

## Research Article

# Representation of a Solution of the Cauchy Problem for an Oscillating System with Multiple Delays and Pairwise Permutable Matrices

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Nonhomogeneous system of linear differential equations of second order with multiple different delays and pairwise permutable matrices defining the linear parts is considered. Solution of corresponding initial value problem is represented using matrix polynomials.

## 1. Introduction

Motivated by delayed exponential representing a solution of a system of differential or difference equations with one or multiple fixed or variable delays [1–6], which has many applications in theory of controllability, asymptotic properties, boundary-value problems, and so forth [3–5, 7–15], we extended representation of a solution of a system of differential equations of second order with delay [1]

$$\ddot{x}(t) = -B^2 x(t - \tau) \quad (1)$$

to the case of two delays

$$\ddot{x}(t) = -B_1^2 x(t - \tau_1) - B_2^2 x(t - \tau_2), \quad (2)$$

where the linear parts were given by permutable matrices [16]. Equations (1), (2), and the below-stated (11) with  $f \equiv 0$  are generalizations of the scalar equation

$$\ddot{x}(t) = -b^2 x(t) \quad (3)$$

representing linear oscillator, to  $N$ -dimensional space with one or multiple fixed delays. Clearly, each solution of the latter equation is oscillating whenever  $0 \neq b \in \mathbb{R}$ . Analogically, (1) with  $x \in \mathbb{R}^N$  can have at least one oscillating solution whenever  $N$  is odd. Indeed, if  $B$  is  $N \times N$  matrix,  $N \geq 3$  is odd, and  $B$  has a simple real nonzero eigenvalue  $\lambda$ , then there exists a regular matrix  $S$  such that  $S^{-1}BS = J = \begin{pmatrix} \lambda & 0 \\ 0 & \tilde{J} \end{pmatrix}$  where  $\tilde{J}$  is  $(N - 1) \times (N - 1)$  matrix. On letting  $x = Sy$ , one gets

$$\ddot{y} = -J^2 y(t - \tau) \quad (4)$$

or rewrites as the system

$$\begin{aligned} \ddot{y}_1 &= -\lambda^2 y_1(t - \tau), \\ \ddot{y}_2 &= -\tilde{J}^2 y_2(t - \tau), \end{aligned} \quad (5)$$

where  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{N-1}$ . Note that the first column  $v$  of  $S$  is the eigenvector of  $B$  corresponding to  $\lambda$ . Clearly, if solution  $y_1$  of (5) is oscillating, then solution  $y$  of (4) is oscillating in the first coordinate whenever its initial

condition satisfies  $\{y(t) \mid t \in [-\tau, 0]\} \subset \mathbb{R} \times \{0\}^{N-1}$ . Consequently, solution  $x$  of (1) is oscillating in  $\text{span}\{v\}$  whenever  $\{x(t) \mid t \in [-\tau, 0]\} \subset \text{span}\{v\}$ . Taking  $y_1(t) = e^{\mu t}$ , one obtains characteristic equation  $\mu^2 = -\lambda^2 e^{-\mu\tau}$  of (5), which has solutions  $\mu_{1,2} = \alpha \pm i\beta \in \mathbb{C}$  with  $\beta \neq 0$ . Thus,  $y_1$  is oscillating.

On the other hand, there can exist a nonoscillating solution of the system (1) whenever  $x \in \mathbb{R}^N$  and  $N$  is even. For instance, if  $N = 2$  and  $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , then (1) has the form

$$\ddot{x}(t) = x(t - \tau) \tag{6}$$

with  $x \in \mathbb{R}^2$ , which, obviously, does not have an oscillating solution satisfying nonoscillating initial condition. Similarly, it can be shown that system with odd dimension can possess a nonoscillating solution satisfying an appropriate initial condition.

For simplicity, we call the generalizations (1), (2), and (11) with  $f \equiv 0$ , of scalar equation (3), *oscillating* although their solutions do not always have to be oscillating. Nevertheless, at the end of this paper, in Corollary 8 we state the representation of a solution of more general system (86) without squares of matrices.

We note that the delayed matrix exponential from [1–5] as well as the representation of a solution of second-order differential equations derived in [1, 16] and in this paper can lead to new results in nonlinear boundary value problems for impulsive functional differential equations considered in [17] or stochastic delayed differential equations from [18].

So, in the present paper, we extend our result from [16] to three and more delays by the assumption of pairwise permutable matrices defining linear parts. By such an assumption, we are able to construct matrix functions solving homogeneous system of differential equations of second order with any number of fixed delays, and, consequently, we use these functions to represent a solution of the corresponding nonhomogeneous initial value problem. As will be shown in the next sections, extending from two to more delays brings many technical difficulties, for example, the use of multinomial coefficients. Naturally, the results of the present paper hold with one or two different delays as well. However, these cases can be studied in a simpler way, which was already done in [1, 16]. Thus, we focus our attention on the case of three and more different delays.

First, we recall our result from [16].

**Theorem 1.** *Let  $\tau_1, \tau_2 > 0$ ,  $\tau := \max\{\tau_1, \tau_2\}$ , and  $\varphi \in C^1([-\tau, 0], \mathbb{R}^N)$ . Let  $B_1, B_2$  be  $N \times N$  permutable matrices; that is,  $B_1 B_2 = B_2 B_1$ , and let  $f : [0, \infty) \rightarrow \mathbb{R}^N$  be a given function. Solution  $x(t)$  of*

$$\ddot{x}(t) = -B_1^2 x(t - \tau_1) - B_2^2 x(t - \tau_2) + f(t) \tag{7}$$

satisfying initial condition

$$\begin{aligned} x(t) &= \varphi(t), \\ \dot{x}(t) &= \dot{\varphi}(t), \end{aligned} \quad -\tau \leq t \leq 0 \tag{8}$$

has the form

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ \mathcal{X}(t)\varphi(0) + \mathcal{Y}(t)\dot{\varphi}(0) \\ -B_1^2 \int_0^0 \mathcal{Y}(t - \tau_1 - s)\varphi(s) ds \\ -B_2^2 \int_0^{-\tau_1} \mathcal{Y}(t - \tau_2 - s)\varphi(s) ds \\ + \int_0^t \mathcal{Y}(t - s)f(s) ds, & 0 \leq t, \end{cases} \tag{9}$$

