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Research Article

Starlikeness and Convexity of Generalized Struve Functions

Nihat Yagmur[1] and Halit Orhan[2]

[1] *Department of Mathematics, Faculty of Science and Art, Erzincan University, 24000 Erzincan, Turkey*
[2] *Department of Mathematics, Faculty of Science, Ataturk University, 25240 Erzurum, Turkey*

Correspondence should be addressed to Nihat Yagmur; nhtyagmur@gmail.com

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We give sufficient conditions for the parameters of the normalized form of the generalized Struve functions to be convex and starlike in the open unit disk.

1. Introduction and Preliminary Results

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian, and Kummer hypergeometric functions and the Bessel functions. Many authors have determined sufficient conditions on the parameters of these functions for belonging to a certain class of univalent functions, such as convex, starlike, and close-to-convex functions. More information about geometric properties of special functions can be found in [1–9]. In the present investigation our goal is to determine conditions of starlikeness and convexity of the generalized Struve functions. In order to achieve our goal in this section, we recall some basic facts and preliminary results.

Let $\mathscr{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathscr{U} = \{z : |z| < 1\}$. Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ which are univalent in $\mathscr{U}$. Also let $\mathscr{S}^*(\alpha)$ and $\mathscr{C}(\alpha)$ denote the subclasses of $\mathscr{A}$ consisting of functions which are, respectively, starlike and convex of order α in $\mathscr{U}$ $(0 \leq \alpha < 1)$. Thus, we have (see, for details, [10]),

$$\begin{gathered}\mathscr{S}^*(\alpha) = \left\{f : f \in \mathscr{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathscr{U};\ 0 \leq \alpha < 1)\right\},\\ \mathscr{C}(\alpha) = \left\{f : f \in \mathscr{A} \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathscr{U};\ 0 \leq \alpha < 1)\right\},\end{gathered} \tag{2}$$

where, for convenience,

$$\mathscr{S}^*(0) = \mathscr{S}^*, \qquad \mathscr{C}(0) = \mathscr{C}. \tag{3}$$

We remark that, according to the Alexander duality theorem [11], the function $f : \mathscr{U} \to \mathbb{C}$ is convex of order α, where $0 \leq \alpha < 1$ if and only if $z \to zf'(z)$ is starlike of order α. We note that every starlike (and hence convex) function of the form (1) is univalent. For more details we refer to the papers in [10, 12, 13] and the references therein.

Denote by $\mathscr{S}_1^*(\alpha)$, where $\alpha \in [0, 1)$, the subclass of $\mathscr{S}^*(\alpha)$ consisting of functions f for which

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha, \tag{4}$$

for all $z \in \mathcal{U}$. A function f is said to be in $\mathcal{C}_1(\alpha)$ if $zf' \in \mathcal{S}_1^*(\alpha)$.

Lemma 1 (see [4]). *If $f \in \mathcal{A}$ and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\beta} \left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < (1-\alpha)^{1-2\beta}\left(1 - \frac{3\alpha}{2} + \alpha^2\right)^{\beta}, \tag{5}$$

for some fixed $\alpha \in [0, 1/2]$ and $\beta \geq 0$, and for all $z \in \mathcal{U}$, then f is in the class $\mathcal{S}^*(\alpha)$.

Lemma 2 (see [14]). *Let $\alpha \in [0, 1)$. A sufficient condition for $f(z) = z + \sum_{n\geq 2} a_n z^n$ to be in $\mathcal{S}_1^*(\alpha)$ and $\mathcal{C}_1(\alpha)$, respectively, is that*

$$\begin{aligned} &\sum_{n\geq 2} (n-\alpha)\left|a_n\right| \leq 1-\alpha, \\ &\sum_{n\geq 2} n(n-\alpha)\left|a_n\right| \leq 1-\alpha, \end{aligned} \tag{6}$$

respectively.

