < 1, \\ x(0) &= 0, \qquad x'(1) = \alpha x'(\eta). \end{aligned} \tag{3.27}$$

We apply Theorem 3.3 to obtain a condition for the existence of a solution for the three-point boundary value problem (3.27). Here $p(t) = t^{-1/4}$, $q(t) = At^{-1/4}$. Now, $\|p(t)\|_2 = \sqrt{2}$ and $\|q(t)\|_2 = \sqrt{2}|A|$. Now the existence condition required to apply Theorem 3.3 is

$$C(\alpha,\eta)\left(\frac{2\sqrt{2}}{\pi} + \sqrt{2}|A|\right) < 1. \tag{3.28}$$

Since we have $C(-2,1/3) = \sqrt{11/54}$, we get from (3.28)

$$\frac{2\sqrt{22}}{\sqrt{54}\pi} + \sqrt{\frac{22}{54}}|A| < 1. \tag{3.29}$$

Accordingly, we see from Theorem 3.3 that a solution for the three-point boundary value problem (3.27) exists if $|A| < \sqrt{54/22}(1 - 2\sqrt{22}/(\sqrt{54}\pi)) = 0.930079132$. Next, we apply Corollary 3.5 to the three-point boundary value problem (3.27). Now, we see that $P(t) = \int_t^1 u^{-1/4}du = 4/3 - 4/3(\sqrt[4]{t})^3$, so that

$$\begin{aligned} \left\|\sqrt{2t}P(t)\right\|_2^2 &= \int_0^1 \left(\sqrt{2t}\left(\frac{4}{3} - \frac{4}{3}\left(\sqrt[4]{t}\right)^3\right)\right)^2 dt = 0.20779, \\ \left\|\sqrt{2t}\int_t^1 q^2(s)\,ds\right\|_2^2 &= 8A^4\int_0^1 t\left(1-\sqrt{t}\right)^2 dt = \frac{4}{15}A^4, \end{aligned} \tag{3.30}$$

so that a solution to the three-point boundary value problem (3.27) exists if

$$\sqrt{0.20779}+\left(\frac{4}{15}\right)^{0.25}|A|<1 \tag{3.31}$$

or equivalently, if $|A| < (15/4)^{0.25}(1-\sqrt{0.20779}) = 0.7572417038$ for every $\eta \in (0,1)$. So we see that Corollary 3.5 does not give a better result than Theorem 3.3. On the other hand, if we apply Theorem 3.3 when $\alpha = -0.1,\ \eta \in (0,1)$ so that $C(-0.1,\eta) = 2/\pi$ we see that a solution to the three-point boundary value problem (3.27) exists if $|A| < 0.4741009622$, which is not as good as that given by Corollary 3.5.

Example 3.8. Let $\alpha = -2,\ \eta = 1/3$, and $A \in \mathbb{R}$. For $e(t) \in L^2(0,1)$, we, next, consider the three-point boundary value problem

$$\begin{gathered} x''(t) = t^{-15/32}|x(t)| + At\left|x'(t)\right| + e(t), \quad 0<t<1, \\ x(0)=0, \qquad x'(1) = \alpha x'(\eta). \end{gathered} \tag{3.32}$$

We apply Theorem 3.3 to obtain a condition for the existence of a solution for the three-point boundary value problem (3.32). Here $p(t) = t^{-15/32},\ q(t) = At$. Now, $\|p(t)\|_2 = 4$ and $\|q(t)\|_2 = (1/\sqrt{3})|A|$. Now the existence condition required to apply Theorem 3.3 is

$$C(\alpha,\eta)\left(\frac{8}{\pi}+\frac{1}{\sqrt{3}}|A|\right) < 1. \tag{3.33}$$

Since, $C(-2,1/3) = \sqrt{11/54}$ and we get from (3.33)

$$\frac{8\sqrt{11}}{\sqrt{54\pi}} + \sqrt{\frac{11}{162}}|A| < 1, \tag{3.34}$$

which is impossible. Now, to apply Theorem 3.2 we see that $\|tp(t)\|_1 = \int_0^1 t^{17/32}dt = 32/49$ and $\|q(t)\|_1 = (1/2)|A|$. Accordingly, we see using Theorem 3.2 a solution for the three-point boundary value problem (3.32) exists if

$$\frac{32}{49}+\frac{1}{2}|A| < 1, \tag{3.35}$$

or, equivalently, if

$$|A| < 2\left(1-\frac{32}{49}\right) = \frac{34}{49} = 0.69387751. \tag{3.36}$$

Next, we apply Corollary 3.5 to the three-point boundary value problem (3.32). Now, we see that $P(t) = \int_t^1 u^{-15/32}du = 32/17 - (32/17)(\sqrt[32]{t})^{17}$, so that

$$\begin{gathered} \|\sqrt{2t}P(t)\|_2^2 = \int_0^1 \left(\sqrt{2t}\left(\frac{32}{17}-\frac{32}{17}\left(\sqrt[32]{t}\right)^{17}\right)\right)^2 dt = 0.258, \\ \left\|\sqrt{2t}\int_t^1 q^2(s)\,ds\right\|_2^2 = \frac{2A^4}{9}\int_0^1 t\left(1-t^3\right)^2 dt = \frac{1}{20}A^4, \end{gathered} \tag{3.37}$$

so that a solution to the three-point boundary value problem (3.32) exists if

$$\sqrt{0.258}+\left(\frac{1}{20}\right)^{0.25}|A|<1 \tag{3.38}$$

or equivalently, if $|A|<(20)^{0.25}(1-\sqrt{0.258})=1.040586544$. Clearly, Corollary 3.5 gives a better result than Theorem 3.2.

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