

WEAKLY HYPERBOLIC EQUATIONS WITH TIME DEGENERACY IN SOBOLEV SPACES

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ABSTRACT. The theory of nonlinear weakly hyperbolic equations was developed during the last decade in an astonishing way. Today we have a good overview about assumptions which guarantee local well posedness in spaces of smooth functions (C^∞ , Gevrey). But the situation is completely unclear in the case of Sobolev spaces. Examples from the linear theory show that in opposite to the strictly hyperbolic case we have in general no solutions valued in Sobolev spaces. If so-called Levi conditions are satisfied, then the situation will be better. Using sharp Levi conditions of C^∞ -type leads to an interesting effect. The linear weakly hyperbolic Cauchy problem has a Sobolev solution if the data are sufficiently smooth. The loss of derivatives will be determined in essential by special lower order terms. In the present paper we show that it is even possible to prove the existence of Sobolev solutions in the quasilinear case although one has the finite loss of derivatives for the linear case. Some of the tools are a reduction process to problems with special asymptotical behaviour, a Gronwall type lemma for differential inequalities with a singular coefficient, energy estimates and a fixed point argument.

1. INTRODUCTION

In this paper we want to prove a local existence result in Sobolev spaces with respect to the spatial variables for the weakly hyperbolic Cauchy problem

$$(1.1) \quad u_{tt} - \lambda^2(t) \Delta u = f(t, x, u, u_t, \mu_i(t) \partial_{x_i} u),$$

$$(1.2) \quad u(x, 0) = u_0(x), u_t(x, 0) = u_1(x).$$

Here and in the following $f(\dots, \mu_i(t) \partial_{x_i} u)$ means that f depends on $\mu_i(t) \partial_{x_i} u$, $i = 1, \dots, n$. The problem becomes weakly hyperbolic if $\lambda(t) = 0$ (time degeneracy). As a model case we suppose for the function $\lambda = \lambda(t)$

1991 *Mathematics Subject Classification.* 35L80, 35B65, 35B30.

Key words and phrases. Quasilinear weakly hyperbolic equations, time degeneracy, local existence, Levi conditions, Sobolev spaces, energy method.

Received: September 26, 1997.

- (A1) $\lambda(t) \in C^1[0, T]$, $\lambda'(t) \geq 0$ for $t > 0$;
 (A2) $\lambda(0) = 0$, $\lambda'(0) = 0$, $\lambda(t) > 0$ for $t > 0$;

where T is a positive constant. If we restrict ourselves to Gevrey classes of order ≤ 2 , then we can use ideas of [6] to prove a local existence result for

$$\begin{aligned} u_{tt} - \lambda^2(t) \Delta u &= f(t, x, u, u_t, \nabla_x u), \\ u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x). \end{aligned}$$

To overcome the critical order 2 we need so-called Levi conditions. These are relations between $\lambda = \lambda(t)$ and the derivative of f with respect to the argument $q = \nabla_x u$. The exact choice of the Levi condition has an important influence on the qualitative properties of the solutions of (1.1), (1.2). Let us illustrate it by the aid of two examples from the linear theory. In both examples we suppose that a is a real large constant.

Example 1. We consider the Cauchy problem $u_{tt} - t^{2l}u_{xx} + at^{l-1}u_x = 0$, $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$ for a natural number l . One can prove that there exists a uniquely determined solution $u(\cdot, t) \in H^{s_0 - \frac{|a|-l}{2(l+1)}}(R^1)$ for all $t > 0$ if $u_0 \in H^{s_0}(R^1)$ and $u_1 \in H^{s_0-1}(R^1)$. In [11] this loss of Sobolev derivatives was shown for $l = 1$ and $a = 4n + 1$ by an explicit representation of the solution.

Example 2. We consider the Cauchy problem $u_{tt} - \lambda^2(t)u_{xx} - a\frac{\lambda^2(t)}{\Lambda(t)}u_x = 0$, $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$, where $\Lambda(t) = \exp(-1/t)$, $\lambda(t) = \Lambda(t)^l$. One can prove that $u(\cdot, t) \in H^{s_0 - \frac{|a|-1}{2}}(R^1)$ for all $t > 0$ if $u_0 \in H^{s_0}(R^1)$ and $u_1 \in H^{s_0-1}(R^1)$.

Both examples can be studied by using the theory of special functions (for the first see [14], for the second one [1],[12],[16]). In both examples the coefficient $a(t)$ of u_x behaving as $O(\lambda'(t))$ ($\lambda(t) = t^l$ in the first example) determines the loss of Sobolev derivatives.

The Levi condition $a(t) = O(\lambda'(t))$ is sharp in the following sense:

If we weaken it to $a(t) = o(\lambda'(t)^s)$, $s \in (0, 1)$, then there doesn't exist a distributional solution (see [5]).

If we sharpen it to $a(t) = o(\lambda'(t))$, then the term of lower order has no influence on the loss of Sobolev regularity.

Various nonlinear generalizations of the above Levi condition are known, for example, $|\partial_{q_i} f(t, x, u, p, q_1, \dots, q_n)| \leq C\lambda'(t)$ in a suitable domain of definition. But if we want to study model equations of the form

$$(1.3) \quad u_{tt} - \lambda^2(t) \Delta u = f(u, u_t, \nabla_x u)$$

under this Levi condition, then one has to define and to work in spaces of solutions with special asymptotics in t . Otherwise the Levi condition is not satisfied. Another nonlinear generalization is to suppose that

f depends on $\mu_i(t)\partial_{x_i}u$, where

$$(A3) \quad \limsup_{t \rightarrow +0} \frac{|\mu_i(t)|}{\lambda'(t)} = Q, \quad \mu_i(t) \in C[0, T].$$

Allowing this kind of sharp Levi condition we have to take into consideration the following question.

Is it possible to connect the quasilinear structure of our problem (1.1), (1.2) with the loss of derivatives of the solution which appears even in the linear case?

If not, then it may turn out to be hard to prove local existence of Sobolev solutions for our starting problem. The goal of this paper is to show how to answer this question, to overcome these difficulties, respectively.

Let us mention some more references related to the object of this paper. In [9] one can find a global existence result of classical solutions for

$$\begin{aligned} u_{tt} - \lambda^2(t)u_{xx} - a(x, t)u_x - b(x, t)u_t - c(x, t)u &= f(x, t), \\ u(x, 0) = u_0(x), u_t(x, 0) &= u_1(x), \end{aligned}$$

under the Levi condition

$$\limsup_{t \rightarrow +0} \frac{|a(x, t)|}{\lambda'(t)} \leq Q < \infty.$$

This Levi condition is sharp. The loss of derivatives depends on Q . Weakly hyperbolic Cauchy problems of the form

$$\begin{aligned} u_{tt} - \sum_{i,j=1}^n (a_{ij}(x, t)u_{x_i})_{x_j} + \sum_{i=1}^n b_i(x, t)u_{x_i} + b_0(x, t)u_t + c(x, t)u &= f(x, t), \\ u(x, 0) = u_0(x), u_t(x, 0) &= u_1(x), \end{aligned}$$

were studied in [10] under the Levi condition

$$ct \left(\sum_{i=1}^n b_i(t)\xi_i \right)^2 \leq A \left(\sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j \right) + \partial_t \left(\sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j \right).$$

In the case of time degeneracy this Levi condition is only sharp if we have a degeneracy of finite order (compare with the Levi condition from [9]). Quasilinear weakly hyperbolic equations of higher order are studied in [7]. There are various papers ([2],[3],[8]) concerning local existence in C^∞ for special quasilinear weakly hyperbolic model equations. But the goal of these papers is another one. The authors are more interested in equations having a main part which differs from that one of (1.1). The authors allow spatial degeneracy, too, and allow even a dependence of the coefficients on the solution itself. But there is no nonlinear dependence on $\nabla_x u$ or the Levi condition is not sharp for time degeneracies of infinite order.