Lemma 3 (see [14]). *Let $\alpha \in [0, 1)$. Suppose that $f(z) = z - \sum_{n\geq 2} a_n z^n$, $a_n \geq 0$. Then a necessary and sufficient condition for f to be in $\mathcal{S}_1^*(\alpha)$ and $\mathcal{C}_1(\alpha)$, respectively, is that*

$$\begin{aligned} &\sum_{n\geq 2} (n-\alpha)\left|a_n\right| \leq 1-\alpha, \\ &\sum_{n\geq 2} n(n-\alpha)\left|a_n\right| \leq 1-\alpha, \end{aligned} \tag{7}$$

respectively. In addition $f \in \mathcal{S}_1^*(\alpha) \Leftrightarrow f \in \mathcal{S}^*(\alpha)$, $f \in \mathcal{C}_1(\alpha) \Leftrightarrow f \in \mathcal{C}(\alpha)$, and $f \in \mathcal{S}^* \Leftrightarrow f \in \mathcal{S}$.

2. Starlikeness and Convexity of Generalized Struve Functions

Let us consider the second-order inhomogeneous differential equation [15, page 341]

$$z^2 w''(z) + zw'(z) + \left(z^2 - p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + 1/2)} \tag{8}$$

whose homogeneous part is Bessel's equation, where p is an unrestricted real (or complex) number. The function H_p, which is called the Struve function of order p, is defined as a particular solution of (8). This function has the form

$$H_p(z) = \sum_{n\geq 0} \frac{(-1)^n}{\Gamma(n + 3/2)\Gamma(p + n + 3/2)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in \mathbb{C}. \tag{9}$$

The differential equation

$$z^2 w''(z) + zw'(z) - \left(z^2 + p^2\right) w(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + 1/2)}, \tag{10}$$

which differs from (8) only in the coefficient of w. The particular solution of (10) is called the modified Struve function of order p and is defined by the formula [15, page 353]

$$\begin{aligned} L_p(z) &= -ie^{-ip\pi/2} H_p(iz) \\ &= \sum_{n\geq 0} \frac{1}{\Gamma(n + 3/2)\Gamma(p + n + 3/2)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in \mathbb{C}. \end{aligned} \tag{11}$$

Now, let us consider the second-order inhomogeneous linear differential equation [16],

$$\begin{aligned} &z^2 w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1-b)p\right] w(z) \\ &\quad = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p + b/2)}, \end{aligned} \tag{12}$$

where $b, c, p \in \mathbb{C}$. If we choose $b = 1$ and $c = 1$, then we get (8), and if we choose $b = 1$ and $c = -1$, then we get (10). So this generalizes (8) and (10). Moreover, this permits to study the Struve and modified Struve functions together. A particular solution of the differential equation (12), which is denoted by $w_{p,b,c}(z)$, is called the generalized Struve function [16] of order p. In fact we have the following series representation for the function $w_{p,b,c}(z)$:

$$\begin{aligned} &w_{p,b,c}(z) \\ &\quad = \sum_{n\geq 0} \frac{(-1)^n c^n}{\Gamma(n+3/2)\Gamma(p+n+(b+2)/2)}\left(\frac{z}{2}\right)^{2n+p+1}, \quad \forall z \in \mathbb{C}. \end{aligned} \tag{13}$$

Although the series defined in (13) is convergent everywhere, the function $w_{p,b,c}(z)$ is generally not univalent in $\mathcal{U}$. Now, consider the function $u_{p,b,c}(z)$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \sqrt{\pi}\Gamma\left(p + \frac{b+2}{2}\right) z^{(-p-1)/2} w_{p,b,c}\left(\sqrt{z}\right). \tag{14}$$

By using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda) = \lambda(\lambda+1)\cdots(\lambda+n-1)$, we obtain for the function $u_{p,b,c}(z)$ the following form:

$$\begin{aligned} u_{p,b,c}(z) &= \sum_{n\geq 0} \frac{(-c/4)^n}{(3/2)_n (\kappa)_n} z^n \\ &= b_0 + b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots, \end{aligned} \tag{15}$$

where $\kappa = p + (b+2)/2 \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order inhomogeneous differential equation

$$\begin{aligned} &4z^2 u''(z) + 2(2p + b + 3) zu'(z) \\ &\quad + (cz + 2p + b) u(z) = 2p + b. \end{aligned} \tag{16}$$

Orhan and Yağmur [16] have determined various sufficient conditions for the parameters p, b, and c such that the functions $u_{p,b,c}(z)$ or $z \to zu_{p,b,c}(z)$ to be univalent, starlike, convex, and close to convex in the open unit disk. In this section, our aim is to complete the above-mentioned results.