where

$$\begin{aligned} \mathcal{X}(t) &= \mathcal{X}_{\tau_1, \tau_2}^{B_1^2, B_2^2}(t) \\ &:= \sum_{\substack{i, j \geq 0 \\ i\tau_1 + j\tau_2 \leq t}} (-1)^{i+j} \\ &\quad \times \binom{i+j}{i} B_1^{2i} B_2^{2j} \frac{(t - i\tau_1 - j\tau_2)^{2(i+j)}}{(2(i+j))!}, \end{aligned} \tag{10}$$

$$\begin{aligned} \mathcal{Y}(t) &= \mathcal{Y}_{\tau_1, \tau_2}^{B_1^2, B_2^2}(t) \\ &:= \sum_{\substack{i, j \geq 0 \\ i\tau_1 + j\tau_2 \leq t}} (-1)^{i+j} \\ &\quad \times \binom{i+j}{i} B_1^{2i} B_2^{2j} \frac{(t - i\tau_1 - j\tau_2)^{2(i+j)+1}}{(2(i+j)+1)!}. \end{aligned}$$

We will denote  $\Theta$  and  $E$  the  $N \times N$  zero and identity matrix, respectively.

## 2. Systems with Multiple Delays

In this section, we derive the representation of a solution of

$$\ddot{x}(t) = -B_1^2 x(t - \tau_1) - \dots - B_n^2 x(t - \tau_n) + f(t) \tag{11}$$

satisfying the initial condition (8), where  $n \geq 3$ ,  $\tau_1, \dots, \tau_n > 0$ ,  $\tau := \max_{i=1, \dots, n} \tau_i$ ,  $B_1, \dots, B_n$  are  $N \times N$  pairwise permutable matrices; that is,  $B_i B_j = B_j B_i$  for each  $i, j \in \{1, \dots, n\}$ ,  $\varphi \in C^1([-\tau, 0], \mathbb{R}^N)$ , and  $f : [0, \infty) \rightarrow \mathbb{R}^N$  are given functions. The solution  $x(t)$  will be represented using matrix functions analogical to (10) and will be stated in Section 3. We note that the same problems with  $n = 1, 2$  were studied in [1, 16].

From now on, we assume the property of empty sum and empty product; that is,

$$\begin{aligned} \sum_{i \in \emptyset} f(i) &= 0, & \sum_{i \in \emptyset} F(i) &= \Theta, \\ \prod_{i \in \emptyset} f(i) &= 1, & \prod_{i \in \emptyset} F(i) &= E \end{aligned} \tag{12}$$

for any function  $f$  and matrix function  $F$ , whether they are defined or not for indicated argument.

We recall that  $(j_1, \dots, j_n)!$  is a multinomial coefficient [19] given by

$$(j_1, \dots, j_n)! = \frac{(j_1 + \dots + j_n)!}{j_1! \dots j_n!}. \quad (13)$$

Note that if  $n = 2$ , then  $(j_1, j_2) = \binom{j_1 + j_2}{j_1}$  and (20) coincides with (10).

We will need a property of multinomial coefficients described in the next lemma.

**Lemma 2.** *Let  $n \geq 2$  be fixed. Then*

$$\begin{aligned} (i_1, i_2, \dots, i_n)! &= (i_1 - 1, i_2, \dots, i_n)! \\ &+ (i_1, i_2 - 1, i_3, \dots, i_n)! + (i_1, \dots, i_{n-1}, i_n - 1)! \end{aligned} \quad (14)$$

for any  $i_1, \dots, i_n \geq 1$ .

*Proof.* If  $n = 2$ , then the statement follows from the property of binomial coefficients:

$$\begin{aligned} (i_1, i_2)! &= \binom{i_1 + i_2}{i_1} = \binom{i_1 + i_2 - 1}{i_1 - 1} + \binom{i_1 + i_2 - 1}{i_1} \\ &= (i_1 - 1, i_2)! + (i_1, i_2 - 1)!. \end{aligned} \quad (15)$$

Let the statement be true for  $n - 1$ . Next, we use the property of multinomial coefficient

$$(i_1, i_2, i_3, \dots, i_n)! = (i_1 + i_2, i_3, \dots, i_n)! (i_1, i_2)! \quad (16)$$

with inductive hypothesis to derive

$$\begin{aligned} (i_1, i_2, i_3, \dots, i_n)! &= [(i_1 + i_2 - 1, i_3, \dots, i_n)! + (i_1 + i_2, i_3 - 1, \dots, i_n)! \\ &+ \dots + (i_1 + i_2, i_3, \dots, i_n - 1)!] (i_1, i_2)!. \end{aligned} \quad (17)$$

Clearly, from (16), we get

$$\begin{aligned} (i_1 + i_2, i_3 - 1, \dots, i_n)! (i_1, i_2)! &= (i_1, i_2, i_3 - 1, \dots, i_n)!, \\ &\vdots \end{aligned} \quad (18)$$

$$(i_1 + i_2, i_3, \dots, i_n - 1)! (i_1, i_2)! = (i_1, i_2, i_3, \dots, i_n - 1)!.$$

Applying the case  $n = 2$  (property of binomial coefficient) and (16), we get

$$\begin{aligned} (i_1 + i_2 - 1, i_3, \dots, i_n)! (i_1, i_2)! &= (i_1 + i_2 - 1, i_3, \dots, i_n)! [(i_1 - 1, i_2)! + (i_1, i_2 - 1)!] \\ &= (i_1 - 1, i_2, i_3, \dots, i_n)! + (i_1, i_2 - 1, i_3, \dots, i_n)!. \end{aligned} \quad (19)$$

Putting (18) and (19) in (17), we obtain that the statement holds for  $n$  and the proof is complete.  $\square$

In further work, we write  $(\{j \mid j \in M\})!$  for the multinomial coefficient of elements of the finite set  $M$ , and  $(i, \{j \mid j \in M\})!$  for the multinomial coefficient of  $i$  and elements of the finite set  $M$ ; for example, if  $M = \{1, 2\}$ , then  $(a, \{j \mid j \in M\})! = (a, 1, 2)!$ . For the completeness, we define  $(\{j \mid j \in \emptyset\})! := 1$ .

