ON A PROBLEM OF LOWER LIMIT IN THE STUDY OF NONRESONANCE

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Abstract. We prove the solvability of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f(u) + h & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega \end{cases}$$

for every given h, under a condition involving only the asymptotic behaviour of the potential F of f with respect to the first eigenvalue of the p-Laplacian Δ_p. More general operators are also considered.

1. Introduction

This paper is concerned with the existence of solutions for the problem

$$(\mathcal{P}_p) \begin{cases} -\Delta_p u = f(u) + h & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, Δ_p denotes the p-Laplacian $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and h is a given function on Ω.

A classical result, essentially due to Hammerstein [H], asserts that if f satisfies a suitable polynomial growth restriction connected with the Sobolev imbeddings and if

$$(F_1) \qquad \limsup_{s\to\pm\infty} \frac{2F(s)}{|s|^2} < \lambda_1,$$

then problem $(\mathcal{P}_2)$ is solvable for any h. Here F denotes the primitive $F(s) = \int_0^s f(t)\,dt$ and λ_1 is the first eigenvalue of -Δ on $H_0^1(\Omega)$. Several improvements of this result have been considered in the recent years.

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In 1989, the case N=1 and p=2 was considered in [Fe,O,Z]. It was shown there that $(\mathcal{P}_2)$ is solvable for any $h \in L^\infty(\Omega)$ if

$$(F_2) \qquad \liminf_{s\to\pm\infty} \frac{2F(s)}{|s|^2} < \lambda_1.$$

If $N \geq 1$ and p=2, [F,G,Z] showed later that $(\mathcal{P}_2)$ is solvable for any $h \in L^\infty(\Omega)$ if

$$(F_3) \qquad \liminf_{s\to\pm\infty} \frac{2F(s)}{|s|^2} < (\frac{\pi}{2R(\Omega)})^2,$$

where $R(\Omega)$ denotes the radius of the smallest open ball $B(\Omega)$ containing Ω. This result was extended to the p-laplacian case in [E,G.1], where solvability of $(\mathcal{P}_p)$ was derived under the condition

$$(F_4) \qquad \liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < (p-1)\{\frac{1}{R(\Omega)}\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\}^p.$$

Note that (F_4) reducer to (F_3) when $p = 2$.

The question now naturally arises whether $(p-1)\{\frac{1}{R(\Omega)}\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\}^p$ can be replaced by λ_1 in (F_4), where λ_1 denotes the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ (cf. Anane [A]).

Observe that for $N > 1$ and $p = 2$, $(\frac{\pi}{2R(\Omega)})^2 < \lambda_1$, and a similar strict inequality holds when $1 < p < \infty$. One of our purposes in this paper is to show that the constants in (F_3) and (F_4) can be inproved a little bit.

Denote by $l(\Omega) = l$ the length of the smallest edge of an arbitrary parallelepiped containing Ω. In the first part of the paper we assume

$$(F_5) \qquad \liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < C_p(l)$$

where $C_p(l) = C_p = (p-1)\{\frac{2}{l}\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\}^p$.

Observe that for $N = 1$, $C_p = \lambda_1$ is the first eigenvalue of -Δ_p on $\Omega =]0, l[$. In particular: $C_2 = (\frac{\pi}{l})^2$, and we recover the result of [Fe,O,Z]. It is clear that (F_5) is a weaker hypothesis than (F_4). The difference between (F_5) and (F_4) is particularly important when Ω is a rectangle or a triangle. However $C_p(l) < \lambda_1$ when $N > 1$, and the question raised above remains open.

In the second part of the paper we investigate the question of replacing Δ_p by the second order elliptic operator

$$A_p(u) = \sum_{1\leq i,j\leq N} \frac{\partial}{\partial x_i}(|\nabla u|_a^{p-2} a_{ij}(x)\frac{\partial u}{\partial x_j}),$$

where $|\nabla u|_a^2 = \sum_{1\leq i,j\leq N} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$. Observe that the method used in [F,G,Z], and [E,G.1] exploits the symmetry of the Laplacian or p-laplacian. It is not clear whether it can be adapted to more general second order elliptic operators like A_p above.

