A RESULT ON THE BIFURCATION FROM THE PRINCIPAL EIGENVALUE OF THE Ap-LAPLACIAN

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ABSTRACT. We study the following bifurcation problem in any bounded domain Ω in $\mathbf{R}^N$:

$$\begin{cases} A_p u := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[\left(\sum_{m,k=1}^{N} a_{mk}(x)\frac{\partial u}{\partial x_m}\frac{\partial u}{\partial x_k}\right)^{\frac{p-2}{2}} a_{ij}(x)\frac{\partial u}{\partial x_j}\right] = \\ \qquad \lambda g(x)|u|^{p-2}u + f(x,u,\lambda), \\ u \in W_0^{1,p}(\Omega). \end{cases}$$

We prove that the principal eigenvalue λ_1 of the eigenvalue problem

$$\begin{cases} A_p u = \lambda g(x)|u|^{p-2}u, \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

is a bifurcation point of the problem mentioned above.

1. INTRODUCTION

In this paper we study the *bifurcation problem*

$$\begin{cases} A_p u = \lambda g(x)|u|^{p-2}u + f(u,u,\lambda), \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$; $g \in L^\infty_{\rm loc}(\Omega) \cap L^r(\Omega)$ is an indefinite weight function, with $r = r(N,p)$ satisfying the conditions

$$\begin{cases} r > Np' & \text{for } 1 < p \leq N, \\ r = 1 & \text{for } p > N. \end{cases} \tag{1.2}$$

We assume that $|\Omega^+| \neq 0$ with $\Omega^+ = \{x \in \Omega; g(x) > 0\}$. The so–called *$A_p$-Laplacian* is defined by

$$\begin{aligned} A_p u &= -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left[\left(\sum_{m,k=1}^N a_{mk}(x)\frac{\partial u}{\partial x_m}\frac{\partial u}{\partial x_k}\right)^{\frac{p-2}{2}} a_{ij}(x)\frac{\partial u}{\partial x_j}\right] \\ &= -\operatorname{div}(|\nabla u|_a^{p-2} A(\nabla u)), \end{aligned}$$

where $A = (a_{ij}(x))_{1\leq i,j\leq N}$ is a matrix satisfying the conditions

$$\begin{cases} a_{ij} \equiv a_{ji} \in L^\infty(\Omega) \cap C^{1,\gamma}_{\rm loc}(\Omega), \quad 0 < \gamma \leq 1, \forall i,j = 1,\dots,N, \\ |\xi|_a^2 := \displaystyle\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Nonlinearity f is a function satisfying some conditions to be specified later.

Problems involving the A_p-Laplacian, have been studied in [M, L-T, T, E, E-Li-T]. We note that bifurcation problem is not considered there.

Bifurcation problem of the type (1.1), with $a_{ij} \equiv \delta_{ij}$, $\forall i,j = 1,\dots,N$, and other conditions on g and f, were studied on bounded domains by [B-H], [D1, D2] and [D-M]. The later authors consider the regular bounded domain with $\partial\Omega$ of class $C^{2,\beta}$ for some $\beta \in]0,1[$ and $g \equiv 1$. This result was extended for the bounded domain having the segment property and $g \in L^\infty(\Omega)$ by [E, E-La-T]. The case $\Omega = \mathbb{R}^N$ was studied by [D-H] (cf. also [D-K-N]) under some appropriate conditions on f and g.

In this work we investigate the situation improving the conditions on f and g for any bounded domain. This paper is organized as follows: in Section 2, we introduce some assumptions and notations which we use later and prove some technical preliminaries. In Section 3, we verify that the topological degree is well defined for our operators. We also show that the topological degree has a jump when λ crosses λ_1, which implies the *bifurcation result.*

2. Assumptions, Definitions and Preliminaries

We first introduce some basic definitions, assumptions and notations. For every x fixed in Ω denote

$$\langle \xi, \eta \rangle_a = \sum_{i,j=1}^N a_{ij}(x)\xi_i\eta_j, \forall \xi, \eta \in \mathbb{R}^N.$$