Define the functions  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}, \mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2} : \mathbb{R} \rightarrow L(\mathbb{R}^N)$  as

$$\begin{aligned} \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) &:= \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 \tau_1 + \dots + j_n \tau_n \leq t}} (-1)^{j_1 + \dots + j_n} (j_1, \dots, j_n)! \\ &\times \prod_{i=1}^n B_i^{2j_i} \frac{(t - j_1 \tau_1 - \dots - j_n \tau_n)^{2(j_1 + \dots + j_n)}}{(2(j_1 + \dots + j_n))!}, \end{aligned} \quad (20)$$

$$\begin{aligned} \mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) &:= \sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 \tau_1 + \dots + j_n \tau_n \leq t}} (-1)^{j_1 + \dots + j_n} (j_1, \dots, j_n)! \\ &\times \prod_{i=1}^n B_i^{2j_i} \frac{(t - j_1 \tau_1 - \dots - j_n \tau_n)^{2(j_1 + \dots + j_n) + 1}}{(2(j_1 + \dots + j_n) + 1)!} \end{aligned}$$

for any  $t \in \mathbb{R}$ .

We will need functions  $\mathcal{X}_\tau^{B^2}, \mathcal{Y}_\tau^{B^2} : \mathbb{R} \rightarrow L(\mathbb{R}^N)$  for  $\tau > 0$  and  $N \times N$  complex matrix  $B$  (cf. [16]) defined as

$$\mathcal{X}_\tau^{B^2}(t) := \sum_{\substack{i \geq 0 \\ i\tau \leq t}} (-1)^i B^{2i} \frac{(t - i\tau)^{2i}}{(2i)!}, \quad (21)$$

$$\mathcal{Y}_\tau^{B^2}(t) := \sum_{\substack{i \geq 0 \\ i\tau \leq t}} (-1)^i B^{2i} \frac{(t - i\tau)^{2i+1}}{(2i+1)!}$$

with the properties

$$\begin{aligned} \dot{\mathcal{X}}_\tau^{B^2}(t) &= -B^2 \mathcal{Y}_\tau^{B^2}(t - \tau), & \dot{\mathcal{Y}}_\tau^{B^2}(t) &= -B^2 \mathcal{X}_\tau^{B^2}(t - \tau), \\ \ddot{\mathcal{X}}_\tau^{B^2}(t) &= \mathcal{X}_\tau^{B^2}(t), & \ddot{\mathcal{Y}}_\tau^{B^2}(t) &= -B^2 \mathcal{Y}_\tau^{B^2}(t - \tau) \end{aligned} \quad (22)$$

for any  $t \in \mathbb{R}$ , considering the one-sided derivatives at  $-\tau, 0$ .

Some of properties of functions  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}$  and  $\mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}$  are concluded in Lemma 4, but to prove it we will need the next lemma.

**Lemma 3.** *Let  $n \geq 1$  and  $\tau_1, \dots, \tau_n > 0$ . Let  $B_1, \dots, B_n$  be  $N \times N$  pairwise permutable matrices, that is,  $B_i B_j = B_j B_i$  for each  $i, j \in \{1, \dots, n\}$ . Then for any  $t \in \mathbb{R}$ ,*

$$\begin{aligned} \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) &= \sum_{M \subset \{1, \dots, n\}} S_M(t), \\ \mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) &= \sum_{M \subset \{1, \dots, n\}} \tilde{S}_M(t), \end{aligned} \quad (23)$$

where the sums are taken over all subsets of  $\{1, \dots, n\}$  including the trivial ones, and

$$S_M(t) := \sum_{\substack{j_i \geq 1, i \in M \\ \sum_{i \in M} j_i \tau_i \leq t}} (-1)^{\sum_{i \in M} j_i} (\{j_i \mid i \in M\})! \times \prod_{i \in M} B_i^{2j_i} \frac{(t - \sum_{i \in M} j_i \tau_i)^{2 \sum_{i \in M} j_i}}{(2 \sum_{i \in M} j_i)!}, \tag{24}$$

$$\tilde{S}_M(t) := \sum_{\substack{j_i \geq 1, i \in M \\ \sum_{i \in M} j_i \tau_i \leq t}} (-1)^{\sum_{i \in M} j_i} (\{j_i \mid i \in M\})! \times \prod_{i \in M} B_i^{2j_i} \frac{(t - \sum_{i \in M} j_i \tau_i)^{2 \sum_{i \in M} j_i + 1}}{(2 \sum_{i \in M} j_i + 1)!}. \tag{25}$$

*Proof.* Denote  $\mathbb{N}_0, \mathbb{N}$  the set of all nonnegative, positive integers, respectively; that is,  $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ . Thus, we have the trivial identity

$$\underbrace{\mathbb{N}_0 \times \dots \times \mathbb{N}_0}_n = \left( \{0\} \times \underbrace{\mathbb{N}_0 \times \dots \times \mathbb{N}_0}_{n-1} \right) \cup \left( \mathbb{N} \times \underbrace{\mathbb{N}_0 \times \dots \times \mathbb{N}_0}_{n-1} \right) = \dots = \bigcup_{M_1, \dots, M_n \in \{\{0\}, \mathbb{N}\}} M_1 \times \dots \times M_n. \tag{26}$$

Analogically, for any  $t \in \mathbb{R}$  each  $n$ -tuple  $j_1, \dots, j_n \geq 0$  such that  $\sum_{i=1}^n j_i \tau_i \leq t$  can be divided in two distinct sets of  $i$ -s so that  $j_i \geq 1$  if  $i \in M \subset \{1, \dots, n\}$  and  $j_i = 0$  if  $i \in \{1, \dots, n\} \setminus M$ . That is,  $M$  denotes the set of all indices  $i$  such that  $j_i = 0$ . Moreover,  $\sum_{i=1}^n j_i \tau_i = \sum_{i \in M} j_i \tau_i$ . Accordingly, we can write

$$\left\{ (j_1, \dots, j_n) \in \mathbb{N}_0^n \mid \sum_{i=1}^n j_i \tau_i \leq t \right\} = \bigcup_{M \subset \{1, \dots, n\}} \left\{ (j_1, \dots, j_n) \in \mathbb{N}_0^n \mid j_i = 0 \ \forall i \notin M, \sum_{i \in M} j_i \tau_i \leq t \right\}, \tag{27}$$

where the union is taken over all subsets of  $\{1, \dots, n\}$  including the trivial ones. So, in the view of definition (20), the statement for  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}$  follows.

