EXISTENCE OF A POSITIVE SOLUTION FOR AN NTH ORDER BOUNDARY VALUE PROBLEM FOR NONLINEAR DIFFERENCE EQUATIONS

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Abstract. The nth order eigenvalue problem:

$$\Delta^n x(t) = (-1)^{n-k}\lambda f(t, x(t)), \quad t \in [0, T],$$

$$x(0) = x(1) = \cdots = x(k-1) = x(T+k+1) = \cdots = x(T+n) = 0,$$

is considered, where $n \geq 2$ and $k \in \{1, 2, \ldots, n-1\}$ are given. Eigenvalues λ are determined for f continuous and the case where the limits $f_0(t) = \lim_{n \to 0^+} \frac{f(t,u)}{u}$ and $f_\infty(t) = \lim_{n \to \infty} \frac{f(t,u)}{u}$ exist for all $t \in [0, T]$. Guo's fixed point theorem is applied to operators defined on annular regions in a cone.

1. Introduction

Define the operator Δ to be the forward difference

$$\Delta u(t) = u(t+1) - u(t),$$

and then define

$$\Delta^i u(t) = \Delta(\Delta^{i-1} u(t)), i \geq 1.$$

For $a < b$ integers define the discrete interval $[a, b] = \{a, a+1, \ldots, b\}$. Let the integers n, $T \geq 2$ be given, and choose $k \in \{1, 2, \ldots, n-1\}$. Consider the nth order nonlinear difference equation

$$\Delta^n x(t) = (-1)^{n-k}\lambda f(t, x(t)), t \in [0, T], \tag{1}$$

satisfying the boundary conditions

$$x(0) = x(1) = \cdots = x(k-1) = x(T+k+1) = \cdots = x(T+n) = 0. \tag{2}$$

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We determine eigenvalues λ that yield a solution to (1) and (2), where

$$(A) f : [0, T] \times \mathfrak{R}^+ \to \mathfrak{R}^+$$

is continuous, where $\mathfrak{R}^+$ denotes the nonnegative reals,

(B) For all $t \in [0, T], f_0(t) = \lim_{u \to 0^+} \frac{f(t, u)}{u}$ and $f_\infty(t) = \lim_{n \to \infty} \frac{f(t, u)}{u}$ both exist.

We apply Guo's fixed point theorem using cone methods, Guo and Lakshmikantham [14], and Krasnosel'skiǐ [19], to accomplish this. This method was first applied to differential equations in the landmark paper by Erbe and Wang [12]. Our proof will follow along the lines of those in Henderson [16], Lauer [17], and Merdivenci [20], additionally utilizing techniques from Peterson [21], Hartman [15], Eloe and Kaufmann [11], Agarwal and Wong [6,7], Agarwal and Henderson [1], and Agarwal, Henderson and Wong [2]. A key to applying this fixed point theorem involves discrete concavity of solutions of the boundary value problem in conjunction with a lower bound on an appropriate Green's function. Extensive use of the results by Eloe [8] concerning a lower bound for the Green's function will be made. Related results for nth order differential equation may be found in Agarwal and Wong [3,4], Eloe and Henderson [9,10], and Fang [13].

2. Preliminaries

Let $G(t, s)$ be the Green's funtion for the disconjugate boundary value problem

$$Lx(t) \equiv \Delta^n x(t) = 0, t \in [0, T], \tag{3}$$

and satisfying (2), where, as shown in Kelly and Peterson [18], $G(t, s)$ is the unique function satisfying:

(a) $G(t, s)$ is defined for all $t \in [0, T + n], s \in [0, T]$
(b) $LG(t, s) = \delta_{ts}$ for all $t \in [0, T]$, $s \in [0, T]$ where $\delta_{ts} = 1$ if $t = s, \delta_{ts} = 0$ if $t \neq s$,
(c) For all $s \in [0, T]$, $G(t, s)$ satisfies the boundary conditions (2) in t.

We will use $G(t, s)$ as the kernel of an integral operator preserving a cone in a Banach space. This is the setting for our fixed point theorem.