While this paper was being completed, we learned of a work by P.Omari and Grossinho (Cf. [GR,O.1], [GR,O.2]), where a result of the same type as ours is established in the case of the linear operator $A_2(u)$. The authors in [GR,O.2] also consider parabolic operators.

2. The case of the p-laplacian

In this section we will consider the problem $(\mathcal{P}_p)$ where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, $1 < p < \infty$, f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and $h \in L^\infty(\Omega)$.

Denote by $[AB]$ the smallest edge of an arbitrary parallelepiped containing Ω. Making an orthogonal change of variables, we can always suppose that AB is parallel to one of the axis of $\mathbb{R}^N$. So $\Omega \subset P = \prod_{j=1}^N [a_j, b_j]$ with, for some i, $|AB| = b_i - a_i = \min_{1 \leq j \leq N}\{b_j - a_j\}$, a quantity which we denote by $l(\Omega) = l$.

Theorem 1. *Assume*

$$(F) \qquad \liminf_{s \to \pm\infty} \frac{pF(s)}{|s|^p} < C_p,$$

where $C_p = C_p(l)$ is defined in the introduction. Then for any $h \in L^\infty(\Omega)$ $(\mathcal{P}_p)$ has a solution $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$.

Definition 1. An upper solution for $(\mathcal{P}_p)$ is defined as a function $\beta : \overline{\Omega} \to \mathbb{R}$ such that:

- $\beta \in C^1(\overline{\Omega})$
- $\Delta_p \beta \in C(\overline{\Omega})$
- $-\Delta_p \beta(x) \geq f(\beta(x)) + h(x) \quad$ a.e.x in Ω

A lower solution α is defined by reversing the inequalities above.

Lemma 1. *Assume that $(\mathcal{P}_p)$ admits an upper solution β and a lower solution α with $\alpha(x) \leq \beta(x)$ in Ω. Then $(\mathcal{P}_p)$ admits a solution $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$, with $\alpha(x) \leq u(x) \leq \beta(x)$ in Ω.*

Proof. This lemma is well known when $p = 2$ (see, e.g., [F.G.Z]). We sketch a proof in the general case $1 < p < \infty$.

Define

$$\tilde{f}(x,s) = \begin{cases} f(\beta(x)) & \text{if} \quad s \geq \beta(x) \\ f(s) & \text{if} \quad \alpha(x) \leq s \leq \beta(x) \\ f(\alpha(x)) & \text{if} \quad s \leq \alpha(x). \end{cases}$$

By a simple fixed point argument and the results of Di Benedetto [B], there is a solution $u \in W_0^{1,p}(\Omega) \cap C^1(\Omega)$ of

$$(\tilde{\mathcal{P}}) \begin{cases} -\Delta_p u &= \tilde{f}(x,u) + h(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases}$$

We claim that $\alpha(x) \leq u(x) \leq \beta(x)$ in Ω, which clearly implies the conclusion. To prove the first inequality, one multiplies the equation $(\tilde{\mathcal{P}})$ by $w = u - u_\alpha$, where $u_\alpha(x) = \max(u(x), \alpha(x))$, integrates by parts and uses the fact that α is a lower solution we obtain $\langle (-\Delta_p u) - (-\Delta_p(u - w)), w \rangle \leq 0$, which implies $w = 0$ (since $-\Delta_p$ is strictly monotone). ∎