The symbol $|\cdot|_a$ denotes the norm induced by $\langle \cdot,\cdot \rangle_a$. We use $W_0^{1,p}(\Omega)$-norm defined by

$$\|v\|_{1,p} = \||\nabla v|_a\|_p = \left(\int_\Omega |\nabla v(x)|_a^p \, dx\right)^{\frac{1}{p}}.$$

Denote for $t \in]1, +\infty[$, $t' = \dfrac{t}{t-1}$; $t^* = \dfrac{Nt}{N-t}$ if $1 < t < N$ and $t^* = \infty$ if $N \leq t < \infty$.

2.1. **Assumptions.** We assume that

(f_1) $f : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory's conditions in the first two variables and

$$f(x, s, \lambda) = o(|s|^{p-1}) \text{ for } s \to 0 \tag{2.1}$$

uniformly a.e. with respect to x and uniformly with respect to λ in bounded sets of $\mathbb{R}$;

(f_2) there is a $q \in]p, p^*[$ such that

$$\lim_{|s| \to +\infty} \frac{|f(x, s, \lambda)|}{|s|^{q-1}} = 0, \tag{2.2}$$

uniformly a.e. with respect to x and uniformly with respect to λ in bounded sets.

2.2. **Definitions.** 1. By a *solution* of (1.1) we understand a pair (λ, u) in $\mathbb{R} \times W_0^{1,p}(\Omega)$ satisfying (1.1) in the weak sense, i.e., such that

$$\int_\Omega |\nabla u|_a^{p-2} \langle \nabla u, \nabla v \rangle_a \, dx = \int_\Omega [\lambda g(x)|u|^{p-2}u + f(x, u, \lambda)]v \, dx, \tag{2.3}$$

for all $v \in W_0^{1,p}(\Omega)$. We note that the pair $(\lambda, 0)$ is a solution of (1.1) for every $\lambda \in \mathbb{R}$. The pairs of this form will be called the *trivial solutions* of (1.1). We say that $P = (\overline{\lambda}, 0)$ is a *bifurcation point* of (1.1) if in any neighborhood of P in $\mathbb{R} \times W_0^{1,p}(\Omega)$ there exists a nontrivial solution of (1.1).

2. Let X be a real reflexive Banach space and let X^* stand for its dual with respect to the pairing $\langle \cdot, \cdot \rangle$. We shall deal with mappings T acting form X into X^*. The strong convergence in X (and in X^*) is denoted by $\to$ and the weak convergence by $\rightharpoonup$, respectively. T is *demicontinuous* at u in X, if $u_n \to u$ in X, implies that $Tu_n \rightharpoonup Tu$ in X^*. T is said to belong to the *class* (S_+), if for any sequence $\{u_n\}$ in X with $u_n \rightharpoonup u$ and $\limsup_{n \to +\infty} \langle Tu_n, u_n - u \rangle \leq 0$, it follows that $u_n \to u$ in X. We write $T \in (S_+)$.

2.3. **Degree theory.** If $T \in (S_+)$ and T is demicontinuous, then it is possible to define the *degree* $\operatorname{Deg}[T; D, 0]$, where $D \subset X$ is a bounded open set such that $Tu \neq 0$ for any $u \in \partial D$. Its properties are analogous to the ones of the Leray-Schauder degree (cf. [B], [S] or [B-P]).

A point $u_0 \in X$ will be called a *critical point* of T if $Tu_0 = 0$. We say that u_0 is an *isolated critical point* of T if there exists $\varepsilon > 0$ such that for any $u \in B_\varepsilon(u_0)$, $Tu \neq 0$ if $u \neq u_0$. Then the limit

$$\operatorname{Ind}(T, u_0) = \lim_{\varepsilon \to 0^+} \operatorname{Deg}[T; B_\varepsilon(u_0), 0]$$

exists and is called the *index of the isolated critical point* u_0.