Statement for  $\mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}$  can be proved in a similar way.  $\square$

**Lemma 4.** Let  $n \geq 3$  and  $\tau_1, \dots, \tau_n > 0$ . Let  $B_1, \dots, B_n$  be  $N \times N$  pairwise permutable matrices; that is,  $B_i B_j = B_j B_i$  for each  $i, j \in \{1, \dots, n\}$ . Then the following holds for any  $t \in \mathbb{R}$ :

(1) if  $B_i = \Theta$  for some  $i \in \{1, \dots, n\}$ , then

$$\mathcal{X}_{\tau_1, \dots, \tau_{i-1}, \tau_i, \tau_{i+1}, \dots, \tau_n}^{B_1^2, \dots, B_{i-1}^2, B_i^2, B_{i+1}^2, \dots, B_n^2}(t) = \mathcal{X}_{\tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \dots, \tau_n}^{B_1^2, \dots, B_{i-1}^2, B_{i+1}^2, \dots, B_n^2}(t), \tag{28}$$

(2) if  $\tau_i = \tau_k$  for  $i < k, i, k \in \{1, \dots, n\}$ , then

$$\mathcal{X}_{\tau_1, \dots, \tau_{i-1}, \tau_i, \tau_{i+1}, \dots, \tau_{k-1}, \tau_k, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{i-1}^2, B_i^2, B_{i+1}^2, \dots, B_{k-1}^2, B_k^2, B_{k+1}^2, \dots, B_n^2}(t) = \mathcal{X}_{\tau_1, \dots, \tau_{i-1}, \tau_i, \tau_{i+1}, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{i-1}^2, B_i^2 + B_k^2, B_{i+1}^2, \dots, B_{k-1}^2, B_{k+1}^2, \dots, B_n^2}(t), \tag{29}$$

(3) for any bijective mapping  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$  we get

$$\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = \mathcal{X}_{\tau_{\sigma(1)}, \dots, \tau_{\sigma(n)}}^{B_{\sigma(1)}^2, \dots, B_{\sigma(n)}^2}(t), \tag{30}$$

(4) taking the one-sided derivatives at  $0, \tau_1, \dots, \tau_n$ , then

$$\ddot{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = -B_1^2 \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_1) - \dots - B_n^2 \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_n), \tag{31}$$

(5) considering the one-sided derivatives at  $0$  (they both equal  $\Theta$ ), then

$$\dot{\mathcal{Y}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t). \tag{32}$$

Statements (1)–(4) hold with  $\mathcal{Y}$  instead of  $\mathcal{X}$ .

*Proof.* Statement (1) follows easily from definition of  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}$ , because  $\Theta^{2i} = E$  if  $i = 0$  and  $\Theta^{2i} = \Theta$  whenever  $i > 0$ . Next, if  $\tau_i = \tau_k$ , then

$$\sum_{\substack{j_1, \dots, j_n \geq 0 \\ j_1 \tau_1 + \dots + j_n \tau_n \leq t}} F(j_1, \dots, j_n) = \sum_{\substack{j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_{k-1}, j_{k+1}, \dots, j_n \geq 0 \\ j_1 \tau_1 + \dots + j_{i-1} \tau_{i-1} + j_{i+1} \tau_{i+1} + \dots + j_{k-1} \tau_{k-1} + j_{k+1} \tau_{k+1} + \dots + j_n \tau_n \leq t}} \sum_{\substack{j_i, j_k \geq 0 \\ j_i + j_k = l}} F(j_1, \dots, j_n) \tag{33}$$

for any matrix function  $F$ . Thus, using the property of multinomial coefficient (see (16))

$$(j_1, \dots, j_n)! = (j_1, \dots, j_{i-1}, j_i + j_k, j_{i+1}, \dots, j_{k-1}, j_{k+1}, \dots, j_n)! (j_i, j_k)!, \tag{34}$$

for (2), we obtain

$$\begin{aligned}
 & \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) \\
 &= \sum_{\substack{j_1, \dots, j_{i-1}, l, j_{i+1}, \dots, j_{k-1}, j_{k+1}, \dots, j_n \geq 0 \\ j_1 \tau_1 + \dots + j_{i-1} \tau_{i-1} + l \tau_i + j_{i+1} \tau_{i+1} + \dots + j_{k-1} \tau_{k-1} + j_{k+1} \tau_{k+1} + \dots + j_n \tau_n \leq t}} (-1)^{\sum_{s \in \{1, \dots, n\}, s \neq i, k} j_s + l} \\
 & \quad \times (j_1, \dots, j_{i-1}, l, j_{i+1}, \dots, j_{k-1}, j_{k+1}, \dots, j_n)! \\
 & \quad \times \left( \sum_{\substack{j_i, j_k \geq 0 \\ j_i + j_k = l}} (j_i, j_k)! B_i^{2j_i} B_k^{2j_k} \right) \\
 & \quad \times \prod_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} B_s^{2j_s} \frac{\left( t - \sum_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} j_s \tau_s - l \tau_i \right)^{2 \left( \sum_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} j_s + l \right)}}{\left( 2 \left( \sum_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} j_s + l \right) \right)!} \\
 &= \sum_{\substack{j_1, \dots, j_{i-1}, l, j_{i+1}, \dots, j_{k-1}, j_{k+1}, \dots, j_n \geq 0 \\ j_1 \tau_1 + \dots + j_{i-1} \tau_{i-1} + l \tau_i + j_{i+1} \tau_{i+1} + \dots + j_{k-1} \tau_{k-1} + j_{k+1} \tau_{k+1} + \dots + j_n \tau_n \leq t}} (-1)^{\sum_{s \in \{1, \dots, n\}, s \neq i, k} j_s + l} \\
 & \quad \times (j_1, \dots, j_{i-1}, l, j_{i+1}, \dots, j_{k-1}, j_{k+1}, \dots, j_n)! (B_i^2 + B_k^2)^l \\
 & \quad \times \prod_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} B_s^{2j_s} \frac{\left( t - \sum_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} j_s \tau_s - l \tau_i \right)^{2 \left( \sum_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} j_s + l \right)}}{\left( 2 \left( \sum_{\substack{s \in \{1, \dots, n\} \\ s \neq i, k}} j_s + l \right) \right)!} \\
 &= \mathcal{X}_{\tau_1, \dots, \tau_{i-1}, \tau_i, \tau_{i+1}, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{i-1}^2, B_i^2 + B_k^2, B_{i+1}^2, \dots, B_{k-1}^2, B_{k+1}^2, \dots, B_n^2}(t). \tag{35}
 \end{aligned}$$

Property (3) is trivial.