Lemma 2. *Let $a < b$ and $M > 0$, and assume*

$$(F^+) \qquad \liminf_{s\to+\infty} \frac{pF(s)}{|s|^p} < C_p(b-a).$$

Then there exists $\beta_1 \in C^1(I)$ such that $\Delta_p\beta_1 \in C(I)$ and

$$\begin{cases} -\Delta_p\beta_1(t) & \geq & f(\beta_1(t)) + M & \forall t \in I, \\ \beta_1(t) & \geq & 0 & \forall t \in I, \end{cases}$$

where $I = [a, b]$

Lemma 3. *Assume*

$$(F^-) \qquad \liminf_{s\to-\infty} \frac{pF(s)}{|s|^p} < C_p(b-a).$$

Then there exists $\alpha_1 \in C^1(I)$ such that $\Delta_p\alpha_1 \in C(I)$ and

$$\begin{cases} -\Delta_p\alpha_1(t) & \leq & f(\alpha_1(t)) - M & \forall t \in I, \\ \alpha_1(t) & \leq & 0 & \forall t \in I. \end{cases}$$

Accepting for a moment the conclusion of these two lemmas, let us turn to the

Proof of Theorem 1. By Lemma 1 it suffices to show the existence of an upper solution and a lower solution for $(\mathcal{P}_p)$. Let us describe the construction of the upper solution (that of the lower solution is similar).
Let $M > \|h\|_\infty$ and $i \in \{1, 2, ..., N\}$ such that $b - a = b_i - a_i = \min_{1\leq j\leq N} b_j - a_j$. By Lemma 2 there exists $\beta_1 : I \to \mathbb{R}$ such that $\beta_1 \in C^1(I)$, $\Delta_p\beta_1 \in C(I)$ and

$$\begin{cases} -\Delta_p\beta_1(t) & \geq & f(\beta_1(t)) + M & \forall t \in I \\ \beta_1(t) & \geq & 0 & \forall t \in I. \end{cases}$$

Writing $\beta(x) = \beta_1(x_i)$ for all $x \in \overline{\Omega}$, it is clear that $\beta \in C^1(\overline{\Omega})$, $-\Delta_p\beta(x) = -\Delta_p\beta_1(x_i) \in C(\overline{\Omega})$, and we have:

$$\begin{aligned} -\Delta_p\beta(x) &= -\Delta_p\beta_1(x_i) \\ &\geq f(\beta_1(x_i)) + M \\ &= f(\beta(x)) + M \\ &\geq f(\beta(x)) + h(x) \quad a.e. x \in \Omega. \end{aligned}$$

The proof of Theorem 1 is thus complete. ∎

Proof of Lemma 2. The proof of Lemma 3 follows simiarly.

First case.
Suppose $\inf_{s\geq 0} f(s) = -\infty$. Then $\exists \beta \in \mathbb{R}_+^*$ such that $f(\beta) < -M$, and the constant function β provides a solution to the problem in Lemma 2.

Second case.
Suppose now $\inf_{s\geq 0} f(s) > -\infty$. Let $K > M$ such that $\inf_{s\geq 0} f(s) > -K + 1$. Thus $f(s) + K \geq 1$ for all $s \geq 0$. Define $g : \mathbb{R} \to \mathbb{R}$ by:

$$g(s) = \begin{cases} f(s) + K & if \quad s \geq 0, \\ f(0) + K & if \quad s < 0, \end{cases}$$

and denote $G(s) = \int_0^s g(t)\,dt$ for all s in $\mathbb{R}$. It is easy to see that $g(s) \geq 1\ \forall s \in \mathbb{R}$ and that

$$\liminf_{s\to+\infty} \frac{pG(s)}{s^p} = \liminf_{s\to+\infty} \frac{pF(s)}{s^p} < C_p.$$

Now it is clearly sufficient to prove the existence of a function $\beta_1 : I \to \mathbb{R}$ such that $\beta_1 \in C^1(I),\ \Delta_p\beta_1 \in C(I)$ and

$$\begin{cases} -\Delta_p\beta_1(t) &= g(\beta_1(t)) \quad \forall t \in I, \\ \beta_1(t) &\geq 0 \quad \forall t \in I. \end{cases}$$

For that purpose we will need the following three lemmas.