Assume, furthermore, that T is a *potential operator*, i.e. for some continuously differentiable functional $\Phi : X \to \mathbb{R}$, $\Phi'(u) = Tu$, $u \in X$. Then we have the following two lemmas which we can find in [D1], [D2] or [D-H].

Lemma 2.1. *Let u_0 be a local minimum of Φ and an isolated critical point of T. Then*

$$\mathrm{Ind}\,(T, u_0) = 1.$$

Lemma 2.2. *Assume that $\langle Tu, u\rangle > 0$ for all $u \in X$, $\|u\|_X = \rho$. Then*

$$\mathrm{Deg}\,[T; B_\rho(0), 0] = 1.$$

2.4. **Preliminaries.** Define operators A_p, $G : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and $F : \mathbb{R} \times W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega)$, by

$$\begin{aligned}\langle A_p u, v\rangle &= \int_\Omega |\nabla u(x)|_a^{p-2}\langle \nabla u(x), \nabla v(x)\rangle_a dx\\ \langle Gu, v\rangle &= \int_\Omega g(x)|u(x)|^{p-2}u(x)v(x)\ dx\\ \langle F(\lambda, u), v\rangle &= \int_\Omega f(x, u(x), \lambda)v(x)\ dx\end{aligned}$$

for any $u, v \in W_0^{1,p}(\Omega)$.

Remark 2.3. *(i) Due to (2.3) the function u is a weak solution of (1.1) if and only if*

$$A_p u - \lambda Gu - F(\lambda, u) = 0 \quad \text{in } W^{-1,p'}(\Omega). \tag{2.4}$$

(ii) The operator A_p has the following properties: A_p is odd, $(p-1)$-homogeneous, strictly monotone, i.e.,

$$\langle A_p u - A_p v, u - v\rangle > 0 \text{ for all } u \neq v, \tag{2.5}$$

and $A_p \in (S_+)$ (cf. [T]). We have also

$$\|A_p u\|_{W^{-1,p'}(\Omega)} = \||\nabla u|_a\|_p^{p-1} \tag{2.6}$$

for any $u \in W_0^{1,p}(\Omega)$.

Lemma 2.4. *G is compact, odd and $(p-1)$-homogeneous.*

Proof. Step 1 Definition of G.
First case: if $1 < p < N$, $r > Np'$. Let $u, v \in W_0^{1,p}(\Omega)$. By Hölder's inequality, we have

$$\left|\int_\Omega g(x)|u(x)|^{p-2}u(x)v(x)dx\right| \le \|g\|_r \|u\|_s^{p-1}\|v\|_{p^*}\ ,$$

where s is given by

$$\frac{p-1}{s} + \frac{1}{p^*} + \frac{1}{r} = 1.$$

Therefore

$$\frac{p-1}{s} = 1 - \frac{1}{r} - \frac{1}{p^*} > 1 - \frac{1}{Np'} - \frac{1}{p^*} > 1 - \frac{p}{N} - \frac{1}{p^*} = \frac{p-1}{p^*}.$$

i.e.

$$p - 1 < s < p^*.$$

Then it suffices that

$$\max(1, p-1) < s < p^* \tag{2.7}$$

and G is well defined.
Second case: if $p = N$, $r > NN' = N + N'$. In this case

$$W_0^{1,N}(\Omega) \hookrightarrow L^{\overline{q}}(\Omega),$$

for any $\overline{q} \in [1, +\infty[$. Since $r > N$, there is $\overline{q} > 1$ such that $\frac{1}{\overline{q}} + \frac{1}{r} + \frac{1}{N'} = 1$.
We obtain that

$$\overline{q} = \frac{1}{1 - \left(\frac{r+N'}{rN'}\right)}. \tag{2.8}$$

By Hölder's inequality, we arrive at

$$\left| \int_\Omega g(x)|u(x)|^{N-2}u(x)v(x)dx \right| \leq \|g\|_r \|u\|_N^{N-1} \|v\|_{\overline{q}} \ ,$$

for any u, v in $W_0^{1,N}(\Omega)$. Then in this case G is well defined.
Third case: if $p > N$, $r = 1$. In this case

$$W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \cap L^\infty(\Omega).$$

Then for any $u, v \in W_0^{1,p}(\Omega)$, we have

$$\left| \int_\Omega g(x)|u(x)|^{p-2}u(x)v(x)dx \right| < \infty,$$

with $g \in L^1(\Omega)$, and G is well defined also in this case.
Step 2 Compactness of G. Let $(u_n) \subset W_0^{1,p}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$. We must show that $Gu_n \to Gu$ strongly in $W_0^{1,p}(\Omega)$, i.e.