For convenience, we use the notations: $w_{p,b,c}(z) = w_p(z)$ and $u_{p,b,c}(z) = u_p(z)$.

Proposition 4 (see [16]). *If $b, c, p \in \mathbb{C}$, $\kappa = p + (b+2)/2 \neq 1, 0, -1, -2, \ldots$, and $z \in \mathbb{C}$, then for the generalized Struve function of order p the following recursive relations hold:*

(i) $zw_{p-1}(z) + czw_{p+1}(z) = (2\kappa - 3)w_p(z) + 2(z/2)^{p+1}/\sqrt{\pi}\Gamma(\kappa)$;

(ii) $zw'_p(z) + (p+b-1)w_p(z) = zw_{p-1}(z)$;

(iii) $zw'_p(z) + czw_{p+1}(z) = pw_p(z) + 2(z/2)^{p+1}/\sqrt{\pi}\Gamma(\kappa)$;

(iv) $[z^{-p}w_p(z)]' = -cz^{-p}w_{p+1}(z) + 1/2^p\sqrt{\pi}\Gamma(\kappa)$;

(v) $u_p(z) + 2zu'_p(z) + (cz/2\kappa)u_{p+1}(z) = 1$.

Theorem 5. *If the function u_p, defined by (15), satisfies the condition*

$$\left|\frac{zu'_p(z)}{u_p(z)}\right| < 1 - \alpha, \tag{17}$$

where $\alpha \in [0, 1/2]$ and $z \in \mathcal{U}$, then $zu_p \in \mathcal{S}^*(\alpha)$.

Proof. If we define the function $g : \mathcal{U} \to \mathbb{C}$ by $g(z) = zu_p(z)$ for $z \in \mathcal{U}$. The given condition becomes

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1 - \alpha, \tag{18}$$

where $z \in \mathcal{U}$. By taking $\beta = 0$ in Lemma 1, we thus conclude from the previous inequality that $g \in \mathcal{S}^*(\alpha)$, which proves Theorem 5. □

Theorem 6. *If the function u_p, defined by (15), satisfies the condition*

$$\left|\frac{zu''_p(z)}{u'_p(z)}\right| < \frac{1 - 3\alpha/2 + \alpha^2}{1-\alpha}, \tag{19}$$

where $\alpha \in [0, 1/2]$ and $z \in \mathcal{U}$, then it is starlike of order α with respect to 1.

Proof. Define the function $h : \mathcal{U} \to \mathbb{C}$ by $h(z) = [u_p(z) - b_0]/b_1$. Then $h \in \mathcal{A}$ and

$$\left|\frac{zh''(z)}{h'(z)}\right| = \left|\frac{zu''_p(z)}{u'_p(z)}\right| < \frac{1 - 3\alpha/2 + \alpha^2}{1-\alpha}, \tag{20}$$

where $\alpha \in [0, 1/2]$ and $z \in \mathcal{U}$. By taking $\beta = 1$ in Lemma 1, we deduce that $h \in \mathcal{S}^*(\alpha)$; that is, h is starlike of order α with respect to the origin for $\alpha \in [0, 1/2]$. So, Theorem 6 follows from the definition of the function h, because $b_0 = 1$. □

Theorem 7. *If for $\alpha \in [0, 1/2]$ and $c \neq 0$ one has*

$$\left|\frac{zu'_{p+1}(z)}{u_{p+1}(z)}\right| < 1 - \alpha, \tag{21}$$

for all $z \in \mathcal{U}$, then $u_p + 2zu'_p$ is starlike of order α with respect to 1.

Proof. Theorem 5 implies that $zu_{p+1} \in \mathcal{S}^*(\alpha)$. On the other hand, the part (v) of Proposition 4 yields

$$u_p(z) + 2zu'_p(z) = \frac{-c}{2\kappa}zu_{p+1}(z) + 1. \tag{22}$$

Since the addition of any constant and the multiplication by a nonzero quantity do not disturb the starlikeness. This completes the proof. □