Lemma 4. *Define*

$$\tau_G(d) = \int_0^d \frac{dt}{(p\{G(d) - G(s)\})^{\frac{1}{p}}}$$

for $d > 0$. *Then*

$$\limsup_{d\to+\infty} \tau_G(d) \geq \Big(\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\Big)\Big(\liminf_{s\to+\infty} \frac{pG(s)}{|s|^p}\Big)^{\frac{-1}{p}}.$$

In particular (F^+) *implies* $\limsup_{d\to+\infty} \tau_G(d) > (p-1)^{-\frac{1}{p}}\frac{(b-a)}{2}$.

Proof. Let be a positive number such that $\liminf_{s\to+\infty} \frac{pG(s)}{s^p} < \rho < C_p$. Then $\limsup_{s\to+\infty}(K(s)) = +\infty$ where $K(s) = \rho|s|^p - pG(s)$. Let w_n be the smallest number in $[0,n]$ such that $\max_{0\leq s\leq n} K(s) = K(w_n)$; it is easily seen that w_n is increasing with respect to n. Since $\rho|s|^p - pG(s) < \rho w_n^p - pG(w_n)\ \ \forall s \in [0, w_n[$, we have

$$\begin{aligned} \tau_G(w_n) &> \rho^{-\frac{1}{p}} \int_0^{w_n} \frac{dt}{(w_n^p - s^p)^{\frac{1}{p}}} \\ &= \rho^{-\frac{1}{p}} \int_0^1 \frac{dt}{(1-s^p)^{\frac{1}{p}}} \end{aligned}$$

and therefore

$$\limsup_{d\to+\infty} \tau_G(d) \geq \rho^{-\frac{1}{p}} \int_0^1 \frac{dt}{(1-s^p)^{\frac{1}{p}}}$$

for all ρ such that $\liminf_{s\to+\infty} \frac{pG(s)}{s^p} < \rho < C_p$, which clearly implies the lemma. ∎

Lemma 5. *Let* $d > 0$ *and consider the mapping* T_d *defined by*

$$T_d(u)(t) = d - \int_a^t \Big([\int_a^r g(u(s))\,ds]^{\frac{1}{p-1}}\Big)\,dr$$

in the Banach space $C(I)$. *Then* T_d *has a fixed point.*

Proof. Clearly by Ascoli's theorem T_d is compact. The proof of Lemma 5 uses a homotopy argument based on the Leray Schauder topological degree. So T will have a fixed point if the following condition holds:
$\exists r > 0$ such that $(I - \lambda T_d)(u) \neq 0\ \forall u \in \partial B(0,r)\ \forall\lambda \in [0,1]$, where $\partial B(0,r) = \{u \in C(I);\ \|u\|_\infty = r\}$.

To prove that this condition holds, suppose by contradiction that $\forall n = 1, 2, \ldots\ \exists u_n \in \partial B(0, n),\ \exists \lambda_n \in [0, 1]$ such that: $u_n = \lambda_n T_d(u_n)$. The latter relation means

$$u_n = \lambda_n d - \lambda_n \int_a^t (\{\int_a^r g(u_n(s))\, ds\}^{\frac{1}{p-1}})\, dr. \tag{1}$$

Therefore $u_n \in C^1(I)$ and we have successively

$$\begin{cases} u_n'(t) &= -\lambda_n\{\int_a^t g(u_n(s))\, ds\}^{\frac{1}{p-1}} \leq 0, \\ u_n'(a) &= 0, \end{cases} \tag{2}$$

$\Delta_p u_n \in C(I)$ and

$$\begin{aligned} -\Delta_p u_n(t) &= -(|u_n'(t)|^{p-2} u_n'(t))' \\ &= ((-u_n'(t))^{p-1})' \\ &= \lambda_n^{p-1} g(u_n(t)). \end{aligned} \tag{3}$$

Note that by (2), $u_n'(t) < 0$ in $]a, b]$, so that u_n is decreasing.
Hence, for $n > d$, $u_n(b) = -n$. Multiplying the equation (3) by $u_n'(t)$, we obtain

$$-\frac{p-1}{p} \frac{d}{dt}(-u_n'(t))^p = \lambda_n^{p-1} \frac{d}{dt} G(u_n(t)). \tag{4}$$