Now, we prove the statement (4). If  $\tau := \tau_1 = \dots = \tau_n$ , then

$$\begin{aligned}
 \check{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) &= \check{\mathcal{X}}_{\tau}^{B_1^2 + \dots + B_n^2}(t) \\
 &= - (B_1^2 + \dots + B_n^2) \mathcal{X}_{\tau}^{B_1^2 + \dots + B_n^2}(t - \tau) \\
 &= -B_1^2 \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_1) \\
 & \quad - \dots - B_n^2 \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_n)
 \end{aligned} \tag{36}$$

by (2) and from the property of  $\mathcal{X}_{\tau}^{B_1^2 + \dots + B_n^2}(t)$  (see (22)).

Hence, without any loss of generality, we assume that  $\tau_i \neq \tau_j$  for each  $i \neq j$ ,  $i, j \in \{1, \dots, n\}$  (in the other case, we collect matrices as stated in (2)). Note the case  $n = 2$  was proved in [16, Lemma 2.3.] Now, assume that  $\mathcal{X}_{\tau_1, \dots, \tau_{n-1}}^{B_1^2, \dots, B_{n-1}^2}(t)$  solves

$$\ddot{x}(t) = -B_1^2 x(t - \tau_1) - \dots - B_{n-1}^2 x(t - \tau_{n-1}), \tag{37}$$

that is, that the statement is fulfilled for  $n - 1$  different delays.

Let  $\tau_k := \max_{i=1, \dots, n} \tau_i$ . If  $t < \tau_k$ , then  $t - \tau_k < 0$ , that is,

$$\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_k) = \Theta, \tag{38}$$

and from definition (20) it holds

$$\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = \mathcal{X}_{\tau_1, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{k-1}^2, B_{k+1}^2, \dots, B_n^2}(t) \tag{39}$$

for such  $t$ . Consequently,

$$\begin{aligned}
 \check{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) &= \check{\mathcal{X}}_{\tau_1, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{k-1}^2, B_{k+1}^2, \dots, B_n^2}(t) \\
 &= - \sum_{\substack{i=1, \dots, n \\ i \neq k}} B_i^2 \mathcal{X}_{\tau_1, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{k-1}^2, B_{k+1}^2, \dots, B_n^2}(t - \tau_i) \\
 &= - \sum_{i=1}^n B_i^2 \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_i)
 \end{aligned} \tag{40}$$

by the inductive hypothesis.

Now, let  $t \geq \max_{i=1, \dots, n} \tau_i$ . Applying Lemma 3, we get

$$\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = \sum_{M \subset \{1, \dots, n\}} S_M(t) \tag{41}$$

with  $S_M(t)$  given by (24) and the sum taken over all subsets of  $\{1, \dots, n\}$  including the trivial ones. Note that

$$\begin{aligned}
 S_{\emptyset}(t) &= \sum_{\substack{j_i \geq 1, i \in \emptyset \\ 0 \leq t}} (-1)^0 (\{j_i \mid i \in \emptyset\})! E \frac{(t - 0)^0}{0!} \\
 &= \sum_{0 \leq t} E = E \chi_{[0, \infty)}(t)
 \end{aligned} \tag{42}$$

with a characteristic function  $\chi_{\bar{M}}$  of a set  $\bar{M}$  given by

$$\chi_{\bar{M}}(t) = \begin{cases} 1, & t \in \bar{M}, \\ 0, & t \notin \bar{M}. \end{cases} \tag{43}$$

Since each  $M \subset \{1, \dots, n\}$  is a finite set, Lemma 2 yields

$$(\{j_i \mid i \in M\})! = \sum_{i \in M} (j_i - 1, \{j_k \mid k \in M \setminus \{i\}\})!. \tag{44}$$

We apply this identity to derive a formula for the second derivative of  $S_M$  for any  $\emptyset \neq M \subset \{1, \dots, n\}$ :

$$\begin{aligned}
 S_M''(t) &= \sum_{\substack{j_i \geq 1, i \in M \\ \sum_{i \in M} j_i \tau_i \leq t}} (-1)^{\sum_{i \in M} j_i} (\{j_i \mid i \in M\})! \\
 & \quad \times \prod_{i \in M} B_i^{2j_i} \frac{\left( t - \sum_{i \in M} j_i \tau_i \right)^{2 \left( \sum_{i \in M} j_i - 1 \right)}}{\left( 2 \left( \sum_{i \in M} j_i - 1 \right) \right)!} \\
 &= \sum_{i \in M} \sum_{\substack{j_k \geq 1, k \in M \\ \sum_{k \in M} j_k \tau_k \leq t}} (-1)^{\sum_{k \in M} j_k} (j_i - 1, \{j_k \mid k \in M \setminus \{i\}\})! \\
 & \quad \times \prod_{k \in M} B_k^{2j_k} \frac{\left( t - \tau_i - \sum_{k \in M \setminus \{i\}} j_k \tau_k - (j_i - 1) \tau_i \right)^{2 \left( \sum_{k \in M} j_k - 1 \right)}}{\left( 2 \left( \sum_{k \in M} j_k - 1 \right) \right)!}.
 \end{aligned} \tag{45}$$

Next, for any fixed  $i \in \{1, \dots, n\}$  we split the second sum to  $j_i = 1$  and  $j_i \geq 2$ , that is,

$$\begin{aligned} & \sum_{\substack{j_k \geq 1, k \in M \\ \sum_{k \in M} j_k \tau_k \leq t}} F(j_1, \dots, j_{i-1}, j_i, j_{i+1}, \dots, j_n) \\ &= \sum_{\substack{j_k \geq 1, k \in M \setminus \{i\} \\ \sum_{k \in M \setminus \{i\}} j_k \tau_k \leq t - \tau_i}} F(j_1, \dots, j_{i-1}, 1, j_{i+1}, \dots, j_n) \\ &+ \sum_{\substack{j_k \geq 1, k \in M \setminus \{i\} \\ j_i \geq 2 \\ \sum_{k \in M} j_k \tau_k \leq t}} F(j_1, \dots, j_{i-1}, j_i, j_{i+1}, \dots, j_n), \end{aligned} \tag{46}$$

and use the equality

$$\begin{aligned} & \sum_{\substack{j_k \geq 1, k \in M \setminus \{i\} \\ j_i \geq 2 \\ \sum_{k \in M} j_k \tau_k \leq t}} F(j_1, \dots, j_{i-1}, j_i, j_{i+1}, \dots, j_n) \\ &= \sum_{\substack{j_k \geq 1, k \in M \\ \sum_{k \in M} j_k \tau_k \leq t - \tau_i}} F(j_1, \dots, j_{i-1}, j_i + 1, j_{i+1}, \dots, j_n) \end{aligned} \tag{47}$$

since

$$\sum_{k \in M} j_k \tau_k \leq t \iff \sum_{k \in M \setminus \{i\}} j_k \tau_k + (j_i - 1) \tau_i \leq t - \tau_i. \tag{48}$$