Let $\mathcal{B}$ be a Banach space and let $\mathcal{P} \subset \mathcal{B}$ be such that $\mathcal{P}$ is closed and non-empty. Then $\mathcal{P}$ is a *cone* provided (i) $au + bv \in \mathcal{P}$ for all $u, v \in \mathcal{P}$ and for all $a, b \geq 0$, and (ii) $u, -u \in \mathcal{P}$ implies $u = 0$.

Applying the following fixed point theorem from Guo, Guo and Lakshmikantham [14], will yield solutions of (1), (2) for certain λ.

Theorem 1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let Ω_1 and Ω_2 be two bounded open sets in $\mathcal{B}$ such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$H : \mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i) $\|Hx\| \leq \|x\|, x \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Hx\| \geq \|x\|, x \in \mathcal{P} \cap \partial\Omega_2$,
or

(ii) $\|Hx\| \geq \|x\|, x \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Hx\| \leq \|x\|, x \in \mathcal{P} \cap \partial\Omega_2$.
Then H *has a fixed point in* $\mathcal{P} \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

We now apply Theorem 1 to the eigenvalue problem (1), (2), following along the lines of methods incorporated by Henderson [16]. Note that $x(t)$ is a solution of (1), (2) if, and only if,

$$x(t) = (-1)^{n-k}\lambda \sum_{s=0}^{T} G(t,s) f(s, x(s)), \quad t \in [0, T].$$

Hartman [15] extensively studied the boundary value problem (1), (2), with $(-1)^{n-k}\lambda f(t,u) \geq 0$. We begin by stating three Lemmas from Hartman.

Lemma 1. *Let* $G(t,s)$ *denote the Green's function of (3), (2). Then*

$$(-1)^{n-k} G(t,s) \geq 0, \quad (t,s) \in [k, T+k] \times [0, T].$$

Lemma 2. *Assume that* u *satisfies the difference inequality* $(-1)^{n-k}\Delta^n u(t) \geq 0, t \in [0,T]$, *and the homogeneous boundary conditions, (2). Then* $u(t) \geq 0, t \in [0, T+k]$.

Lemma 3. *Suppose that the finite sequence* $u(0), \ldots, u(j)$ *has* N_j *nodes and the sequence* $\Delta u(0), \ldots, \Delta u(j-1)$ *has* M_j *nodes. Then* $M_j \geq N_j - 1$.

Eloe [8] employed these three lemmas to arrive at the following theorem that gives a lower bound for the solution to the class of boundary value problems studied by Hartman.

Theorem 2. *Assume that* u *satisfies the difference inequality* $(-1)^{n-k}\Delta^n u(t) \geq 0, t \in [0,T]$, *and the homogeneous boundary conditions, (2). Then for* $t \in [k, T+k]$,

$$(-1)^{n-k} u(t) \geq \frac{\nu!}{[(T+1)\cdots(T+\nu)]} \|u\|,$$

where $\|u\| = \max_{t \in [k, T+k]} |u(t)|$ *and* $u = \max\{k, n-k\}$.

We remark that Agarwal and Wong [5] have recently sharpened the inequality of Theorem 2. However, this sharper inequality is of little consequence for this work.

Eloe also contributed the following corollary.

Corollary 1. *Let* $G(t,s)$ *denote the Green's function for the boundary value problem, (3), (2). Then for all* $s \in [0,T], t \in [k, T+k]$,

$$(-1)^{n-k} G(t,s) \geq \frac{\nu!}{[(T+1)\cdots(T+\nu)]} \|G(\cdot, s)\|,$$

where $\|G(\cdot,s)\| = \max_{t \in [k, T+k]} |G(t,s)|$ *and* $\nu = \max\{k, n-k\}$.