Indeed

$$\begin{aligned} ((-u_n'(t))^p)' &= (((-u_n'(t))^{p-1})^{\frac{p}{p-1}})' \\ &= \tfrac{p}{p-1}((-u_n'(t))^{p-1})^{\frac{p}{p-1}-1}((-u_n'(t))^{p-1})' \\ &= -\tfrac{p}{p-1} u_n'(t)((-u_n'(t))^{p-1})'. \end{aligned}$$

By (4) we have

$$\begin{aligned} (p-1)(-u_n'(t))^p &= \lambda_n^{p-1} p[G(\lambda_n d) - G(u_n(t))] \\ &\leq p[G(d) - G(u_n(t))] \end{aligned}$$

since G is increasing. Hence $(p-1)^{\frac{1}{p}}(-u_n'(t))\{p[G(d) - G(u_n(t))]\}^{-\frac{1}{p}} \leq 1$.
Integrating from a to b and changing variable $s = u_n(t)$ ($u_n(a) = \lambda_n d$ and $u_n(b) = -n$), we obtain

$$(p-1)^{\frac{1}{p}} \int_{-n}^{\lambda_n d} [p(G(d) - G(s))]^{-\frac{1}{p}}\, ds \leq b - a,$$

i.e.

$$\begin{aligned} 0 &\leq (p-1)^{\frac{1}{p}} \textstyle\int_0^{\lambda_n d} [p(G(d) - G(s))]^{-\frac{1}{p}}\, ds \\ &= (b-a) + (p-1)^{\frac{1}{p}} \textstyle\int_0^{-n} [p(G(d) - G(s))]^{-\frac{1}{p}}\, ds. \end{aligned}$$

Since $G(s) = sg(0)$ for $s \leq 0$, we obtain

$$\begin{aligned} 0 &\leq (b-a) + (p-1)^{\frac{1}{p}} \textstyle\int_0^{-n} [p(G(d) - sg(0))]^{-\frac{1}{p}}\, ds \\ &= (b-a) - \frac{(p-1)^{\frac{1}{p}}}{(p-1)g(0)} [p(G(d) + ng(0))]^{\frac{p-1}{p}} + \frac{(pG(d))^{\frac{p-1}{p}}}{(p-1)g(0)}. \end{aligned}$$

Letting $n \to +\infty$, we get a contradiction. ■

Let us denote by $u_d \in C(I)$ a fixed point of the mapping T_d of Lemma 5.

Lemma 6. $\exists d > 0$ *such that* $u_d(t) \geq 0 \;\; \forall t \in [a, \frac{a+b}{2}[$.

Proof. We know that u_d is decreasing and that $u_d(a) = d$ for all $d > 0$. Let us distinguish two cases. First if $\exists d > 0$ such that $u_d(b) \geq 0$, then the conclusion of Lemma 6 clearly follows.
So we can assume that $\forall d > 0: \;\; u_d(b) < 0$. Since $u_d(a) = d > 0, \;\; \exists \delta_d \in]a, b[$ such that $u_d(\delta_d) = 0$. It is clear that $u_d(t) \geq 0 \;\; \forall t \in [a, \delta_d[$, and so it is sufficient to show that $\limsup_{d\to+\infty} \delta_d > \frac{a+b}{2}$.
Processing as in the proof of Lemma 5 we obtain

$$(p-1)^{\frac{1}{p}}(-u_d'(t))\{p(G(d) - G(u_d(t)))\}^{-\frac{1}{p}} = 1.$$

Integrating from a to δ_d and changing variable $s = u_d(t)$, one gets, $(p-1)^{\frac{1}{p}}\tau_G(d) = \delta_d - a$, and consequently

$$\limsup_{d\to+\infty} \delta_d = a + (p-1)^{\frac{1}{p}} \limsup_{d\to+\infty} \tau_G(d).$$