2. MAIN RESULTS AND SOME IMPORTANT TOOLS

To formulate the main results we suppose for the function $f = f(t, x, u, p, \vec{r})$, $p = u_t$, $\vec{r} = (r_1, \dots, r_n)$, $r_i = \mu_i(t)q_i$, $q_i = \partial_{x_i}u$, the condition

$$(A4) \quad f \in C([0, T], C^\infty(R_u^1 \times R_p^1 \times R_r^n) \times H^s(R^n))$$

for all $s \in N$.

Theorem 1. (Local existence result)

Let us consider the weakly hyperbolic Cauchy problem

$$(2.1) \quad \begin{aligned} u_{tt} - \lambda^2(t) \Delta u &= f(t, x, u, u_t, \mu_i(t)\partial_{x_i}u), \\ u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), \end{aligned}$$

where the data u_0, u_1 belong to $H^{s_0}(R^n), H^{s_0-1}(R^n)$, respectively. The natural number s_0 satisfies $s_0 \geq 2(Q + 4) + n/2 + 1 + r, r \in N$, where the nonnegative constant Q is chosen from the relation

$$(2.2) \quad \limsup_{t \rightarrow +0} |\partial_{q_k} f(t, x, u_0(x), u_1(x), \mu_i(t)\partial_{x_i}u_0(x))|/\lambda'(t) = Q$$

for all $(x, t) \in R^n \times [0, T]$ and $i, k = 1, \dots, n$. Under the assumptions (A1) to (A4) there exists a locally defined Sobolev solution

$$\begin{aligned} u \in & C([0, T^*], H^{4+[n/2]+r}(R^n)) \cap C^1([0, T^*], H^{3+[n/2]+r}(R^n)) \\ & \cap C^2([0, T^*], H^{2+[n/2]+r}(R^n)). \end{aligned}$$

Theorem 2. (Uniqueness result)

The solution of (2.1) is uniquely determined in

$$C([0, T], H^{s_0}(R^n)) \cap C^1([0, T], H^{s_0-1}(R^n)) \cap C^2([0, T], H^{s_0-2}(R^n))$$

under the assumptions of Theorem 1 and the condition

$$(2.3) \quad s_0 > \max(n/2 + 1, 5 + 2[(Q + 1)/2]).$$

Remark 1. There is a gap between the order of Sobolev classes for which one has local existence to those in which one has additionally uniqueness of solutions. This is a typical effect in the weakly hyperbolic theory if one uses the sharp Levi condition (A3). The question whether the solution of Theorem 1 is uniquely determined seems difficult.

If $r > 1 + 2[(Q + 1)/2] - [n/2]$, then Theorem 2 implies the uniqueness of the solution from Theorem 1.

An important tool for our considerations will be the following

Lemma 1. (Lemma of Nersesjan - generalization of the well-known Gronwall's lemma to differential inequalities with a singular coefficient [9])

Let us consider the differential inequality

$$(2.4) \quad y'(t) \leq K(t)y(t) + f(t)$$

for $t \in (0, T)$, where the functions $K = K(t)$ and $f = f(t)$ belong to $C(0, T), T > 0$. Under the assumptions

$$\int_0^\delta K(\tau) d\tau = \infty, \int_\delta^T K(\tau) d\tau < \infty,$$

$$\lim_{\delta \rightarrow +0} \int_\delta^t \exp\left(\int_s^t K(\tau) d\tau\right) f(s) ds \quad \text{exists},$$

$$\lim_{\delta \rightarrow +0} y(\delta) \exp\left(\int_\delta^t K(\tau) d\tau\right) = 0$$

for all $t \in (0, T)$ and $\delta \in (0, t)$ every solution belonging to $C[0, T] \cap C^1(0, T)$ satisfies

$$(2.5) \quad y(t) \leq \int_0^t \exp\left(\int_s^t K(\tau) d\tau\right) f(s) ds.$$

This lemma can be applied to the study of the linear weakly hyperbolic Cauchy problem

$$(2.6) \quad w_{tt} - \lambda^2(t) \Delta w = f(x, t), w(x, 0) = w_t(x, 0) = 0,$$

where $\lambda = \lambda(t)$ satisfies the assumptions (A1) and (A2). In [15] it was proved that for every function $f = f(x, t)$ satisfying $\lambda^{-d} f(x, t) \in C([0, T]; Y_{+0}^s)$ there exists a uniquely determined solution $w = w(x, t)$ satisfying $\lambda^{-d} w(x, t) \in C^2([0, T]; Y_{+0}^s)$, where d is a suitable positive constant. Here we use the notation

$$Y_{+0}^s = \{u \in C^\infty(R_x^n) : \|\partial_x^\alpha u\|_{L_2(R^n)} \rho^{|\alpha|s} \leq C|\alpha|!^s\}$$

with suitable constants ρ and C depending on u . Using Lemma 1 and the result from [15] one obtains the next result.

Corollary 1. *If $f \in C([0, T], H^{N-1}(R^n))$ satisfies $\|f(x, t)/(\lambda(t)^{d-1} \lambda'(t))\|_{H^{N-1}(R^n)} \leq C_N, N \geq 4, d > 1$, then there exists a uniquely determined Sobolev solution $w \in C([0, T], H^N(R^n)) \cap C^1([0, T], H^{N-1}(R^n)) \cap C^2([0, T], H^{N-2}(R^n))$ of the linear weakly hyperbolic Cauchy problem*

$$w_{tt} - \lambda^2(t) \Delta w = f(x, t), w(x, 0) = w_t(x, 0) = 0.$$

The solution fulfils $E_N(w)(t) \leq C_N \lambda(t)^d$, where we use C_N as an universal constant (for the definition of $E_N(w)(t)$ see the proof).

Proof. a) Let $\chi = \chi(x)$ be a Gevrey function from $Y_{+0}^s \cap C_0^\infty(R^n)$ with $\chi(x) = 0$ if $|x| \geq 1, \chi \geq 0$ and $\int_{R^n} \chi(x) dx = 1$. Then the functions

$\lambda'^{-1}\lambda^{-(d-1)}f_\varepsilon = \lambda'^{-1}\lambda^{-(d-1)}f * \chi_\varepsilon$ (Friedrichs mollifier of $\lambda'^{-1}\lambda^{-(d-1)}f$) belong to $C([0, T], Y_{+0}^s)$, too. This follows from

$$\begin{aligned} \lambda'^{-1}\lambda^{-(d-1)}\partial_x^\alpha f_\varepsilon &= \lambda'^{-1}\lambda^{-(d-1)}f * \partial_x^\alpha \chi_\varepsilon, \\ \|\lambda'^{-1}\lambda^{-(d-1)}\partial_x^\alpha f_\varepsilon\|_{L_2(R^n)} &\leq \|\lambda'^{-1}\lambda^{-(d-1)}f\|_{L_2(R^n)} \|\partial_x^\alpha \chi_\varepsilon\|_{L_1(R^n)} \end{aligned}$$

and the fact that χ_ε has compact support for each fixed $\varepsilon > 0$. Moreover, the regularizations fulfil $\lim_{\varepsilon \rightarrow +0} \|\lambda'^{-1}\lambda^{-(d-1)}f_\varepsilon - \lambda'^{-1}\lambda^{-(d-1)}f\|_{H^{N-1}(R^n)} = 0$.