Lemma 8. *If $b, p \in \mathbb{R}$, $c \in \mathbb{C}$, and $\kappa = p + (b+2)/2$ such that $\kappa > |c|/2$, then the function $u_p : \mathcal{U} \to \mathbb{C}$ satisfies the following inequalities:*

$$\frac{6\kappa - 2|c|}{6\kappa - |c|} \le |u_p(z)| \le \frac{6\kappa}{6\kappa - |c|}, \tag{23}$$

$$\frac{|c|(2\kappa - |c|)}{3\kappa(4\kappa - |c|)} \le |u'_p(z)| \le \frac{2|c|}{3(4\kappa - |c|)}, \tag{24}$$

$$|zu''_p(z)| \le \frac{|c|^2}{4\kappa(4\kappa - |c|)}. \tag{25}$$

Proof. We first prove the assertion (23) of Lemma 8. Indeed, by using the well-known triangle inequality:

$$|z_1 + z_2| \le |z_1| + |z_2|, \tag{26}$$

and the inequalities $(3/2)_n \ge (3/2)^n$, $(\kappa)_n \ge \kappa^n$ $(n \in \mathbb{N})$, we have

$$\begin{aligned} |u_p(z)| &= \left|1 + \sum_{n\ge1}\frac{(-c/4)^n}{(3/2)_n(\kappa)_n}z^n\right| \\ &\le 1 + \sum_{n\ge1}\left(\frac{|-c/4|}{(3/2)\kappa}\right)^n \\ &= 1 + \frac{|c|}{6\kappa}\sum_{n\ge1}\left(\frac{|c|}{6\kappa}\right)^{n-1} \\ &= \frac{6\kappa}{6\kappa - |c|}, \quad \left(\kappa > \frac{|c|}{6}\right). \end{aligned} \tag{27}$$

Similarly, by using reverse triangle inequality:

$$|z_1 - z_2| \ge ||z_1| - |z_2||, \tag{28}$$

and the inequalities $(3/2)_n \geq (3/2)^n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, then we get

$$\begin{aligned}\left|u_p(z)\right| &= \left|1+\sum_{n\geq 1}\frac{(-c/4)^n}{(3/2)_n(\kappa)_n}z^n\right| \\ &\geq 1-\sum_{n\geq 1}\left(\frac{|-c/4|}{(3/2)\kappa}\right)^n \\ &= 1-\frac{|c|}{6\kappa}\sum_{n\geq 1}\left(\frac{|c|}{6\kappa}\right)^{n-1} \\ &= \frac{6\kappa-2|c|}{6\kappa-|c|}, \quad \left(\kappa>\frac{|c|}{6}\right),\end{aligned} \tag{29}$$

which is positive if $\kappa > |c|/3$.

In order to prove assertion (24) of Lemma 8, we make use of the well-known triangle inequality and the inequalities $(3/2)_n \geq (3/2)n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, and we obtain

$$\begin{aligned}\left|u_p'(z)\right| &= \left|\sum_{n\geq 1}\frac{n(-c/4)^n}{(3/2)_n(\kappa)_n}z^{n-1}\right| \\ &\leq \frac{2}{3}\sum_{n\geq 1}\left(\frac{|c|}{4\kappa}\right)^n \\ &= \frac{2}{3}\frac{|c|}{4\kappa}\sum_{n\geq 1}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{2|c|}{3(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right).\end{aligned} \tag{30}$$

Similarly, by using the reverse triangle inequality and the inequalities $(3/2)_n \geq (3/2)n$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, we have

$$\begin{aligned}\left|u_p'(z)\right| &= \left|\sum_{n\geq 1}\frac{n(-c/4)^n}{(3/2)_n(\kappa)_n}z^{n-1}\right| \\ &\geq \frac{|c|}{6\kappa}-\frac{2}{3}\left(\frac{|c|}{4\kappa}\right)^2\sum_{n\geq 2}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{|c|(2\kappa-|c|)}{3\kappa(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right),\end{aligned} \tag{31}$$

which is positive if $\kappa > |c|/2$.