To fulfill the hypotheses of Theorem 1, let

$$\mathcal{B} = \{u : [0, T+n] \to \mathfrak{R} \quad u(0) = u(1) = \cdots = u(k-1) \\ = u(T+k+1) = \cdots = u(T+n) = 0\},$$

with $\|u\| = \max_{t \in [t, T+k]} |u(t)|$. Now, $(\mathcal{B}, \|\cdot\|)$ is a Banach space.

Let

$$\sigma = \frac{\nu!}{[(T+1)\cdots(T+\nu)]}, \tag{4}$$

and define a cone

$$\mathcal{P} = \{u \in \mathcal{B} | u(t) \geq 0 \text{ on } [0, T+n] \text{ and } \min_{t \in [k, T+k]} u(t) \geq \sigma \|u\|\}.$$

Also choose $\tau, \eta \in [k, T+k]$ such that

$$(-1)^{n-k} \sum_{s=k}^{T} G(\tau, s) f_\infty(s) = \max_{t \in [k, T+k]} \sum_{s=k}^{T} G(t, s) f_\infty(s), \tag{5}$$

$$(-1)^{n-k} \sum_{s=k}^{T} G(\eta, s) f_0(s) = \max_{t \in [k, T+k]} .(-1)^{n-k} \sum_{s=k}^{T} G(t, s) f_0(s), \tag{6}$$

3. Main Results

Theorem 3. *Assume conditions (A) and (B) are satisfied. Then, for each λ satisfying*

$$\frac{1}{\sigma(-1)^{n-k} \sum_{s=0}^{T} G(\tau, s) f_\infty(s)} < \lambda < \frac{1}{\sum_{s=k}^{T} \|G(\cdot, s)\| f_0(s)}.$$

there exists at least one solution of (1), (2) in $\mathcal{P}$.

Proof. Let λ be given as in Theorem 3. Let $\epsilon > 0$ be such that

$$\frac{1}{\sigma(-1)^{n-k} \sum_{s=k}^{T} G(\tau, s)(f_\infty(s) - \epsilon)} \geq \lambda \geq \frac{1}{\sum_{s=0}^{T} \|G(\cdot, s)\|(f_0(s) + \epsilon)}.$$

Define a summation operator $H : \mathcal{P} \to \mathcal{B}$ by

$$Hx(t) = (-1)^{n-k} \lambda \sum_{s=0}^{T} G(t, s) f(s, x(s)), \qquad x \in \mathcal{P}. \tag{7}$$

We seek a fixed point of H in the cone $\mathcal{P}$. By the nonnegativity of f and $(-1)^{n-k}G, Hx(t) \geq 0$ on $[0, T+n]$, and from the properties of G, Hx

satisfies the boundary conditions. Now if we choose $x \in \mathcal{P}$, we have

$$\begin{aligned} Hx(t) &= (-1)^{n-k}\lambda \sum_{s=0}^{T} G(t,s)f(s,x(s)) \\ &\leq \lambda \sum_{0=0}^{T} \|G(\cdot,s)\| f(s,x(s)), t \in [k, T+k]. \end{aligned}$$

So

$$\|Hx\| = \max_{t\in[k,T+k]} |Hx(t)| \leq \lambda \sum_{s=0}^{T} \|G(\cdot,s)\| f(s,x(s)).$$

Hence, if $x \in \mathcal{P}$, $(-1)^{n-k}G(t,s) \geq \sigma\|G(\cdot,s)\|$, for $t \in [k, T+k]$ and $s \in [0,T]$, and thus,

$$\begin{aligned} \min_{t\in[k,T+k]} Hx(t) &= \min_{k,T+k]} (-1)^{n-k}\lambda \sum_{s=0}^{T} G(t,s)f(s,x(s)) \\ &\geq \sigma\lambda \sum_{s=0}^{T} \|G(\cdot,s)\| f(s,x(s)) \\ &\geq \sigma\|Hx\|. \end{aligned}$$

Thus $H : \mathcal{P} \to \mathcal{P}$. Additionally, H is completely continuous.