$$\sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \||\nabla v|_a\|_p \leq 1}} \left| \int_\Omega g[|u_n|^{p-2}u_n - |u|^{p-2}u]v \ dx \right| = 0(1), \quad n \to +\infty.$$

If $1 < p < N$, $r > Np'$: Let s be as in (2.7). Then

$$\begin{aligned}
&\sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \||\nabla v|_a\|_p \leq 1}} \left| \int_\Omega g[|u_n|^{p-2}u_n - |u|^{p-2}u]v \ dx \right| \\
&\leq \sup_{\substack{v \in W_0^{1,p}(\Omega) \\ \||\nabla v|_a\|_p \leq 1}} \left[\|g\|_r \| |u_n|^{p-2}u_n - |u|^{p-2}u \|_{\frac{s}{p-1}} \|v\|_{p^*} \right] \\
&\leq c\|g\|_r \| |u_n|^{p-2}u_n - |u|^{p-2}u \|_{\frac{s}{p-1}} \ ,
\end{aligned}$$

where c is the constant of Sobolev's embedding. We have

$$\||u_n|^{p-2}u_n - |u|^{p-2}u\|_{\frac{s}{p-1}} = o(1), \text{ as } n \to +\infty$$

due to the continuity of Nemytskii's operator $u \mapsto |u|^{p-2}u$ from $L^s(\Omega)$ into $L^{\frac{s}{p-1}}(\Omega)$. Rellich's theorem yields that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ implies that $u_n \to u$ strongly in $L^s(\Omega)$ because $\max(1, p-1) < s < p^*$. The compactness of G then follows.

If $p = N$, $r > N + N' = NN'$:

$$\left|\int_\Omega g[|u_n|^{N-2}u_n - |u|^N u]v \, dx\right| \le \|g\|_r \||u_n|^{N-2}u_n - |u|^{N-2}u\|_N^{N-1} \|v\|_{\overline{q}} \,,$$

where $\overline{q}$ is given by (2.8). By Sobolev's embedding, there is $c > 0$ such that

$$\|v\|_{\overline{q}} \le c\||\nabla v|_a\|_N, \quad \forall v \in W_0^{1,N}(\Omega).$$

Thus

$$\sup_{\substack{\||\nabla v|_a\|_N \le 1 \\ v \in W_0^{1,p}(\Omega)}} \left|\int_\Omega g[|u_n|^{N-2}u_n - |u|^{N-2}u]v \, dx\right|$$
$$\le C\|g\|_r \||u_n|^{N-2}u_n - |u|^{N-2}u\|_N^{N-1}.$$

From the continuity of $u \mapsto |u|^{N-2}u$ from $L^N(\Omega)$ into $L^{N'}(\Omega)$, and from the compact embedding of $W_0^{1,N}(\Omega)$ in $L^N(\Omega)$, we have the desired result.

If $p > N$, $r = 1$. By Rellich's embedding theorem of $W_0^{1,p}(\Omega)$ into $C(\overline{\Omega})$, we obtain

$$\sup_{\substack{\||\nabla v|_a\|_p \le 1 \\ v \in W_0^{1,p}(\Omega)}} \left|\int_\Omega g[|u_n|^{p-2}u_n - |u|^{p-2}u]v \, dx\right|$$
$$\le C\|g\|_1 \sup_{\overline{\Omega}} \left||u_n|^{p-2}u_n - |u|^{p-2}u\right|,$$

where C is the constant given by embedding of $W_0^{1,p}(\Omega)$ in $C(\overline{\Omega}) \cap L^\infty(\Omega)$. It is clear that

$$\sup_{\overline{\Omega}} \left||u_n|^{p-2}u_n - |u|^{p-2}u\right| = o(1), \text{ as } n \to +\infty.$$