Let us now consider the auxiliary problems

$$w_{tt} - (\lambda(t) + \varepsilon_k)^2 \Delta w = f_{\varepsilon_k}(x, t), \quad w(x, 0) = w_t(x, 0) = 0.$$

We can choose the sequence $\{\varepsilon_k\}, \varepsilon_k \rightarrow +0$, in such a way that

$$\|\lambda'^{-1}\lambda^{-(d-1)}f_{\varepsilon_k} - \lambda'^{-1}\lambda^{-(d-1)}f_{\varepsilon_{k+1}}\|_{H^{N-1}(R^n)} \leq \frac{1}{2^k} \quad \text{for all } k \in N.$$

We define energies which take into consideration the degeneracy at $t = 0$ of our starting problem, namely, the partial energies ($j \geq 1, t \in [0, T]$)

$$(2.7) \quad e_{j,\varepsilon}^2(w)(t) = \sum_{|\alpha|=j-1} \int_{R^n} ((\lambda(t) + \varepsilon)^2 |\partial_x^\alpha \nabla_x w|^2 + |\partial_x^\alpha w_t|^2 + |\partial_x^\alpha w|^2) dx$$

and the energies of finite order ($N \geq 1, t \in [0, T]$)

$$(2.8) \quad E_{N,\varepsilon}(w)(t) = \sum_{j=1}^N e_{j,\varepsilon}(w)(t).$$

Due to [6] we have for f_{ε_k} a uniquely determined solution $w_{\varepsilon_k} \in C^2([0, T]; Y_{+0}^s)$. Now let us introduce $g_k = f_{\varepsilon_{k+1}} - f_{\varepsilon_k}$ and $v_k = w_{\varepsilon_{k+1}} - w_{\varepsilon_k}$. After differentiation of (2.7), (2.8) we get

$$\begin{aligned} e'_{j,\varepsilon_{k+1}}(v_k)(t) &\leq \frac{\lambda'(t)}{(\lambda(t) + \varepsilon_{k+1})} e_{j,\varepsilon_{k+1}}(v_k)(t) + e_{j,\varepsilon_{k+1}}(v_k)(t) \\ &\quad + \sum_{|\alpha|=j-1} \|\partial_x^\alpha g_k\|_{L_2(R^n)}, \\ E'_{N,\varepsilon_{k+1}}(v_k)(t) &\leq \frac{\lambda'(t)}{(\lambda(t) + \varepsilon_{k+1})} E_{N,\varepsilon_{k+1}}(v_k)(t) + E_{N,\varepsilon_{k+1}}(v_k)(t) \\ &\quad + \|g_k\|_{H^{N-1}(R^n)}, \end{aligned}$$

respectively. After application of Gronwall's inequality we have

$$\begin{aligned} E_{N,\varepsilon_{k+1}}(v_k)(t) &\leq (\lambda(t) + \varepsilon_{k+1})^{1+\eta} \int_0^t (\lambda(s) + \varepsilon_{k+1})^{-1-\eta} \|g_k\|_{H^{N-1}}(s) ds \\ &\leq (\lambda(t) + \varepsilon_{k+1})^{1+\eta} \frac{1}{2^k(d-1-\eta)} \int_0^t ds (\lambda(s) + \varepsilon_{k+1})^{d-1-\eta} \\ &\leq \frac{(\lambda(t) + \varepsilon_{k+1})^d}{2^k(d-1-\eta)} \end{aligned}$$

for each $d > 1$. Using (2.7), (2.8) and $w_{\varepsilon_k} \in C^2([0, T]; Y_{+0}^s)$, then $\{w_{\varepsilon_k}\} = \{w_{\varepsilon_0} + \sum_{i=0}^{n-1} v_i\}$ is a Cauchy sequence in $C^1([0, T], H^{N-1}(R^n))$. The limit function $w \in C^1([0, T], H^{N-1}(R^n))$ satisfies $E_N(w)(t) \leq C\lambda(t)^d$, $d > 1$. Here one has to use that $E_{N, \varepsilon_k}(w_{\varepsilon_k})(t) \leq C(\lambda(t) + \varepsilon)^d$ for all $t \in [0, T]$, where C is independent of N . Hence, $w \in C([0, T], H^N(R^n))$, too. Thus the existence part is proved.

b) To prove uniqueness we define the energies ($j \geq 1, N \geq 1, t \in [0, T]$)

$$(2.9) \quad f_j^2(w)(t) = \sum_{|\alpha|=j-1} \int_{R^n} (|\partial_x^\alpha w_t|^2 + |\partial_x^\alpha w|^2) dx, \quad F_N(w)(t) = \sum_{j=1}^N f_j(w)(t).$$

With these energies we obtain

$$F'_N(w)(t) \leq F_N(w)(t) + \lambda^2(t) F_{N+2}(w)(t), \quad F_N(w)(0) = 0.$$

The application of Gronwall's inequality leads to

$$F_N(w)(t) \leq C_N \lambda^2(t) \max_{t \in [0, T]} F_{N+2}(w)(t).$$

If we use the energies E_N and the relation $E_N(w)(t) \leq C_N F_{N+1}(w)(t)$, then $E_N(w)(t) \leq C_{N+1} \lambda^2(t) \max_{t \in [0, T]} F_{N+3}(w)(t)$. As in the part a) we derive

$$E'_N(w)(t) \leq \frac{\lambda'(t)}{\lambda(t)} E_N(w)(t) + E_N(w)(t).$$

The assumption $N \geq 4$ implies the existence of $F_4(w)(t)$. Using the last both inequalities we conclude with Lemma 1 (we have to use it with the notations $K(t) = (1 + \eta)d_t \ln \lambda(t)$, $\eta > 1$, $y(t) = E_N(w)(t)$) the relation $E_1(w)(t) \equiv 0$, that is, $w \equiv 0$ in $R^n \times [0, T]$. The corollary is completely proved. ■

3. PROOF OF THE THEOREMS

3.1. Proof of Theorem 1. a) Reduction process.

In this step we reduce (1.1) to an equivalent system of nonlinear ordinary differential equations and a quasilinear weakly hyperbolic equation. A special asymptotical behaviour of the nonlinear right-hand side of the weakly hyperbolic equation allows us to apply Corollary 1.