So we obtain

$$S''_M(t) = - \sum_{i \in M} B_i^2 (S_{M \setminus \{i\}}(t - \tau_i) + S_M(t - \tau_i)) \tag{49}$$

for each  $\emptyset \neq M \subset \{1, \dots, n\}$ . Obviously,  $S''_0(t) = \Theta$ . Consequently,

$$\begin{aligned} & \ddot{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) \\ &= - \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 (S_{M \setminus \{i\}}(t - \tau_i) + S_M(t - \tau_i)) \\ &= - \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &- \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 S_M(t - \tau_i). \end{aligned} \tag{50}$$

Now, we add and subtract

$$\sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_M(t - \tau_i) \tag{51}$$

to the right-hand side of (50) to get

$$\begin{aligned} & \ddot{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) \\ &= - \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i=1}^n B_i^2 S_M(t - \tau_i) \\ &+ \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_M(t - \tau_i) \\ &- \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \end{aligned} \tag{52}$$

and apply  $M = M \setminus \{i\}$  whenever  $i \notin M$ :

$$\begin{aligned} & \ddot{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) \\ &= - \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i=1}^n B_i^2 S_M(t - \tau_i) \\ &+ \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &- \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i). \end{aligned} \tag{53}$$

Denoting  $\#M$  the number of elements of the set  $M$ , we split the last two terms of the right-hand side of the latter equality with respect to

$$\begin{aligned} & \sum_{\emptyset \neq M \subset \{1, \dots, n\}} = \sum_{\substack{M \subset \{1, \dots, n\} \\ 1 \leq \#M \leq n-1}} + \sum_{\substack{M \subset \{1, \dots, n\} \\ \#M = n}} \\ &= \sum_{\substack{M \subset \{1, \dots, n\} \\ \#M = 1}} + \sum_{\substack{M \subset \{1, \dots, n\} \\ 2 \leq \#M \leq n}}. \end{aligned} \tag{54}$$

Hence, we have

$$\begin{aligned} & \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &- \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &= \sum_{\substack{M \subset \{1, \dots, n\} \\ 1 \leq \#M \leq n-1}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &+ \sum_{\substack{M = \{1, \dots, n\} \\ i \notin M}} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &- \sum_{M \in \{\{1\}, \dots, \{n\}\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &- \sum_{\substack{M \subset \{1, \dots, n\} \\ 2 \leq \#M \leq n}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i). \end{aligned} \tag{55}$$

Now, we show that

$$\begin{aligned} & \sum_{\substack{M \subset \{1, \dots, n\} \\ 1 \leq \#M \leq n-1}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ &= \sum_{\substack{M \subset \{1, \dots, n\} \\ 2 \leq \#M \leq n}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i). \end{aligned} \tag{56}$$

Let  $M \subset \{1, \dots, n\}$ , and let  $i \notin M$  be arbitrary and fixed such that  $1 \leq \#M \leq n - 1$ . Then, clearly,

$$B_i S_{M \setminus \{i\}}(t - \tau_i) = B_i S_{(M \cup \{i\}) \setminus \{i\}}(t - \tau_i) \tag{57}$$

and  $2 \leq \#(M \cup \{i\}) \leq n$ ,  $i \in M \cup \{i\}$ . Moreover, if  $M_1, M_2 \subset \{1, \dots, n\}$ ,  $i \notin M_{1,2}$  are such that  $M_1 \neq M_2$ ,  $1 \leq \#M_{1,2} \leq n - 1$ , then  $M_1 \cup \{i\} \neq M_2 \cup \{i\}$ .

On the other side, if  $M \subset \{1, \dots, n\}$ ,  $i \in M$  are arbitrary and fixed such that  $2 \leq \#M \leq n$ , then

$$B_i S_{M \setminus \{i\}}(t - \tau_i) = B_i S_{(M \setminus \{i\}) \setminus \{i\}}(t - \tau_i) \quad (58)$$

and  $1 \leq \#(M \setminus \{i\}) \leq n - 1$ ,  $i \notin M \setminus \{i\}$ . Furthermore, if  $M_1, M_2 \subset \{1, \dots, n\}$ ,  $i \in M_{1,2}$  are such that  $M_1 \neq M_2$ ,  $2 \leq \#M_{1,2} \leq n$ , then,  $M_1 \setminus \{i\} \neq M_2 \setminus \{i\}$ . In conclusion, there is 1-1 correspondence between the terms on the left-hand side of (56) and the terms on the right-hand side. So (56) is valid.

Putting (56) in (55) we obtain

$$\begin{aligned} & \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ & - \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ & = \sum_{M=\{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ & - \sum_{M \in \{\{1\}, \dots, \{n\}\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i). \end{aligned} \quad (59)$$

Next, by the property of empty sum, we get

$$\sum_{M=\{1, \dots, n\}} \sum_{i \notin M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) = \sum_{M=\{1, \dots, n\}} \Theta = \Theta. \quad (60)$$

Moreover, it holds

$$\begin{aligned} & \sum_{M \in \{\{1\}, \dots, \{n\}\}} \sum_{i \in M} B_i^2 S_{M \setminus \{i\}}(t - \tau_i) \\ & = \sum_{i=1}^n B_i^2 S_{\emptyset}(t - \tau_i) = \sum_{M=\emptyset}^n \sum_{i=1}^n B_i^2 S_M(t - \tau_i). \end{aligned} \quad (61)$$

Therefore, putting (60) and (61) in (59) and the result in (53), we obtain

$$\begin{aligned} \ddot{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) & = - \sum_{\emptyset \neq M \subset \{1, \dots, n\}} \sum_{i=1}^n B_i^2 S_M(t - \tau_i) \\ & - \sum_{M=\emptyset}^n \sum_{i=1}^n B_i^2 S_M(t - \tau_i) \\ & = - \sum_{i=1}^n B_i^2 \sum_{M \subset \{1, \dots, n\}} S_M(t - \tau_i) \\ & = - \sum_{i=1}^n B_i^2 \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_i). \end{aligned} \quad (62)$$

Hence,  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$  solves (31) for all  $t \geq 0$ . Clearly, the same is true for  $t < 0$ .