We now prove assertion (25) of Lemma 8 by using again the triangle inequality and the inequalities $(3/2)_n \geq n(n-1)$, $(\kappa)_n \geq \kappa^n$ $(n \in \mathbb{N})$, and we arrive at the following:

$$\begin{aligned}\left|zu_p''(z)\right| &= \left|\sum_{n\geq 2}\frac{n(n-1)(-c/4)^n}{(3/2)_n(\kappa)_n}z^{n-1}\right| \\ &\leq \frac{|c|}{4\kappa}\sum_{n\geq 2}\left(\frac{|c|}{4\kappa}\right)^{n-1} \\ &= \frac{|c|^2}{4\kappa(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right).\end{aligned} \tag{32}$$

Thus, the proof of Lemma 8 is completed. □

Theorem 9. *If $b, p \in \mathbb{R}$, $c \in \mathbb{C}$ and $\kappa = p + (b+2)/2$, then the following assertions are true.*

(i) *If $\kappa > (7/8)|c|$, then $u_p(z)$ is convex in $\mathcal{U}$.*

(ii) *If $\kappa > ((11+\sqrt{41})/24)|c|$, then $zu_p(z)$ is starlike of order $1/2$ in $\mathcal{U}$, and consequently the function $z \to z^{-p}w_p(z)$ is starlike in $\mathcal{U}$.*

(iii) *If $\kappa > ((11+\sqrt{41})/24)|c| - 1$, then the function $z \to u_p(z) + 2zu_p'(z)$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathcal{U}$.*

Proof. (i) By combining the inequalities (24) with (25), we immediately see that

$$\left|\frac{zu_p''(z)}{u_p'(z)}\right| \leq \frac{3|c|}{4(2\kappa-|c|)}. \tag{33}$$

So, for $\kappa > \left(\frac{7}{8}\right)|c|$, we have

$$\left|\frac{zu_p''(z)}{u_p'(z)}\right| < 1. \tag{34}$$

This shows $u_p(z)$ is convex in $\mathcal{U}$.

(ii) If we let $g(z) = zu_p(z)$ and $h(z) = zu_p(z^2)$, then

$$\begin{aligned}h(z) &= \frac{g(z^2)}{z} = 2^p\sqrt{\pi}\Gamma(\kappa)z^{-p}w_{p,b,c}(z), \\ \frac{zh'(z)}{h(z)}-1 &= 2\left[\frac{z^2g'(z^2)}{g(z^2)}-1\right] = 2\frac{z^2u_p'(z^2)}{u_p(z^2)},\end{aligned} \tag{35}$$

so that

$$\left|\frac{zh'(z)}{h(z)}-1\right| < 1, \quad \forall z \in \mathcal{U}, \tag{36}$$

if and only if

$$\left|\frac{z^2u_p'(z^2)}{u_p(z^2)}\right| < \frac{1}{2}, \quad \forall z \in \mathcal{U}. \tag{37}$$

It follows that $zu_p(z)$ is starlike of order $1/2$ if (37) holds. From (24) and (23), we have

$$\left|z^2u_p'(z^2)\right| \leq \frac{2|c|}{3(4\kappa-|c|)}, \quad \left(\kappa>\frac{|c|}{4}\right), \tag{38}$$

$$\frac{6\kappa-2|c|}{6\kappa-|c|} \leq \left|u_p(z^2)\right|, \quad \left(\kappa>\frac{|c|}{3}\right), \tag{39}$$

respectively.

By combining the inequalities (38) with (39), we see that

$$\left|\frac{z^2u_p'(z^2)}{u_p(z^2)}\right| \leq \frac{|c|(6\kappa-|c|)}{3(3\kappa-|c|)(4\kappa-|c|)}, \tag{40}$$

where $\kappa > |c|/3$, and the above bound is less than or equal to $1/2$ if and only if $\kappa > ((11+\sqrt{41})/24)|c|$. It follows that zu_p is starlike of order $1/2$ in $\mathcal{U}$ and $z^{-p}w_{p,b,c}$ is starlike in $\mathcal{U}$.

(iii) The part (ii) of Theorem 9 implies that for $\kappa > ((11+\sqrt{41})/24)|c|-1$, the function $z \to zu_{p+1}(z)$ is starlike of order $1/2$ in $\mathcal{U}$. On the other hand, the part (v) of Proposition 4 yields

$$u_p(z) + 2zu_p'(z) = \frac{-c}{2\kappa} zu_{p+1}(z) + 1. \tag{41}$$

So the function $z \to u_p(z) + 2zu_p'(z)$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathcal{U}$.

This completes the proof. □

Struve Functions. Choosing $b = c = 1$, we obtain the differential equation (8) and the Struve function of order p, defined by (9), satisfies this equation. In particular, the results of Theorem 9 are as follows.