Let us define the iterates $u^{(i)}$, $i = 0, \dots, p$, in the following way (see [13] in a special case):

$$(3.1) \quad \begin{aligned} u_{tt}^{(0)} &= f(t, x, u^{(0)}, u_t^{(0)}, 0, \dots, 0), \\ u^{(0)}(x, 0) &= u_0(x), \quad u_t^{(0)}(x, 0) = u_1(x), \end{aligned}$$

(3.2)

$$\begin{aligned}
 u_{tt}^{(i)} &= f(t, x, \sum_{k=0}^i u^{(k)}, \sum_{k=0}^i u_t^{(k)}, \mu_1(t) \sum_{k=0}^{i-1} \partial_{x_1} u^{(k)}, \dots, \mu_n(t) \sum_{k=0}^{i-1} \partial_{x_n} u^{(k)}) \\
 &\quad - f(t, x, \sum_{k=0}^{i-1} u^{(k)}, \sum_{k=0}^{i-1} u_t^{(k)}, \mu_1(t) \sum_{k=0}^{i-2} \partial_{x_1} u^{(k)}, \dots, \mu_n(t) \sum_{k=0}^{i-2} \partial_{x_n} u^{(k)}) \\
 &\quad + \lambda^2(t) \Delta u^{(i-1)},
 \end{aligned}$$

(3.3) $u^{(i)}(x, 0) = u_t^{(i)}(x, 0) = 0$

for $i = 1, \dots, p$, $\sum_{k=0}^{-1} \dots \equiv 0$. This system of nonlinear ordinary differential equations has uniquely determined solutions $u^{(i)}(x, \cdot) \in C^2[0, T_0]$ if we choose T_0 sufficiently small. Using additionally the regularity of the data u_0 and u_1 we obtain

$$f'_j(u^{(0)})(t) \leq f_j(u^{(0)})(t) + \sum_{|\alpha|=j-1} \|\partial_x^\alpha f(t, x, u^{(0)}, u_t^{(0)}, 0, \dots, 0)\|_{L_2(R^n)}.$$

Supposing for $u^{(0)}$ the conditions

(3.4) $|u^{(0)} - u_0| \leq \varepsilon_0, |u_t^{(0)} - u_1| \leq \varepsilon_0, |\partial_{x_i}(u^{(0)} - u_0)| \leq \varepsilon_0$

gives after a standard procedure (Leibniz formula, Gagliardo-Nirenberg inequality etc.)

(3.5)
$$\begin{aligned}
 f'_j(u^{(0)})(t) &\leq f_j(u^{(0)})(t) + C_j(1 + F_j(u^{(0)})(t)), \\
 F'_N(u^{(0)})(t) &\leq F_N(u^{(0)})(t) + C_N(1 + F_N(u^{(0)})(t)),
 \end{aligned}$$

respectively. By Gronwall's inequality $F_N(u^{(0)})(t)$ exists for $t \in [0, T_0]$. If we choose data $u_0 \in H^{s_0}(R^n)$, $u_1 \in H^{s_0-1}(R^n)$ with $n/2 + 1 < N \leq s_0$, then (3.4) is satisfied, probably with a smaller T_0 . There are no difficulties to estimate the other iterates. One has only to study

$$u_{tt}^{(i)} = g_i(t, x, u^{(i)}, u_t^{(i)}) + \lambda^2(t) \Delta u^{(i-1)}, u^{(i)}(x, 0) = u_t^{(i)}(x, 0) = 0,$$

where

$$\begin{aligned}
 &g_i(t, x, u^{(i)}, u_t^{(i)}) \\
 &= f(t, x, \sum_{k=0}^i u^{(k)}, \sum_{k=0}^i u_t^{(k)}, \mu_1(t) \sum_{k=0}^{i-1} \partial_{x_1} u^{(k)}, \dots, \mu_n(t) \sum_{k=0}^{i-1} \partial_{x_n} u^{(k)}) \\
 &\quad - f(t, x, \sum_{k=0}^{i-1} u^{(k)}, \sum_{k=0}^{i-1} u_t^{(k)}, \mu_1(t) \sum_{k=0}^{i-2} \partial_{x_1} u^{(k)}, \dots, \mu_n(t) \sum_{k=0}^{i-2} \partial_{x_n} u^{(k)})
 \end{aligned}$$

and repeat the reasoning. The term $\lambda^2(t) \Delta u^{(i-1)}$ worses the Sobolev regularity of $u^{(i)}$ with respect to x compared with that of $u^{(i-1)}$ by 2. Thus we have proved the

Lemma 2. *There exists a positive constant T_p such that the system of non-linear ordinary differential equations (3.1) to (3.3) has uniquely determined solutions*

$u^{(i)} \in C^2([0, T_p], H^{s_0-1-2i}(R^n))$, $i = 1, \dots, p$, where $s_0 - 1 - 2p > n/2 + 1$. Moreover, to given positive constants $\varepsilon_i, i = 0, \dots, p$, there exists an interval $[0, T_p]$ in which the estimates

$$(3.6) \quad \begin{aligned} \max_{j=1, \dots, n} (|u^{(0)} - u_0|, |u_t^{(0)} - u_1|, |\mu_j(t)(u_{x_j}^{(0)} - u_{0,x_j})|) &\leq \varepsilon_0, \\ \max_{j=1, \dots, n} (|u^{(i)}|, |u_t^{(i)}|, |\mu_j(t)u_{x_j}^{(i)}|) &\leq \varepsilon_i \end{aligned}$$

hold for all $(x, t) \in R^n \times [0, T_p], i = 1, \dots, p$.

One expects the statement of this lemma. The sense of our procedure will be expressed by the next

Lemma 3. *Choosing data $u_0 \in H^{s_0}(R^n)$, $u_1 \in H^{s_0-1}(R^n)$ with $n/2 + 1 < N \leq s_0 - 2p$ it holds*

$$(3.7) \quad F_N(u^{(i)})(t) \leq C_{N,i} \lambda^i(t)$$

for all $t \in [0, T_p]$ and $i = 0, \dots, p$.

Proof. Instead of (3.7) we show the stronger statement

$$F_M(u^{(i)})(t) \leq C_{M,i} \lambda^i(t) \quad \text{for } i = 0, \dots, p \quad \text{and } M \leq s_0 - 2i.$$

Recalling (3.5) we find these estimates for $i = 0$ and $M \leq s_0$. Using (3.2) we obtain

$$\begin{aligned} u_{tt}^{(i+1)} &= b_{i+1}(x, t)u_t^{(i+1)} + C_{i+1}(x, t)u^{(i+1)} \\ &\quad + \sum_{k=1}^n \mu_k(t)a_{k,i+1}(x, t)u_{x_k}^{(i)} + \lambda^2(t) \Delta u^{(i)}. \end{aligned}$$

Homogeneous data, (A1) and (A3) imply similar to (3.5)

$$\begin{aligned} F_M(u^{(i+1)})(t) &\leq C_M \int_0^t (\lambda'(\tau)C_{M+1,i} \lambda^i(\tau) + \lambda^2(\tau)C_{M+2,i} \lambda^i(\tau))d\tau \\ &\leq C_{M,i+1} \lambda^{i+1}(t) \end{aligned}$$

for all $t \in [0, T_p]$ and $M \leq s_0 - 2(i + 1)$.

For the solution of (1.1) we choose the ansatz $u = v + \sum_{i=0}^p u^{(i)}$, $p > Q + 1$ (Q from Theorem 1), where the functions $u^{(i)} \in C^2([0, T_p], H^{s_0-1-2i}(R^n))$ are the solutions of (3.1) to (3.3). Taking account of $p > Q + 1$ and $s_0 \geq 2(Q + 4) + n/2 + 1 + r$ we conclude from Theorem 1 that $u^{(i)} \in C^2([0, T_p], H^{s+[n/2]+r}(R^n))$ for $i = 0, 1, \dots, p$. ■

It remains to consider the weakly hyperbolic Cauchy problem

$$(3.8) \quad v_{tt} - \lambda^2(t) \Delta v = F(t, x, v, v_t, \mu_i(t)\partial_{x_i} v), v(x, 0) = v_t(x, 0) = 0,$$

where

$$\begin{aligned} & F(t, x, v, v_t, \mu_i(t) \partial_{x_i} v) \\ &= f(t, x, v + \sum_{k=0}^p u^{(k)}, \partial_t(v + \sum_{k=0}^p u^{(k)}), \mu_i(t) \partial_{x_i}(v + \sum_{k=0}^p u^{(k)})) \\ & \quad - f(t, x, \sum_{k=0}^p u^{(k)}, \sum_{k=0}^{p-1} \partial_t u^{(k)}, \mu_i(t) \sum_{k=0}^{p-1} \partial_{x_i} u^{(k)}) + \lambda^2(t) \Delta u^{(p)}. \end{aligned}$$

Our starting Cauchy problem (1.1),(1.2) is obviously equivalent to (3.1), (3.3), (3.8).