For  $\mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$ , statements (1)–(3) can be proved as for  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$ . Next, if  $\tau := \tau_1 = \dots = \tau_n$ , we apply the point (2) of this lemma and property (22) for  $\mathcal{Y}_{\tau}^{B_1^2 + \dots + B_n^2}(t)$  to see that

$$\begin{aligned} \ddot{\mathcal{Y}}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) & = \ddot{\mathcal{Y}}_{\tau}^{B_1^2 + \dots + B_n^2}(t) \\ & = -(B_1^2 + \dots + B_n^2) \mathcal{Y}_{\tau}^{B_1^2 + \dots + B_n^2}(t - \tau) \\ & = -B_1^2 \mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_1) \\ & \quad - \dots - B_n^2 \mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t - \tau_n). \end{aligned} \quad (63)$$

So,  $\mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$  is a solution of (31) when all delays are the same.

Again, the case  $n = 2$  with different delays was proved in [16]; thus, we assume that the statement is fulfilled for  $n - 1$ ,  $n \geq 3$  and that  $\tau_i \neq \tau_j$  for each  $i \neq j$ ,  $i, j \in \{1, \dots, n\}$ . As before, if  $t < \tau_k$  and  $\tau_k := \max_{i=1, \dots, n} \tau_i$ , then

$$\mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = \mathcal{Y}_{\tau_1, \dots, \tau_{k-1}, \tau_{k+1}, \dots, \tau_n}^{B_1^2, \dots, B_{k-1}^2, B_{k+1}^2, \dots, B_n^2}(t) \quad (64)$$

by definition (20), and the statement follows from the inductive hypothesis. For  $t \geq \max_{i=1, \dots, n} \tau_i$ , we apply Lemma 3 to see that

$$\mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t) = \sum_{M \subset \{1, \dots, n\}} \tilde{S}_M(t) \quad (65)$$

with  $\tilde{S}_M(t)$  given by (25). This time

$$\begin{aligned} S_{\emptyset}(t) & = \sum_{\substack{j_i \geq 1, i \in \emptyset \\ 0 \leq t}} (-1)^0 (\{j_i \mid i \in \emptyset\})! E \frac{(t-0)^1}{0!} \\ & = \sum_{0 \leq t} E t = t E \chi_{[0, \infty)}(t) \end{aligned} \quad (66)$$

and  $S_{\emptyset}''(t) = \Theta$ . The rest proceeds analogically to  $\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$ . The final statement follows directly from definition (20).  $\square$

*Remark 5.* Another proof of statements (1)–(3) of the previous lemma can be made with the aid of statement (4) of the same lemma and uses the uniqueness of a solution of the corresponding initial value problem. For instance in statement (1) of the lemma, both

$$\mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t), \quad \mathcal{X}_{\tau_1, \dots, \tau_{i-1}, \tau_{i+1}, \dots, \tau_n}^{B_1^2, \dots, B_{i-1}^2, B_{i+1}^2, \dots, B_n^2}(t) \quad (67)$$

solve

$$\begin{aligned} \dot{x}(t) & = -B_1^2 x(t - \tau_1) - \dots - B_{i-1}^2 x(t - \tau_{i-1}) \\ & \quad - B_{i+1}^2 x(t - \tau_{i+1}) - \dots - B_n^2 x(t - \tau_n) \end{aligned} \quad (68)$$

with initial condition

$$x(t) = \begin{cases} \Theta, & -\tau \leq t < 0, \\ E, & t = 0, \end{cases} \quad \dot{x}(t) = \Theta, \quad -\tau \leq t \leq 0 \quad (69)$$

and  $\tau = \max_{i=1, \dots, n} \tau_i$ .

We are ready to state and prove our main result.

### 3. Main Result

Here we find a solution of the initial value problem (11), (8) in the sense of the next definition.

*Definition 6.* Let  $\tau_1, \dots, \tau_n > 0$ ,  $\tau := \max_{i=1, \dots, n} \tau_i$ , and  $\varphi \in C^1([-\tau, 0], \mathbb{R}^N)$ , and let  $B_1, \dots, B_n$  be  $N \times N$  matrices, and let  $f : [0, \infty) \rightarrow \mathbb{R}^N$  be a given function. Function  $x : [-\tau, \infty) \rightarrow \mathbb{R}^N$  is a solution of (11) and initial condition (8), if  $x \in C^1([-\tau, \infty), \mathbb{R}^N) \cap C^2([0, \infty), \mathbb{R}^N)$  (taken the second right-hand derivative at 0) satisfies (11) on  $[0, \infty)$  and condition (8) on  $[-\tau, 0]$ .

**Theorem 7.** Let  $n \geq 3$ ,  $\tau_1, \dots, \tau_n > 0$ ,  $\tau := \max_{i=1, \dots, n} \tau_i$ , and  $\varphi \in C^1([-\tau, 0], \mathbb{R}^N)$ , and let  $B_1, \dots, B_n$  be  $N \times N$  pairwise permutable matrices; that is,  $B_i B_j = B_j B_i$  for each  $i, j \in \{1, \dots, n\}$ , and let  $f : [0, \infty) \rightarrow \mathbb{R}^N$  be a given function. Solution  $x(t)$  of (11) satisfying initial condition (8) has the form

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ \mathcal{X}(t)\varphi(0) + \mathcal{Y}(t)\dot{\varphi}(0) - \sum_{i=1}^n B_i^2 \int_{-\tau_i}^0 \mathcal{Y}(t - \tau_i - s)\varphi(s) ds + \int_0^t \mathcal{Y}(t - s)f(s) ds, & 0 \leq t, \end{cases} \quad (70)$$

where  $\mathcal{X}(t) = \mathcal{X}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$  and  $\mathcal{Y}(t) = \mathcal{Y}_{\tau_1, \dots, \tau_n}^{B_1^2, \dots, B_n^2}(t)$ .