Now consider $f_0(t)$. For each $t \in [0,T]$, there exists $k_t > 0$ such that $f(t,u) \leq (f_0(t)+\epsilon)u$ for $0 < u \leq k_t$. Let $K_1 = \min_{t\in[0,T]} k_t$. So, for $x \in \mathcal{P}$ with $\|x\| = K_1$, we have

$$\begin{aligned} Hx(t) &= (-1)^{n-k}\lambda \sum_{s=0}^{T} G(t,s)f(s,x(s)) \\ &\leq \lambda \sum_{s=0}^{T} \|G(\cdot,s)\|(f_0(s)+\epsilon)x(s) \\ &\leq \lambda \sum_{s=0}^{T} \|G(\cdot,s)\|(f_0(s)+\epsilon)\|x\| \\ &\leq \|x\|, \qquad t \in [k, T+k]. \end{aligned}$$

Therefore, $\|H(x)\| \leq \|x\|$. Hence, if we set

$$\Omega_1 = \{u \in \mathcal{B} | \|u\| < K_1\}$$

then

$$\|Hx\| \leq \|x\| \text{ for all } x \in \mathcal{P} \cap \partial\Omega_1. \tag{8}$$

Next consider $f_\infty(t)$. For each $t \in [0,T]$, there exists $\tilde{k}_t > 0$ such that $f(t,u) \geq (f_\infty(t)-\epsilon)u$ for all $u \geq \tilde{k}_t$. Let $\tilde{K}_2 = \max_{t\in[0,T]} \tilde{k}_t$ and $K_2 =$

$\max\{2K_1, \frac{1}{\sigma}\tilde{K}_2\}$. Define

$$\Omega_2 = \{u \in \mathcal{B} | \|u\| < K_2\}$$

If $x \in \mathcal{P}$ with $\|x\| = K_2$, then $\min_{t\in[k,T+k]} x(t) \geq \sigma\|x\| \geq \tilde{K}_2$, and

$$\begin{aligned} Hx(\tau) &= (-1)^{n-k}\lambda \sum_{s=0}^{T} G(\tau,s)f(s,x(s)) \\ &\leq (-1)^{n-k}\lambda \sum_{s=0}^{T} G(\tau,s)f(s,x(s)) \\ &\geq (-1)^{n-k}\lambda \sum_{s=0}^{T} G(\tau,s)f_\infty(s)-\epsilon)x(s)(s,x(s)) \\ &\geq \sigma(-1)^{n-k}\lambda \sum_{s=k}^{T} G(\tau,s)(f_\infty(s)-\epsilon)\|x\| \\ &\geq \|x\|. \end{aligned}$$

Thus, $\|Hx\| \geq \|x\|$, and so

$$\|Hx\| \geq \|x\| \text{ for all } x \in \mathcal{P} \cap \partial\Omega_2 \tag{9}$$

So with (8) and (9) we have shown that H satisfies the first condition of Theorem 1. Thus we can conclude that H has a fixed point $u(t) \in \mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$. This fixed point, $u(t)$, is a solution of (1), (2) corresponding to the given value of λ. ■

Theorem 4. *Assume conditions (A) and (B) are satisfied. Then, for each λ satisfying*

$$\frac{1}{\sigma(-1)^{n-k}\sum_{s=k}^{T} G(\eta,s)f_0(s)} < \lambda < \frac{1}{\sum_{s=0}^{T} \|G(\cdot,s)\| f_\infty(s)},$$

there exists at least solution of (1), (2) in $\mathcal{P}$.

Proof. Let λ be given as stated above. Let $\epsilon > 0$ be such that

$$\frac{1}{\sigma(-1)^{n-k}\sum_{s=k}^{T} G(\eta,s)(f_0(s)-\epsilon)} \leq \lambda \leq \frac{1}{\sum_{s=0}^{T} \|G(\cdot,s)\|(f_\infty(s)+\epsilon)}$$

Let H be the cone preserving, completely continuous operator defined in (7).