Remark 2. Using (3.7) the right-hand side $F(t, x, v, v_t, \mu_i(t) \partial_{x_i} v)$ has in $(t, x, 0, 0, \dots, 0)$ a special asymptotical behaviour, namely, $F(t, x, 0, 0, 0) \sim \lambda^p(t) \lambda'(t)$. In this sense it is reasonable to speak of an improvement of the asymptotical behaviour from iterate to iterate.

b) To proceed further, let us devote to (3.8) and define the successive approximation scheme

$$\begin{aligned} v_{tt}^{(q+1)} - \lambda^2(t) \Delta v^{(q+1)} &= F(t, x, v^{(q)}, v_t^{(q)}, \mu_i(t) \partial_{x_i} v^{(q)}), \\ v^{(q+1)}(x, 0) &= v_t^{(q+1)}(x, 0) = 0, \\ v^{(0)} &\equiv 0. \end{aligned}$$

Then the differences $w^{(q)} = v^{(q+1)} - v^{(q)}$ satisfy

$$\begin{aligned} w_{tt}^{(0)} - \lambda^2(t) \Delta w^{(0)} &= F(t, x, 0, 0, 0), \\ w_{tt}^{(q)} - \lambda^2(t) \Delta w^{(q)} &= F(t, x, v^{(q)}, v_t^{(q)}, \mu_i(t) \partial_{x_i} v^{(q)}) \\ & \quad - F(t, x, v^{(q-1)}, v_t^{(q-1)}, \mu_i(t) \partial_{x_i} v^{(q-1)}) \end{aligned}$$

with homogeneous initial conditions. Now we observe that, by Remark 2 and Corollary 1, we have $w^{(0)} \in C([0, T_p], H^N(R^n))$ and the existence of $E_N(w^{(0)})(t)$ for all $t \in [0, T_p]$,

$N = 4 + [n/2] + r$. Hence the right-hand side in the weakly hyperbolic equation for $w^{(1)}$ belongs to $C([0, T_p], H^{N-1}(R^n))$. Using Hadamard's formula, (A3) and the definition of E_N its asymptotical behaviour is $O(\lambda(t)^{p-1} \lambda'(t))$ for $t \rightarrow +0$. Consequently, Corollary 1 is applicable and yields $w^{(1)} \in C([0, T_p], H^N(R^n))$, $E_N(w^{(1)})(t) \leq C_{N,1} \lambda(t)^p$. With this procedure at hand, it is not difficult to conclude $w^{(q)} \in C([0, T_p], H^N(R^n))$ and $E_N(w^{(q)})(t) \leq C_{N,q} \lambda(t)^p$ for all $q \geq 0$. Thus, all iterates are well defined.

c) The ideas of the second step give us the possibility to estimate $E_N(w^{(q)})(t)$. Let us sketch it for $E_N(w^{(0)})(t)$. We obtain for a small $\varepsilon > 0$

$$\begin{aligned} E'_N(w^{(0)})(t) &\leq (1 + \frac{\varepsilon}{2}) \frac{\lambda'(t)}{\lambda(t)} E_N(w^{(0)})(t) \\ & \quad + \sum_{j=1}^N \sum_{|\alpha|=j-1} \|\partial_x^\alpha F(t, x, 0, 0, 0)\|_{L_2(R^n)}. \end{aligned}$$

To estimate $\partial_x^\alpha F(t, x, 0, 0, 0)$, $|\alpha| \leq N - 1$, the essential terms are the following:

$$\begin{aligned} &\partial_x^\alpha \left(\sum_{i=1}^n \int_0^1 \partial_{r_i} f(t, x, \sum_{k=0}^p u^{(k)}, \sum_{k=0}^p u_t^{(k)}, \mu_1(t) \partial_{x_1} \sum_{k=0}^{p-1} u^{(k)}, \right. \\ &\quad \left. \dots, \mu_i(t) \partial_{x_i} \left(\sum_{k=0}^{p-1} u^{(k)} + \tau u^{(p)} \right), \dots, \mu_n(t) \partial_{x_n} \sum_{k=0}^p u^{(k)} \right) d\tau \mu_i(t) \partial_{x_i} u^{(p)}. \end{aligned}$$

Let us denote the integrand by $\sum_{i=1}^n g_i(t, x, \dots)$. Applying Leibniz formula there are terms of the form

$$\sum_{i=1}^n \int_0^1 g_i(t, x, \dots) d\tau \mu_i(t) \partial_x^\alpha u_{x_i}^{(p)} \quad \text{with } |\alpha| = N - 1.$$

Using (2.2) and (3.4) one can estimate to a given $\varepsilon > 0$ the $L_2(R^n)$ -norm of the sum of all these terms by $(Q + \frac{\varepsilon}{2}) \frac{\lambda'(t)}{\lambda(t)} E_N(u^{(p)})(t)$ for all $t \in [0, T_\varepsilon]$, where $T_\varepsilon \leq T_p$.

Moreover, we have terms of the form

$$\begin{aligned} &\sum_{i=1}^n \int_0^1 \sum_{\beta+\rho \leq \alpha} \sum_{l_1+l_2=|\alpha|-|\beta|-|\rho|} \sum_{\nu_1=0}^{l_1} \sum_{\nu_2=0}^{l_2} g_i^{(\beta, \nu_1, \nu_2)}(t, x, \dots) \\ &\times \left(\sum_{|\gamma_1|+\dots+|\gamma_{\nu_1}|=l_1, |\gamma_i| \geq 1} \partial_x^{\gamma_1} \left(\sum_{k=0}^p u^{(k)} \right) \dots \partial_x^{\gamma_{\nu_1}} \left(\sum_{k=0}^p u^{(k)} \right) \right) \\ &\times \left(\sum_{|\delta_1|+\dots+|\delta_{\nu_2}|=l_2, |\delta_i| \geq 1} \partial_x^{\delta_1} \left(\sum_{k=0}^p u_t^{(k)} \right) \dots \partial_x^{\delta_{\nu_2}} \left(\sum_{k=0}^p u_t^{(k)} \right) \right) d\tau \mu_i(t) \partial_x^\rho u_{x_i}^{(p)}, \end{aligned}$$

with $|\rho| < |\alpha|$. Using (A3),(3.4) and the Gagliardo-Nirenberg inequality the $L_2(R^n)$ -norm of this sum can be estimated by $C_N \lambda'(t) E_N(u^{(p)})(t) E_N(\sum_{k=0}^p u^{(k)})(t)$. Here (3.4) is used to estimate the $L_\infty(R^n)$ -norm of $\sum_{k=0}^p u^{(k)}$ and $\sum_{k=0}^p u_t^{(k)}$. Finally, there appear terms in which g_i will be differentiated to r_i , too. Similar to the above considerations the sum of these terms can be estimated by

$$C_N \frac{\lambda'^2(t)}{\lambda(t)} E_N(u^{(p)})(t) E_N(\sum_{k=0}^p u^{(k)})(t).$$

Consequently,

$$\begin{aligned} E'_N(w^{(0)})(t) &\leq (1 + \frac{\varepsilon}{2}) \frac{\lambda'(t)}{\lambda(t)} E_N(w^{(0)})(t) + (Q + \frac{\varepsilon}{2}) \frac{\lambda'(t)}{\lambda(t)} E_N(u^{(p)})(t) \\ &+ C_N \frac{\lambda'^2(t)}{\lambda(t)} E_N(u^{(p)})(t) E_N(\sum_{k=0}^p u^{(k)})(t) \\ &+ C_N \lambda'(t) E_N(u^{(p)})(t) E_N(\sum_{k=0}^p u^{(k)})(t) + \lambda^2(t) E_{N+2}(u^{(p)})(t). \end{aligned}$$

Using (3.7) and Lemma 1 we have $E_N(w^{(0)})(t) \leq C_{N,0}\lambda(t)^p$ with a suitable constant $C_{N,0}$.