Now one easily deduces from Lemma 4 that $\limsup\limits_{d\to+\infty} \delta_d > a + \frac{b-a}{2} = \frac{a+b}{2}$. ∎

Proof of Lemma 2 Continued. Denoting $u_d(t)$ by $u(t)$, we have $u \in C^1(I)$, $\Delta_p u \in C(I)$ and

$$\begin{cases} -\Delta_p u(t) & = & g(u(t)) & \forall t \in I, \\ u(t) & \geq & 0 & \forall t \in [a, \frac{a+b}{2}], \\ u'(a) & = & 0. \end{cases}$$

Define a function β_1 from $[a, b]$ to $\mathbb{R}$ by

$$\beta_1(t) = \begin{cases} u(\frac{3a+b}{2} - t) & if \quad t \in [a, \frac{a+b}{2}], \\ u(t - \frac{b-a}{2}) & if \quad t \in [\frac{a+b}{2}, b]. \end{cases}$$

We will show that this function β fulfills the conditions of Lemma 2. To see this it is sufficient to show that:
(a) β_1 is nonegative in $[a, b]$,
(b) $\beta_1 \in C^1([a, b])$,
(c) $\Delta_p\beta_1 \in C([a, b])$ and $-\Delta_p\beta_1(t) = g(\beta_1(t)) \qquad \forall t \in [a, b]$.

Proof of (a). If $a \leq t \leq \frac{a+b}{2}$, then $a \leq \frac{3a+b}{2} - t \leq \frac{a+b}{2}$, and if $\frac{a+b}{2} \leq t \leq b$, then $a \leq t - \frac{b-a}{2} \leq \frac{a+b}{2}$, so that the conclusion follows from the sign of u on $[a, \frac{a+b}{2}]$.

Proof of (b). $\beta_1 \in C^1([a, \frac{a+b}{2}[)$, $\beta_1 \in C^1(]\frac{a+b}{2}, b])$, and moreover $\frac{d}{dt^+}\beta_1(\frac{a+b}{2}) = u'(a) = 0$ and $\frac{d}{dt^-}\beta_1(\frac{a+b}{2}) = u'(a) = 0$.

Proof of (c). We know that, $-(|u'(t)|^{p-2}u'(t))' = g(u(t))$ for $t \in [a, b]$ therefore

$$-|u'(t)|^{p-2}u'(t) = \int_a^t g(u(s))\, ds.$$

If $\frac{a+b}{2} \leq t \leq b$ then $a \leq t - \frac{b-a}{2} \leq \frac{a+b}{2}$, which gives

$$-(|u'(t-\frac{b-a}{2})|^{p-2}u'(t-\frac{b-a}{2})) = \int_a^{t-\frac{b-a}{2}} g(u(s))\,ds.$$

Changing variable $u = s + \frac{b-a}{2}$, this implies

$$-|\beta_1'(t)|^{p-2}\beta_1'(t) = \int_{\frac{a+b}{2}}^{t} g(\beta_1(s))\,ds,$$

hence $-\Delta_p\beta_1(t) = g(\beta_1(t))$ for all $t \in [\frac{a+b}{2}, b]$. The proof is similar for all $t \in [a, \frac{a+b}{2}]$. ■

3. The case of a more general operator.

Let Ω be a bounded domain in $\mathbb{R}^N$ and let A_p be an elliptic operator of the form

$$A_p(u) = \sum_{1\leq i,j\leq N} \frac{\partial}{\partial x_i}(|\nabla u|_a^{p-2} a_{ij}(x)\frac{\partial u}{\partial x_j})$$

where $(a_{ij}(x))_{1\leq i,j\leq N}$ are real-valued $L^\infty(\Omega)$

functions verifying $a_{ij}(x) = a_{ji}(x)$ for all i, j and

$$(*) \qquad \sum_{1\leq i,j\leq N} a_{ij}(x)\xi_i\xi_j = |\xi|_a^2 \geq |\xi|^2 \quad a.e.x \in \Omega \quad and\ for\ all\ \xi \in \mathbb{R}^N.$$