The oddness and $(p-1)$-homogeneity of G is obvious. Thus the lemma is proved. ∎

Lemma 2.5. *$F(\lambda, \cdot)$ is compact, $F(\lambda, 0) = 0$ and we have*

$$\lim_{\||\nabla u|_a\|_p \to 0} \frac{F(\lambda, u)}{\||\nabla u|_a\|_p^{p-1}} = 0 \quad \text{in } W^{-1,p'}(\Omega), \tag{2.9}$$

uniformly for λ is in a bounded subset of $\mathbb{R}$.

Proof. (2.1) and (2.2) imply that for any $\varepsilon > 0$, there are two reals $\delta = \delta(\varepsilon)$ and $M = M(\delta)$ such that for a.e. $x \in \Omega$, we have

$$|f(x, s, \lambda)| \le \varepsilon |s|^{p-1} \quad \text{for } |s| \le \delta \tag{2.10}$$

and

$$|f(x,s,\lambda)| \le M|s|^{q-1} \quad \text{for } |s| \ge \delta. \tag{2.11}$$

Therefore, for $0 < \varepsilon \le 1$, we obtain

$$\begin{aligned}
&\int_\Omega |f(x,u(x),\lambda)|^{q'}dx \\
&= \int_{\{x,|u(x)|\le\delta\}} |f(x,u(x),\lambda)|^{q'}dx + \int_{\{x,|u(x)|\ge\delta\}} |f(x,u(x),\lambda)|^{q'}dx \\
&\le \int_\Omega |u(x)|^{q'(p-1)}dx + M\int_\Omega |u(x)|^q dx.
\end{aligned}$$

We have $q'(p-1) \le p'(p-1) = p < q$. So $L^q(\Omega) \hookrightarrow L^{q'(p-1)}(\Omega)$ and there is $c > 0$ such that

$$\int_\Omega |u(x)|^{q'(p-1)}dx \le c\int_\Omega |u(x)|^q dx.$$

We deduce that

$$\int_\Omega |f(x,u(x),\lambda)|^{q'}dx \le (c+M)\int_\Omega |u(x)|^q dx.$$

Hence $u \mapsto F(\lambda,u)$ maps $L^q(\Omega)$ into $L^{q'}(\Omega)$. Moreover, if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, $u_n \to u$ in $L^q(\Omega)$ (because $p < q < p^*$) and $F(\lambda,u_n) \to F(\lambda,u)$ in $L^{q'}(\Omega)$. Since $L^{q'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$, we have $F(\lambda,u_n) \to F(\lambda,u)$ in $W^{-1,p'}(\Omega)$. This proves that $F(\lambda,\cdot)$ is compact. It is clear that $F(\lambda,0) = 0$ for any $\lambda \in \mathbb{R}$.

By (f_2), we have $\dfrac{F(\lambda,u)}{\||\nabla u|_a\|_p^{p-1}} \to 0$ in $L^{q'}(\Omega)$. Indeed, set $v = \dfrac{u}{\||\nabla u|_a\|_p}$. Then

$$\frac{F(\lambda,u)}{\||\nabla u|_a\|_p^{p-1}} = \frac{F(\lambda,u)}{|u|^{p-1}}|v|^{p-1}. \tag{2.12}$$

From (2.12) and Hölder's inequality, we deduce that

$$\int_\Omega \left|\frac{F(\lambda,u)}{\||\nabla u|_a\|_p^{p-1}}\right|^{q'} dx \le \left(\int_\Omega \left|\frac{F(\lambda,u)}{|u|^{p-1}}\right|^{q't} dx\right)^{\frac{1}{t}} \left(\int_\Omega |v|^{(p-1)q't'}dx\right)^{\frac{1}{t'}},$$

for some $t > 0$ which satisfies

$$\frac{q'(q-p)}{p^*} < \frac{1}{t} < \frac{p^* - (p-1)q'}{p^*}. \tag{2.13}$$

This is always possible, since $p < q < p^*$. By (2.10) and (2.11), we obtain that

$$\left\|\left|\frac{F(\lambda,u)}{|u|^{p-1}}\right|^{q'}\right\|_t^t \le \varepsilon|\Omega| + M^{q't}\int_\Omega |u|^{q't(q-p)}dx, \quad \forall \varepsilon > 0.$$