Corollary 10. *Let $\mathscr{H}_p : \mathcal{U} \to \mathbb{C}$ be defined by $\mathscr{H}_p(z) = 2^p\sqrt{\pi}\Gamma(p+3/2)z^{-p-1}H_p(z) = u_{p,1,1}(z^2)$, where H_p stands for the Struve function of order p. Then the following assertions are true.*

(i) *If $p > -5/8$, then $\mathscr{H}_p(z^{1/2})$ is convex in $\mathcal{U}$.*

(ii) *If $p > (-25+\sqrt{41})/24$, then $z\mathscr{H}_p(z^{1/2})$ is starlike of order $1/2$ in $\mathcal{U}$, and consequently the function $z \to z^{-p}H_p(z)$ is starlike in $\mathcal{U}$.*

(iii) *If $p > (-49+\sqrt{41})/24$, then the function $z \to \mathscr{H}_p(z^{1/2}) + 2z\mathscr{H}_p'(z^{1/2})$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathcal{U}$.*

Modified Struve Functions. Choosing $b = 1$ and $c = -1$, we obtain the differential equation (10) and the modified Struve function of order p, defined by (11). For the function $\mathscr{L}_p : \mathcal{U} \to \mathbb{C}$ defined by $\mathscr{L}_p(z) = 2^p\sqrt{\pi}\Gamma(p+3/2)z^{-p-1}L_p(z) = u_{p,1,-1}(z^2)$, where L_p stands for the modified Struve function of order p. The properties are same like for function $\mathscr{H}_p$, because we have $|c| = 1$. More precisely, we have the following results.

Corollary 11. *The following assertions are true.*

(i) *If $p > -5/8$, then $\mathscr{L}_p(z^{1/2})$ is convex in $\mathcal{U}$.*

(ii) *If $p > (-25+\sqrt{41})/24$, then $z\mathscr{L}_p(z^{1/2})$ is starlike of order $1/2$ in $\mathcal{U}$, and consequently the function $z \to z^{-p}L_p(z)$ is starlike in $\mathcal{U}$.*

(iii) *If $p > (-49+\sqrt{41})/24$, then the function $z \to \mathscr{L}_p(z^{1/2}) + 2z\mathscr{L}_p'(z^{1/2})$ is starlike of order $1/2$ with respect to 1 for all $z \in \mathcal{U}$.*

Example 12. If we take $p = -1/2$, then from part (ii) of Corollary 10, the function $z \to z^{1/2}H_{-1/2}(z) = \sqrt{2/\pi}\sin z$ is starlike in $\mathcal{U}$. So the function $f(z) = \sin z$ is also starlike in $\mathcal{U}$. We have the image domain of $f(z) = \sin z$ illustrated by Figure 1.

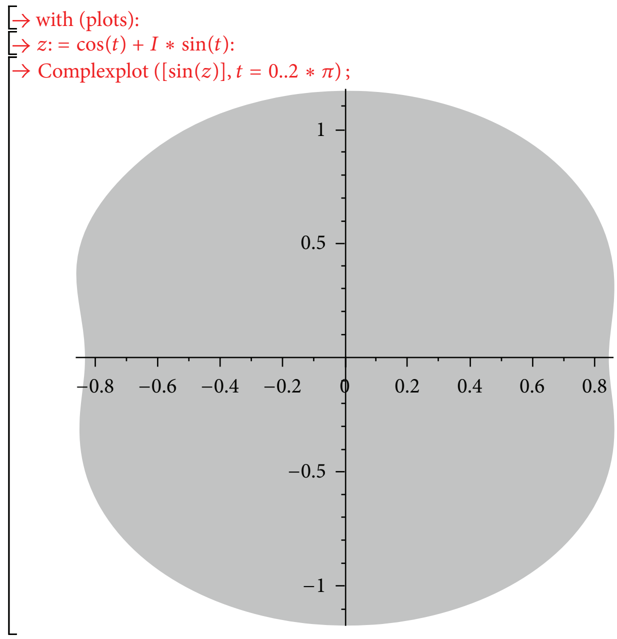

FIGURE 1: $f(z) = \sin z$.

Theorem 13. *If $\alpha \in [0,1)$, $c < 0$, and $\kappa > 0$, then a sufficient condition for zu_p to be in $\mathcal{S}_1^*(\alpha)$ is*

$$u_p(1) + \frac{u_p'(1)}{1-\alpha} \le 2. \tag{42}$$

Moreover, (42) is necessary and sufficient for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{S}_1^*(\alpha)$.