Consider $f_0(t)$. For each $t \in [0,T]$ there exists $k_t > 0$ such that $f(t,u) \geq (f_0(t)-\epsilon)u$ for $0 < u \leq k_t$. Let $K_1 = \min_{t\in[0,T]} k_t$. So, for $x \in \mathcal{P}$ with $\|x\| = K_1$, we have

$$\begin{aligned}
Hx(\eta) &= (-1)^{n-k}\lambda \sum_{s=0}^{T} G(\eta,s)f(s,x(s)) \\
&\geq (-1)^{n-k}\lambda \sum_{s=k}^{T} G(n,x)f(x,x(s)) \\
&\geq (-1)^{n-k}\lambda \sum_{s=0}^{T} G(\eta,s)(f_0(s)-\epsilon)x(s) \\
&\geq \sigma(-1)^{n-k}\lambda \sum_{s=k}^{T} G(\eta,s)(f_0(s)-\epsilon)\|x\| \\
&\geq \|x\|.
\end{aligned}$$

Therefore, $\|Hx\| \geq \|x\|$. Hence, if we set

$$\Omega_1 = \{u \in \mathcal{B} | \|u\| < K_1\},$$

(10) $$\|Hx\| \geq \|x\|, \text{ for all } x \in \mathcal{P} \cap \partial\Omega_1.$$

Next consider $f_\infty(t)$. For each $t \in [0,T]$ there exists $\tilde{k}_t > 2K_1$ such that $f(t,u) \leq (f_\infty(t)+\epsilon)u$ for all $u \geq \tilde{k}_t$. There exists sets $I, J \subset [0,T]$, with $I \cup J = [0,T]$, such that for all $t \in I, f(t,u)$ is bounded as a function of u, and for all $t \in J, f(t,u)$ is unbounded as a function of u.

Choose $M > 0$ such that for all positive u and for all $t \in I, f(t,u) \leq M$. Let

$$\kappa_t = \max\left\{\tilde{k}_t, \frac{M}{f_\infty(t)+\epsilon}\right\}$$

For each $t \in J$ choose $\kappa_t \geq \tilde{k}_t$ such that $f(t,u) \leq f(t,\kappa_t)$, for $0 < u \leq \kappa_t$. Let $K_2 = \max\limits_{t\in[0,T]} \kappa_t$. By the continuity of f, for all $t \in J$ there exists μ_t, where $\kappa_t \leq \mu_t \leq K_2$, such that $f(t,u) \leq f(t,\mu_t)$ for all $0 < u \leq K_2$. Now

$$\begin{aligned}
Hx(t) &= (-1)^{n-k}\lambda \sum_{s=0}^{T} G(t,s)f(s,x(s)) \\
&\leq \lambda \sum_{s\in J} \|G(\cdot,s)\|M + \lambda \sum_{s\in I} \|G(\cdot,s)\|f(s,\mu_s) \\
&\leq \lambda \sum_{s\in I} \|G(\cdot,s)\|(f_\infty(s)+\epsilon)\kappa_s + \lambda \sum_{s\in J} \|G(\cdot,s)\|(f_\infty(s)+\epsilon)\mu_s \\
&\leq \lambda \sum_{s=0}^{T} \|G(\cdot,s)\|(f_\infty(s)+\epsilon)K_2 \\
&= \lambda \sum_{s=0}^{T} \|G(\cdot,s)\|(f_\infty(s)+\epsilon)\|x\| \\
&\leq \|x\| \qquad t \in [k, T+k],
\end{aligned}$$

for $x \in \mathcal{P}$ with $\|x\| = K_2$. Now if we take

$$\Omega_2 = \{u \in \mathcal{B} | \|u\| < K_2\},$$

then

$$\|Hx\| \leq \|x\| \text{ for all } x \in \mathcal{P} \cup \partial\Omega_2. \tag{11}$$

Thus, with (10) and (11), we have shown that H satisfies the hypotheses to Theorem 1(ii), which yields a fixed point of H belonging to $\mathcal{P} \cap (\bar{\Omega}_2 \backslash \Omega_1)$. this fixed point is a solution of (1), (2) corresponding to the given λ. ∎

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