Repeating the above considerations leads to

$$\begin{aligned}
 E'_N(w^{(q)})(t) &\leq (1 + \frac{\varepsilon}{2}) \frac{\lambda'(t)}{\lambda(t)} E_N(w^{(q)})(t) \\
 &\quad + (Q + \frac{\varepsilon}{2}) \frac{\lambda'(t)}{\lambda(t)} E_N(w^{(q-1)})(t) \\
 (3.9) \quad &\quad + \varphi_N \frac{\lambda'^2(t)}{\lambda(t)} E_N(w^{(q-1)})(t) (E_N(v^{(q-1)} + \sum_{k=0}^p u^{(k)}) \\
 &\quad + E_N(v^{(q)} + \sum_{k=0}^p u^{(k)}))(t) ,
 \end{aligned}$$

where the function φ_N increasing in N depends on the $L_\infty(\mathbb{R}^n)$ -norms of

$$v^{(i)} + \sum_{k=0}^p u^{(k)}, v_t^{(i)} + \sum_{k=0}^p u_t^{(k)}, \mu_k(t) \partial_{x_k} (v^{(i)} + \sum_{k=0}^p u^{(k)}), \quad i = q - 1, q .$$

d) Finally, we want to show the existence of constants T^* , D_N and $C_{N,q}$ such that

$$E_N(w^{(q)})(t) \leq C_{N,q}\lambda(t)^p, \quad E_N(v^{(q)} + \sum_{k=0}^p u^{(k)})(t) \leq D_N$$

for all $q \geq 0$, $t \in [0, T^*]$, where the sequence $\{v^{(q)}\}$ converges to a solution valued in Sobolev spaces of (3.8). The application of Lemma 1 to (3.9) gives

$$\begin{aligned}
 E_N(w^{(q)})(t) &\leq \int_0^t \exp(\int_s^t (1 + \frac{\varepsilon}{2}) \frac{\lambda'(\tau)}{\lambda(\tau)} d\tau) ((Q + \frac{\varepsilon}{2}) \frac{\lambda'(s)}{\lambda(s)} E_N(w^{(q-1)})(s) \\
 &\quad + \varphi_N(D_N) \frac{\lambda'^2(s)}{\lambda(s)} E_N(w^{(q-1)})(s) (E_N(v^{(q-1)} + \sum_{k=0}^p u^{(k)})(s) \\
 &\quad + E_N(v^{(q)} + \sum_{k=0}^p u^{(k)})(s))) ds ,
 \end{aligned}$$

respectively, using the induction assumption (the inequalities are fulfilled for $q = 0$)

$$\begin{aligned}
 E_N(w^{(q)})(t) &\leq \lambda(t)^{1+\frac{\varepsilon}{2}} \int_0^t \lambda(s)^{-(1+\frac{\varepsilon}{2})} ((Q + \frac{\varepsilon}{2}) \frac{\lambda'(s)}{\lambda(s)} C_{N,q-1} \lambda(s)^p \\
 &\quad + 2\varphi_N(D_N) \frac{\lambda'^2(s)}{\lambda(s)} C_{N,q-1} \lambda(s)^p D_N) ds \\
 &\leq C_{N,q-1} \frac{Q + \frac{\varepsilon}{2}}{p - 1 - \frac{\varepsilon}{2}} \lambda(t)^p (1 + 2\varphi_N(D_N) D_N \sup_{[0,T]} \lambda'(t)) .
 \end{aligned}$$

Here it is sufficient to estimate the $L_\infty(R^n)$ -norm by D_N . Let us choose $D_N = 1 + \max_{[0, T_\varepsilon]} E_N(\sum_{k=0}^p u^{(k)}(t))$. Taking into consideration (A1) and $p > Q + 1$ we have a positive constant T^* such that

$$\frac{Q + \frac{\varepsilon}{2}}{p - 1 - \frac{\varepsilon}{2}}(1 + \lambda'(T^*)2\varphi_N(D_N)D_N) = b < 1.$$

Consequently, $E_N(w^{(q)}) \leq b^q C_{N,0} \lambda(t)^p$ for all $q \geq 0$ and $t \in [0, T^*]$. An eventually smaller T^* implies that $C_{N,0} \lambda(T^*)^p / (1 - b) \leq 1$, $E_N(v^{(q)})(t) \leq 1$ respectively.

As in part a) of the proof of Corollary 1 we have that $\{v^{(q)}\}$ is a Cauchy sequence in $C([0, T^*], H^N(R^n)) \cap C^1([0, T^*], H^{N-1}(R^n))$, where $N = 4 + [n/2] + r$. The limit function v belonging to $C^2([0, T^*], H^{N-2}(R^n))$, too, solves (3.8). This completes the proof. ■

3.2. Proof of Theorem 2. We only want to sketch the proof because the main ideas one can find in the proofs of Corollary 1 and Theorem 1.

a) The data u_0, u_1 belong to $H^{s_0}(R^n), H^{s_0-1}(R^n)$, respectively. If $s_0 > n/2 + 1$ we obtain $|u_0(x)| \leq C_u, |u_1(x)| \leq C_p$ and $|\partial_{x_i} u_0(x)| \leq C_{q_i}$ for all $x \in R^n, i = 1, \dots, n$, with suitable nonnegative constants C_u, C_p, C_{q_i} . Let v_1, v_2 be two different Sobolev solutions, then $w = v_1 - v_2$ solves

$$(3.10) \quad \begin{aligned} w_{tt} - \lambda^2(t) \Delta w &= \sum_{i=1}^n a_i(t, x, v_1, v_2) \mu_i(t) w_{x_i} + b(t, x, v_1, v_2) w_t \\ &+ c(t, x, v_1, v_2) w, \\ w(x, 0) &= w_t(x, 0) = 0. \end{aligned}$$

To every positive constant ε we can find a constant T_ε such that the arguments of the coefficients fulfil

$$|v_k(x, t)| \leq C_u + \varepsilon, |\partial_t v_k(x, t)| \leq C_p + \varepsilon, |\mu_i(t) \partial_{x_i} v_k(x, t)| \leq C_{q_i} + \varepsilon$$

for $k = 1, 2, i = 1, \dots, n$ and all $(x, t) \in R^n \times [0, T_\varepsilon]$. Using (A4) and (2.2) leads to

$$(3.11) \quad \lim_{t \rightarrow +0} |a_i(t, x, v_1, v_2)| \leq Q$$

for all $x \in R^n$.