We now consider the problem

$$(\mathcal{P}'_p)\begin{cases} -A_p u &= f(u) + h \quad in \quad \Omega, \\ \quad u &= 0 \qquad\qquad on \quad \partial\Omega. \end{cases}$$

Note that A_p is defined from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Note also that $(*)$ implies that for each i, $a_{i,i}(x) > 0$ a.e. in Ω. We suppose that:

$$(A_0) \qquad \begin{cases} \exists i' \in \{1,2,...,N\} \text{ such that } a_{i'i'} = cte \in \mathbb{R} \text{ and} \\ div(a_{1,i'}(x),...,a_{N,i'}(x)) = \sum_{i\neq i'} \frac{\partial}{\partial x_i} a_{i,i'}(x) = 0. \end{cases}$$

We observe that (A_0) holds in particular when $a_{i,i'}$ $i = 1,...,N$, are fixed constants.

Denote by $b = b_{i'}$ and $a = a_{i'}$ where $[a_{i'}, b_{i'}]$ is an edge of an arbitrary parallelepiped containing Ω such that $[a_{i'}, b_{i'}]$ is parallel to the $x_{i'}$-axis and by

$$C_p(b-a) = C_p = (p-1)\{\frac{2}{b-a}\int_0^1 \frac{dt}{(1-t^p)^{\frac{1}{p}}}\}^p$$

Theorem 2. *Assume (A_0) and*

$$(F_2) \qquad \liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < (a_{i'i'})^{\frac{p}{2}} C_p.$$

Then $(\mathcal{P}')$ has a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for any $h \in L^\infty(\Omega)$.

The proof of Theorem 2 follows as in Theorem 1. Upper and lower solutions are defined for A_p in the same way as in definition 1 relative to Δ_p.

Lemma 7. *Assume that ($\mathcal{P}'$) admits an upper solution β and a lower solution α, then ($\mathcal{P}'$) admits a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\alpha(x) \le u(x) \le \beta(x)$.*

The proof of Lemma 7 follows similar lines as Lemma 1. It sufficis to remark that -A_p is strictly monotone.

Proof of Theorem 2. Let us describe the construction of the upper solution (that of lower solution is similar).
Let g be the continuous function defined by

$$g(s) = \frac{f(s)}{(a_{i'i'})^{\frac{p}{2}}} \quad \text{and denote} \quad G(s) = \int_0^s g(t)\,dt.$$

Then (F_2) implies that $\liminf_{s\to\pm\infty} \frac{pG(s)}{|s|^p} < C_p$. By Lemma 2 with $M > \frac{\|h\|_\infty}{(a_{i'i'})^{p-1}}$, there exists $\beta_1 \in C^1(I)$ such that $\Delta_p\beta_1 \in C(I)$ and

$$\begin{cases} -\Delta_p\beta_1(t) & \ge & g(\beta_1(t)) + M & \forall t \in I \\ \beta_1(t) & \ge & 0 & \forall t \in I. \end{cases}$$

Writing $\beta(x) = \beta_1(x_i)$ for all $x \in \overline{\Omega}$, we have $\beta(x) \ge 0 \ \ \forall x \in \overline{\Omega}$, $\beta \in C^1(\Omega)$.
Morever, by (A_0)

$$\begin{aligned} A_p(u) &= \sum_{1\le i,j\le N} \frac{\partial}{\partial x_i}(|\nabla\beta|_a^{p-2} a_{ij}(x)\frac{\partial\beta}{\partial x_j}) \\ &= (a_{i'i'})^{\frac{p}{2}}(|\beta_1'(x_{i'})|^{p-2}\beta_1'(x_{i'}))' + |\beta_1'(x_{i'})|^{p-2}\beta_1'(x_{i'})\sum_{i\ne i'}\frac{\partial}{\partial x_i}a_{ii'}(x) \\ &= (a_{i'i'})^{\frac{p}{2}}\Delta_p\beta(x). \end{aligned}$$