From this inequality and since $u \to 0$ in $W_0^{1,p}(\Omega)$, we have by (2.13) that

$$\left\|\left|\frac{F(\lambda,u)}{|u|^{p-1}}\right|^{q'}\right\|_t^t \to 0, \text{ as } u \to 0 \quad \text{in } W_0^{1,p}(\Omega).$$

On the other hand, v belongs to $L^{p^*}(\Omega)$ (because $\||\nabla v|_a\|_p = 1$). Then we find a constant $c > 0$ so that

$$\left\| |v|^{(p-1)q'} \right\|_{t'} \le c,$$

since $q't'(p-1) < p^*$ by (2.13). This concludes the proof. ∎

Remark 2.6. *Note that every continuous map $T : X \to X^*$ is also demicontinuous. Note also, that if $T \in (S_+)$ then $(T+K) \in (S_+)$ for any compact operator $K : X \to X^*$.*

Remark 2.7. *λ is an* eigenvalue *of*

$$(P) \qquad \begin{cases} A_p u = \lambda g(x)|u|^{p-2}u, \\ u \in W_0^{1,p}(\Omega), \end{cases}$$

if and only if the equation

$$A_p u - \lambda G u = 0 \tag{2.14}$$

has a solution $u \in W_0^{1,p}(\Omega)\backslash\{0\}$.

Now we take $T_\lambda = A_p - \lambda G - F(\lambda, \cdot)$. By Lemma 2.4, Lemma 2.5, Remark 2.3 and Remark 2.6, the degree

$$\mathrm{Deg}\,[T_\lambda; D, 0], \tag{2.15}$$

(where D is a bounded open set in $W_0^{1,p}(\Omega)$ such that $T_\lambda u \neq 0$ for any $u \in \partial D$) is well defined for any $\lambda > 0$.

By the same argument as used in proof of Lemma 2.4, we can show the following proposition.

Proposition 2.8. *If $(\overline{\lambda}, 0)$ is a bifurcation point of problem* (1.1)*, then $\overline{\lambda}$ is an eigenvalue of (P).*

3. Bifurcation from λ_1

We recall that λ_1 can be characterized variationally as follows:

$$\lambda_1 = \min\left\{ \frac{\int_\Omega |\nabla u|_a^p dx}{\int_\Omega g|u|^p dx};\ \ u \in W_0^{1,p}(\Omega),\ \ \int_\Omega g|u|^p dx > 0 \right\}. \tag{3.1}$$

Recall for our problem (P), (cf., [L-T]), that λ_1 is the principal eigenvalue and it is simple and isolated.

Let $E = \mathbb{R} \times W_0^{1,p}(\Omega)$ be equipped with the norm

$$\|(\lambda, u)\| = (|\lambda|^2 + \||\nabla u|_a\|_p^2)^{\frac{1}{2}},\ \ (\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega).$$

Definition 3.1. We say that

$$C = \{(\lambda, u) \in E : (\lambda, u) \text{ solves } (1.1),\ u \neq 0\}$$

is *a continuum of nontrivial solutions* of (1.1), if it is a connected set in E.

Theorem 3.2. *Under the assumptions* (1.2), (1.3), (f_1) *and* (f_2), *the pair* $(\lambda_1, 0)$ *is a bifurcation point of* (1.1). *Moreover, there is a continuum of nontrivial solutions* C *of* (1.1) *such that* $(\lambda_1, 0) \in \overline{C}$ *and* C *is either unbounded in* E *or there is* $\overline{\lambda} \neq \lambda_1$, *an eigenvalue of* (P), *with* $(\overline{\lambda}, 0) \in \overline{C}$.