b) The function

$$w \in C([0, T], H^{s_0}(R^n)) \cap C^1([0, T], H^{s_0-1}(R^n)) \cap C^2([0, T], H^{s_0-2}(R^n))$$

solves

$$\begin{aligned} w_{tt} - b(t, x, v_1, v_2) w_t - c(t, x, v_1, v_2) w &= \sum_{i=1}^n a_i(t, x, v_1, v_2) \mu_i(t) w_{x_i} \\ + \lambda^2(t) \Delta w, \quad w(x, 0) &= w_t(x, 0) = 0. \end{aligned}$$

By (A4) the coefficients belong to $C([0, T], H^{s_0-1}(R^n))$. Similar to part b) of the proof of Corollary 1 it follows

$$\begin{aligned}
 (3.12) \quad F'_N(w)(t) &\leq C_N F_N(w)(t) + \lambda'(t) C_{N+1} F_{N+1}(w)(t) \\
 &\quad + \lambda^2(t) F_{N+2}(w)(t), \\
 F_N(w)(0) &= 0
 \end{aligned}$$

for all $t \in [0, T_\varepsilon]$ and $1 \leq N \leq s_0 - 2$. Applying Gronwall's inequality and repeating the same reasoning $p - 1$ times gives

$$(3.13) \quad F_N(w)(t) \leq C_{N,p} \lambda(t)^p \max_{[0,T]} F_{N+2(1+[p/2])}(w)(t)$$

for all $t \in [0, T_\varepsilon]$ and $1 \leq N \leq s_0 - 2(1 + [p/2])$.

c) Similar to part a) of the proof of Corollary 1 we can show

$$(3.14) \quad E'_N(w)(t) \leq (Q + 1 + \varepsilon) \frac{\lambda'(t)}{\lambda(t)} E_N(w)(t) + C_N E_N(w)(t)$$

for all $t \in [0, T_\varepsilon]$. Using $E_N(w)(t) \leq C_N F_{N+1}(w)(t)$ it follows from (3.13)

$$(3.15) \quad E_N(w)(t) \leq C_{N+1,p} A_{N+1,p} \lambda(t)^p,$$

where $A_{N,p} = \max_{[0,T_\varepsilon]} F_{N+2(1+[p/2])}(w)(t)$.

Finally let us choose $p > Q + 1 + \varepsilon$. Then Lemma 1 can be applied to (3.14). Due to (3.15) the essential assumption

$$\begin{aligned}
 &\lim_{\delta \rightarrow +0} E_N(w)(t) \exp\left(\int_{\delta}^{\tau} (Q + 1 + \varepsilon) \frac{\lambda'(s)}{\lambda(s)} ds\right) = \\
 &\lim_{\delta \rightarrow +0} E_N(w)(\delta) \left(\frac{\lambda(\tau)}{\lambda(\delta)}\right)^{Q+1+\varepsilon} = 0
 \end{aligned}$$

is satisfied for all $\tau > 0$.

Consequently, $E_1(w)(t) = 0$, $w \equiv 0$ in $R^n \times [0, T_\varepsilon]$. Therefore we need the existence of $F_{2(2+[p/2])}(w)(t)$ with $p > Q + 1$. But this follows from the fact that $F_N(w)(t)$ is defined for $N > 5 + 2[(Q + 1)/2]$. Consequently, the special choice of $s_0 > \max(n/2 + 1, 5 + 2[(Q + 1)/2])$ implies the uniqueness of Sobolev solutions of our starting problem (2.1), (2.2) in $[0, T_\varepsilon]$ with respect to t . From the strictly hyperbolic theory we obtain even the uniqueness in $[T_\varepsilon, T]$.

4. FURTHER RESULTS

Corollary 2. *Additionally to the assumptions (A1) to (A4) we suppose that $\lambda(t) \in C^2[0, T]$. If the data u_0, u_1 belong to $\bigcap_{k \in N} H^k(R^n)$, then there exists a locally defined solution $u \in C^2([0, T^*], \bigcap_{k \in N} H^k(R^n))$ of the weakly hyperbolic Cauchy problem (1.1), (1.2).*

Proof. Using (3.5) we obtain $F_N(u^{(0)})(t) \leq A_{N,0} + B_{N,0} \int_0^t F_N(u^{(0)})(\tau) d\tau$ for all $t \in [0, T_0]$, where a sufficiently small T_0 and a suitable N_0 imply the conditions (3.4). Let us fix the set

$$K_0 := \{v \in C^1([0, T_0], \bigcap_{k \in N} H^k(R^n)) : F_N(v)(t) \leq A_{N,0} e^{B_{N,0}t} \quad \forall N\}$$

and consider $u_{tt} = f(t, x, v, v_t, 0, \dots, 0)$, $u(x, 0) = u_0(x)$, $u_t(x, 0) = u_1(x)$, with an arbitrary function v from K_0 . Then we have

$$\begin{aligned} F_N(u)(t) &\leq A_{N,0} + B_{N,0} \int_0^t F_N(v)(\tau) d\tau \\ &\leq A_{N,0} + B_{N,0} \int_0^t A_{N,0} e^{B_{N,0}\tau} d\tau \leq A_{N,0} e^{B_{N,0}t}. \end{aligned}$$

Consequently, $u \in K_0$, too. Using the property of $\bigcap_{k \in N} H^k(R^n)$ to be a Montel space and the uniqueness property of solutions for the Cauchy problem (3.1) we conclude that there exists a uniquely determined solution $u^{(0)} \in C^2([0, T_0], \bigcap_{k \in N} H^k(R^n))$ satisfying the energy estimates $F_N(u^{(0)})(t) \leq A_{N,0} e^{B_{N,0}t}$ for all N .

It is clear that an analogous argument gives locally defined solutions $u^{(i)} \in C^2([0, T_i], \bigcap_{k \in N} H^k(R^n))$ of (3.2), (3.3). From (3.7) we know additionally that

$F_N(u^{(i)})(t) \leq C_{N,i} \lambda^i(t)$ for all $t \in [0, T_p]$ and $i = 0, \dots, p$. Now let us devote to the weakly hyperbolic Cauchy problem (3.8).

Applying Hadamard’s formula and the above cited properties of $u^{(i)}$, especially those for $u^{(p)}$, gives by the same reasoning as in part c) of the proof for Theorem 1

$$\begin{aligned} E'_N(v)(t) &\leq \left(1 + \frac{\varepsilon}{2}\right) \frac{\lambda'(t)}{\lambda(t)} E_N(v)(t) + \left(Q + \frac{\varepsilon}{2}\right) \frac{\lambda'(t)}{\lambda(t)} E_N(v)(t) + \\ &A_N \frac{\lambda'^2(t)}{\lambda(t)} \varphi_N(\|v\|_{L_\infty(R^n)}, \|v_t\|_{L_\infty(R^n)}, \|\nabla_x v\|_{L_\infty(R^n)}) E_N(v)(t) + B_N \lambda(t)^p, \end{aligned}$$

where A_N, B_N and φ_N depend on the energies of $u^{(i)}$, $i = 0, \dots, p$, and φ_N is a monotonously increasing function in its arguments. Now let us define the set

$$K := \{v \in C^1([0, T^*], \bigcap_{k \in N} H^k(R^n)) : E_N(v)(t) \leq D_N \lambda(t)^p e^{L_N t} \quad \forall N\},$$

where the constants D_N and L_N will be determined later. A suitable small T^* and a suitable index N_0 imply for fixed D_{N_0} and L_{N_0} the fulfilment of $\|v\|_{L_\infty(R^n)}, \|v_t\|_{L_\infty(R^n)}, \|\nabla_x v\|_{L_\infty(R^n)} \leq 1$. If we replace on the right-hand

side v by u , then the application of Lemma 1 leads to

$$\begin{aligned}
 E_N(v)(t) &\leq \lambda(t)^{1+\frac{\varepsilon}{2}} \int_0^t \lambda(s)^{-(1+\frac{\varepsilon}{2})} \left((Q + \frac{\varepsilon}{2}) \frac{\lambda'(s)}{\lambda(s)} E_N(u)(s) \right. \\
 &\quad + A_N \frac{\lambda'^2(s)}{\lambda(s)} \varphi_N(\|u\|_{L^\infty(R^n)}, \|u_t\|_{L^\infty(R^n)}, \|\nabla_x u\|_{L^\infty(R^n)}) E_N(u)(s) \\
 &\quad \left. + B_N \lambda(s)^p \right) ds.
 \end{aligned}$$