Proof. Since $zu_p(z) = z + \sum_{n\ge2} b_{n-1}z^n$, according to Lemma 2, we need only show that

$$\sum_{n\ge2}(n-\alpha)b_{n-1} \le 1-\alpha. \tag{43}$$

We notice that

$$\begin{aligned}\sum_{n\ge2}(n-\alpha)b_{n-1} &= \sum_{n\ge2}(n-1)b_{n-1} + \sum_{n\ge2}(1-\alpha)b_{n-1}\\ &= \sum_{n\ge2}\frac{(n-1)(-c/4)^{n-1}}{(3/2)_{n-1}(\kappa)_{n-1}} + (1-\alpha)\left[u_p(1)-1\right]\\ &= u_p'(1) + (1-\alpha)\left[u_p(1)-1\right].\end{aligned} \tag{44}$$

This sum is bounded above by $1-\alpha$ if and only if (42) holds. Since

$$z\left[2-u_p(z)\right] = z - \sum_{n\ge2} b_{n-1}z^n, \tag{45}$$

the necessity of (42) for ψ to be in $\mathcal{S}_1^*(\alpha)$ follows from Lemma 3. □

Corollary 14. *If $c < 0$ and $\kappa > 0$, then a sufficient condition for zu_p to be in $\mathcal{S}_1^*(1/2)$ is*

$$u_{p+1}(1) \le -\frac{2\kappa}{c}. \tag{46}$$

Moreover, (46) *is necessary and sufficient for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{S}_1^*(1/2)$.*

Proof. For $\alpha = 1/2$, the condition (42) becomes $u_p(1) + 2u_p'(1) \le 2$. From the part (v) of Proposition 4 we get

$$u_p(1) + 2zu_p'(1) = 1 - \frac{c}{2\kappa}u_{p+1}(1). \tag{47}$$

So, $u_p(1) + 2u_p'(1) \le 2$ if and only if $1 - (c/2\kappa)u_{p+1}(1) \le 2$. Thus, we obtain the condition (46).

Furthermore, from the proof of Theorem 13, we have necessary and sufficient condition for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{S}_1^*(1/2)$. □

Theorem 15. *If $\alpha \in [0,1)$, $c < 0$ and $\kappa > 0$, then a sufficient condition for zu_p to be in $\mathcal{C}_1(\alpha)$ is*

$$u_p''(1) + (3-\alpha)u_p'(1) + (1-\alpha)u_p(1) - 2\alpha \le 2. \tag{48}$$

Moreover, (48) *is necessary and sufficient for $\psi(z) = z[2 - u_p(z)]$ to be in $\mathcal{C}_1(\alpha)$.*

Proof. In view of Lemma 2, we need only to show that

$$\sum_{n\ge2} n(n-\alpha)b_{n-1} \le 1-\alpha. \tag{49}$$

If we let $g(z) = zu_p(z)$, we notice that

$$\begin{aligned}\sum_{n\ge2} n(n-\alpha)b_{n-1} &= \sum_{n\ge2} n(n-1)b_{n-1} + (1-\alpha)\sum_{n\ge2} nb_{n-1}\\ &= g''(1) + (1-\alpha)\left[g'(1) - 1\right]\\ &= u_p''(1) + (3-\alpha)u_p'(1) + (1-\alpha)u_p(1) - 1 + \alpha.\end{aligned} \tag{50}$$

This sum is bounded above by $1-\alpha$ if and only if (48) holds. Lemma 3 implies that (48) is also necessary for ψ to be in $\mathcal{C}_1(\alpha)$. □

Theorem 16. *If $c < 0$, $\kappa > 0$, and $u_p(1) \le 2$, then $\int_0^z u_p(t)dt \in \mathcal{S}^*$.*

Proof. Since

$$\int_0^z u_p(t)\,dt = \sum_{n\ge0}\frac{b_n}{n+1}z^{n+1} = z + \sum_{n\ge2}\frac{b_{n-1}}{n}z^n, \tag{51}$$

we note that

$$\sum_{n\ge2} n\frac{b_{n-1}}{n} = \sum_{n\ge2} b_{n-1} = u_p(1) - 1 \le 1, \tag{52}$$

if and only if $u_p(1) \le 2$. □

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