Proof. We will give only sketch of the proof since it follows the lines of the proof of Theorem 14.18 in [D2] or Theorem 3.7 in [D-K-N]. The key point in the proof is the fact that the value of

$$\mathrm{Deg}\,[A_p - \lambda G; B_\varepsilon(0), 0] \tag{3.2}$$

changes when λ crosses λ_1. If this fact is proved then the result follows exactly as in the classical bifurcation result of Rabinowitz [R]. Choose $\delta > 0$ such that $(\lambda_1, \lambda_1 + \delta)$ does not contain any eigenvalue of (P). Then the variational characterization (3.1) of λ_1 and Lemma 2.2 yield

$$\mathrm{Deg}\,[A_p - \lambda G; B_\varepsilon(0), 0] = 1, \tag{3.3}$$

when $\lambda \in (\lambda_1 - \delta, \lambda_1)$. To evaluate (3.2) for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ we use the following trick. Fix a number $K > 0$ and define a function $\psi : \mathbb{R} \to \mathbb{R}$ by

$$\psi(t) = \begin{cases} 0 & \text{for } t \leq K, \\ \dfrac{2\delta}{\lambda_1}(t - 2K) & \text{for } t \geq 3K, \end{cases}$$

and ψ is positive and strictly convex in $(K, 3K)$. Define a functional

$$\overline{\Psi}_\lambda(u) = \frac{1}{p}\langle A_p u, u\rangle - \frac{\lambda}{p}\langle Gu, u\rangle + \psi\left(\frac{1}{p}\langle A_p u, u\rangle\right).$$

Then $\overline{\Psi}_\lambda$ is continuously Fréchet differentiable and its critical point $u_0 \in W_0^{1,p}(\Omega)$ corresponds to a solution of the equation

$$A_p u_0 - \frac{\lambda}{1 + \psi'\left(\frac{1}{p}\langle A_p u_0, u_0\rangle\right)} G u_0 = 0.$$

However, since $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the only nontrivial critical points of $\overline{\Psi}'_\lambda$ occur if

$$\psi'\left(\frac{1}{p}\langle A_p u_0, u_0\rangle\right) = \frac{\lambda}{\lambda_1} - 1. \tag{3.4}$$

Due to the definition of ψ we then have

$$\frac{1}{p}\langle A_p u_0, u_0\rangle \in (K, 3K)$$

and due to (3.4) and the simplicity of λ_1, either $u_0 = -u_1$ or $u_0 = u_1$, where u_1 is the principal eigenfunction. So, for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the derivative $\overline{\Psi}'_\lambda$ has precisely three isolated critical points

$$-u_1, 0, u_1.$$

It is not difficult to prove that $\overline{\Psi}_\lambda$ is weakly lower semicontinuous and

$$\lim_{\|u\|_{1,p} \to \infty} \overline{\Psi}_\lambda(u) = \infty$$

due to the definition of ψ. So, $\overline{\Psi}_\lambda$ attains local minima at u_1 and $-u_1$. It follows from Lemma 2.1 that

$$\mathrm{Ind}\,(\overline{\Psi}'_\lambda, u_1) = \mathrm{Ind}\,(\overline{\Psi}'_\lambda, -u_1) = 1. \tag{3.5}$$

Since also

$$\langle \overline{\Psi}'_\lambda(u), u\rangle > 0$$

for $\|u\|_{1,p} = R$, with $R > 0$ sufficiently large, we have according to Lemma 2.2 that

$$\mathrm{Deg}\,[\overline{\Psi}'; B_R(0), 0] = 1. \tag{3.6}$$

Additivity property of the degree, (3.5) and (3.6) yield

$$\mathrm{Deg}\,[A_p - \lambda G; B_\varepsilon(0), 0] = -1 \tag{3.7}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $\varepsilon > 0$ sufficiently small. Since (3.3) and (3.7) establish the "jump" of the degree the proof is complete. ∎

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