If $u \in K$, then

$$\begin{aligned}
 E_N(v)(t) &\leq \lambda(t)^{1+\frac{\varepsilon}{2}} \int_0^t \lambda(s)^{-(1+\frac{\varepsilon}{2})} \left((Q + \frac{\varepsilon}{2}) \frac{\lambda'(s)}{\lambda(s)} D_N \lambda(s)^p e^{L_N s} \right. \\
 &\quad \left. + A_N \frac{\lambda'^2(s)}{\lambda(s)} \varphi_N(1, 1, 1) D_N \lambda(s)^p e^{L_N s} + B_N \lambda(s)^p \right) ds \\
 &\leq \left(\frac{Q + \frac{\varepsilon}{2}}{p - 1 - \frac{\varepsilon}{2}} + \frac{C A_N \varphi_N(1, 1, 1)}{L_N} \right) D_N \lambda(t)^p e^{L_N t} + B_N \lambda(t)^p t.
 \end{aligned}$$

Here we have used $\lambda(t) \in C^2[0, T]$. By (A1) and (A2) the function λ'^2/λ belongs to $C(0, T]$. But the additional regularity assumption yields $\lim_{t \rightarrow +0} \lambda'(t)^2/\lambda(t) = 0$. Consequently, $\lambda'^2(t)/\lambda(t) \leq C$ for $t \in [0, T]$. Let us choose $p \geq 3Q + 2\varepsilon + 1, L_N \geq 3A_N C \varphi_N(1, 1, 1)$ and $D_N \geq 3B_N$. Then a small T^* implies $E_N(v)(t) \leq D_N \lambda(t)^p e^{L_N t}$ for all $t \in [0, T^*]$ and N . Using the property of $\bigcap_{k \in N} H^k(R^n)$ to be a Montel space and the differential equation we obtain a uniquely determined solution $u \in C^2([0, T^*], \bigcap_{k \in N} H^k(R^n))$ by Schauder-Tychonoff's fixed point theorem and Theorem 2.

Remark 3. For the proof of Corollary 2 we have used the additional regularity assumption $\lambda(t) \in C^2[0, T]$. It seems to be possible to avoid this assumption if one uses instead of Schauder-Tychonoff's fixed point theorem the Nash-Moser technique (see [2], [3], [4], [8] for different weakly hyperbolic Cauchy problems). But we want to restrict ourselves to the statement of Corollary 2 because the main goal of this paper is to prove the existence of Sobolev solutions for (1.1),(1.2).

Remark 4. Using Corollary 2 one can prove local existence of C^∞ -solutions for (1.1),(1.2) if the data u_0, u_1 belong to $C^\infty(G), G \subset R^n$ is a bounded domain. Multiplying the data by a cutoff function $\chi \in C_0^\infty(G), \chi \equiv 1$ on $\overline{G'} \subset G$, they belong to $\bigcap_{k \in N} H^k(R^n)$). Recalling Corollary 2 and Theorem 2 there exists a uniquely determined solution $u \in C^2([0, T^*], \bigcap_{k \in N} H^k(R^n))$.

But this solution has the property to have a cone of dependence. This follows from the corresponding property of the solutions for the strictly hyperbolic Cauchy problems ($\varepsilon > 0$)

$$u_{tt} - (\lambda(t) + \varepsilon)^2 \Delta u = f(t, x, u, u_t, \mu_i(t) \partial_{x_i} u), u(x, 0) = u_t(x, 0) = 0$$

and a continuity argument. Consequently, the solution of our auxiliary problem coincides on a small cylinder $\overline{G_1} \times [0, T^*]$ with that one of (1.1),(1.2) with data belonging to $C^\infty(G)$, $\overline{G_1} \subset G'$.

Remark 5. We are now able to prove an existence- and uniqueness result for the solutions of the Cauchy problem for the weakly hyperbolic equations

$$u_{tt} - t^{2l} \Delta u = f(t, x, u, u_t, \mu_i(t) \partial_{x_i} u), \quad \limsup_{t \rightarrow +0} \frac{|\mu_i(t)|}{t^{l-1}} = Q,$$

(degeneracy of finite order at $t = 0$) and

$$u_{tt} - t^{-4} \exp\left(-\frac{2}{t}\right) \Delta u = f(t, x, u, u_t, \mu_i(t) \partial_{x_i} u), \quad \limsup_{t \rightarrow +0} \frac{|\mu_i(t)|}{t^{-2} \exp\left(-\frac{1}{t}\right)} = Q$$

(degeneracy of infinite order at $t = 0$).

If we compare the results for the first equation with those from Example 1 the loss of Sobolev regularity decreases, too, if l increases. In the case of degeneracy of finite order it is not necessary to use the step a) with the improvement of asymptotical behaviour. In this case the energy method can be applied directly to the starting problem (see [10], where the Levi condition is sharp in the case of degeneracy of finite order).

Remark 6. The results of this paper complete the knowledge about the theory for weakly hyperbolic Cauchy problems of the form

$$u_{tt} - \lambda^2(t) \Delta u = f(t, x, u, u_t, \mu_i(t) \partial_{x_i} u), u(x, 0) = u_0(x), u_t(x, 0) = u_1(x).$$

a) If $u_0 \in H^{s_0}(R^n)$, $u_1 \in H^{s_0-1}(R^n)$, then the nonnegative number Q from (2.2) determines the suitable s_0 for which a Sobolev solution exists. The solution has a loss of Sobolev derivatives in comparison with that one of the data.

b) If s_0 is sufficiently large, then the solution is uniquely determined.

The following problem seems to be open under the assumptions (A1),(A2) and (A4):

c) If f depends on $\mu_i(t) \partial_{x_i} u$, where $|\mu_i(t)| = O(\lambda(t)^{1-2/s})$, $s \in (2, \infty)$, then one cannot expect even in the linear case C^∞ -well posedness (see [5]). An interesting problem should be to prove Gevrey-well posedness and to find the critical Gevrey index depending on s . For $s \in [1, 2]$ Levi conditions don't appear ([6]).

Acknowledgment. The research of the author was supported by the project 322-vigoni-dr. The author thanks the referee for the information about the papers by D'Ancona-Manfrin (Ann. Mat. Pura Appl. **148** (1995), 355–372) and Manfrin-Tonin (Rend. Mat. Appl., to appear), which contain results for nonlinear weakly hyperbolic equations and continue the research of the authors as it was mentioned in the introduction.

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