Hence
$A_p\beta \in C(\overline{\Omega})$ and $-A_p\beta(x) = -(a_{i'i'})^{\frac{p}{2}}\Delta_p\beta(x) \ge f(\beta(x)) + h(x)$ a.e.in Ω, which shows that β is an upper solution. ■

4. Comments

1. It is easy to give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that
$$\limsup_{\pm\infty} \frac{pF(s)}{|s|^p} = +\infty \quad \text{and} \quad \liminf_{\pm\infty} \frac{pF(s)}{|s|^p} = 0.$$
(See the work of [Fe.O.Z] in the case $p = 2$).

2. The problem $(\mathcal{P}_p)$ has at least one solution for any given $h \in L^\infty(\Omega)$ if we assume that:

$$(f_0) \qquad \limsup_{\pm\infty} \frac{f(s)}{|s|^{p-2}s} \le \lambda_1$$

and

$$(F_0) \qquad \liminf_{\pm\infty} \frac{pF(s)}{|s|^p} < \lambda_1.$$

This result was proved by Del Santo and Omari [S.O] for $p = 2$, and was generalized by Elhachimi and Gossez [H,G.2] for $p > 1$.
It is clear that (f_0) is not verified in the example of comment 1 above.

3. **Positive density condition.** Let $\eta > 0$ and define
$E = \{s \in \mathbb{R}^*;\ \frac{pF(s)}{|s|^p} < C_p - \eta\}$, $\tilde{E} = \{s \in \mathbb{R}^*;\ \frac{pF(s)}{|s|^p} < \lambda_1 - \eta\}$.

Theorem 3. *(Defigueiredo and Gossez [D,G]) Assume*

$$(f_0) \qquad \exists a, b > 0 \quad \text{such that} \quad |f(s)| \le a|s|^{p-1} + b \qquad \forall s \in \mathbb{R},$$

$$(F_0) \qquad \limsup_{\pm\infty} \frac{pF(s)}{|s|^p} \le \lambda_1,$$

$$(d) \qquad \begin{cases} \liminf\limits_{r\to+\infty} \frac{meas(\tilde{E}\cap[0,r])}{r} & > \quad 0, \\ \liminf\limits_{r\to-\infty} \frac{meas(\tilde{E}\cap[r,0])}{-r} & > \quad 0. \end{cases}$$

Then, for any $h \in W^{-1,p'}(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ solution of $(\mathcal{P}_p)$.

One says that $\tilde{E}$ has a positive density at $+\infty$ and $-\infty$ if **(d)** above is verified. This condition was introduced in [D,G].

The question now naturally arises whether nonresonance still occurs in $(\mathcal{P}_p)$ when the "liminf" condition **(d)** is weakened into a "limsup" condition. We have:

Corollary to Theorem 1. Assume

$$(d') \qquad \begin{cases} \limsup\limits_{r\to+\infty} \frac{meas(E\cap[0,r])}{r} & > \quad 0, \\ \limsup\limits_{r\to-\infty} \frac{meas(E\cap[r,0])}{-r} & > \quad 0. \end{cases}$$

Then, for any $h \in L^\infty(\Omega)$, there exists $u \in W_0^{1,p}(\Omega)$ solution of $(\mathcal{P}_p)$.

Proof. Obviously (d') implies that $E \cap \mathbb{R}_-$ and $E \cap \mathbb{R}_+$ are unbounded, so that (F) is satisfied. ∎

Remarks 1. **(a)** We have not supposed (f_0) nor (F_0) in the corollary.
(b) The question whether we may assume only

$$(\tilde{d}) \qquad \begin{cases} \limsup\limits_{r\to+\infty} \frac{meas(\tilde{E}\cap[0,r])}{r} & > \quad 0, \\ \limsup\limits_{r\to-\infty} \frac{meas(\tilde{E}\cap[r,0])}{-r} & > \quad 0, \end{cases}$$

remains open. Note that the condition $\liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < \lambda_1$ is weaker than $(\tilde{d})$.

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