*Proof.* Obviously,  $x(t)$  satisfies the initial condition on  $[-\tau, 0]$ , and, from definition (20),  $x(0) = \varphi(0)$ . For the derivative, it holds  $\lim_{t \rightarrow 0^-} \dot{x}(t) = \dot{\varphi}(0)$ . Moreover, if  $0 \leq t < \min_{i=1, \dots, n} \tau_i$ , then

$$x(t) = \varphi(0) + t\dot{\varphi}(0) - \sum_{i=1}^n B_i^2 \int_{-\tau_i}^{t-\tau_i} (t - \tau_i - s)\varphi(s) ds + \int_0^t (t - s)f(s) ds \quad (71)$$

since

$$\mathcal{Y}(t - \tau_i - s) = \begin{cases} (t - \tau_i - s)E, & s \in [-\tau_i, t - \tau_i], \\ \Theta, & s \in (t - \tau_i, 0] \end{cases} \quad (72)$$

for each  $i = 1, \dots, n$ . Thus

$$\dot{x}(t) = \dot{\varphi}(0) - \sum_{i=1}^n B_i^2 \int_{-\tau_i}^{t-\tau_i} \varphi(s) ds + \int_0^t f(s) ds \quad (73)$$

and  $\lim_{t \rightarrow 0^+} \dot{x}(t) = \dot{\varphi}(0)$ . Clearly,

$$x \in C^1((-\tau, \infty), \mathbb{R}^N) \cap C^2((0, \infty) \setminus \{\tau_1, \dots, \tau_n\}, \mathbb{R}^N). \quad (74)$$

We show that, although  $\mathcal{X}(t)$  is not  $C^2$  at  $\tau_1, \dots, \tau_n$ , function  $x(t)$  is  $C^2$  at these points and, therefore, in  $(0, \infty)$ . At once, we prove that  $x(t)$  is a solution of (11).

Assume that  $0 \leq t < \min_{i=1, \dots, n} \tau_i$ . Then identities (71) and (73) are valid, and by differentiating (73) for such  $t$  we get

$$\ddot{x}(t) = -\sum_{i=1}^n B_i^2 \varphi(t - \tau_i) + f(t) = -\sum_{i=1}^n B_i^2 x(t - \tau_i) + f(t) \quad (75)$$

since  $x(t - \tau_i) = \varphi(t - \tau_i)$  for each  $i = 1, \dots, n$ .

Now, let  $\emptyset \neq M_{1,2} \subset \{1, \dots, n\}$  be such that  $\tau_i \leq t < \tau_j$  for each  $i \in M_1, j \in M_2$ . Then

$$\mathcal{Y}(t - \tau_j - s) = \begin{cases} \mathcal{Y}(t - \tau_j - s), & s \in [-\tau_j, t - \tau_j], \\ \Theta, & s \in (t - \tau_j, 0] \end{cases} \quad (76)$$

whenever  $j \in M_2$ , and (70) becomes

$$x(t) = \mathcal{X}(t)\varphi(0) + \mathcal{Y}(t)\dot{\varphi}(0) - \sum_{i \in M_1} B_i^2 \int_{-\tau_i}^0 \mathcal{Y}(t - \tau_i - s)\varphi(s) ds - \sum_{j \in M_2} B_j^2 \int_{-\tau_j}^{t-\tau_j} \mathcal{Y}(t - \tau_j - s)\varphi(s) ds + \int_0^t \mathcal{Y}(t - s)f(s) ds. \quad (77)$$

By the point (5) of Lemma 4, we get

$$\dot{x}(t) = \dot{\mathcal{X}}(t)\varphi(0) + \dot{\mathcal{Y}}(t)\dot{\varphi}(0) - \sum_{i \in M_1} B_i^2 \int_{-\tau_i}^0 \dot{\mathcal{Y}}(t - \tau_i - s)\varphi(s) ds - \sum_{j \in M_2} B_j^2 \int_{-\tau_j}^{t-\tau_j} \dot{\mathcal{X}}(t - \tau_j - s)\varphi(s) ds + \int_0^t \dot{\mathcal{X}}(t - s)f(s) ds, \quad (78)$$

and for the second derivative it holds

$$\ddot{x}(t) = \ddot{\mathcal{X}}(t)\varphi(0) + \ddot{\mathcal{Y}}(t)\dot{\varphi}(0) - \sum_{i \in M_1} B_i^2 \int_{-\tau_i}^0 \ddot{\mathcal{Y}}(t - \tau_i - s)\varphi(s) ds - \sum_{j \in M_2} B_j^2 \left( \varphi(t - \tau_j) + \int_{-\tau_j}^{t-\tau_j} \ddot{\mathcal{Y}}(t - \tau_j - s)\varphi(s) ds \right) + f(t) + \int_0^t \ddot{\mathcal{Y}}(t - s)f(s) ds \quad (79)$$

since  $\mathcal{X}(0) = E$ . Now, we apply the property (4) of Lemma 4 together with

$$\mathcal{X}(t - \tau_j) = \mathcal{Y}(t - \tau_j) = \Theta, \quad \forall j \in M_2 \quad (80)$$

to see that both  $\mathcal{X}$  and  $\mathcal{Y}$  are solutions of

$$\ddot{y}(t) = - \sum_{i \in M_1} B_i^2 y(t - \tau_i). \tag{81}$$

Therefore,

$$\begin{aligned} \ddot{x}(t) &= - \sum_{k \in M_1} B_k^2 \left( \mathcal{X}(t - \tau_k) \varphi(0) + \mathcal{Y}(t - \tau_k) \dot{\varphi}(0) \right. \\ &\quad - \sum_{i \in M_1} B_i^2 \int_{-\tau_i}^0 \mathcal{Y}(t - \tau_i - \tau_k - s) \varphi(s) ds \\ &\quad - \sum_{j \in M_2} B_j^2 \int_{-\tau_j}^{t - \tau_j} \mathcal{Y}(t - \tau_j - \tau_k - s) \varphi(s) ds \\ &\quad \left. + \int_0^t \mathcal{Y}(t - \tau_k - s) f(s) ds \right) \\ &\quad - \sum_{j \in M_2} B_j^2 \varphi(t - \tau_j) + f(t) \\ &= - \sum_{i \in M_1} B_i^2 x(t - \tau_i) - \sum_{j \in M_2} B_j^2 \varphi(t - \tau_j) + f(t). \end{aligned} \tag{82}$$

In fact, this is exactly formula (11) since  $x(t - \tau_j) = \varphi(t - \tau_j)$  for each  $j \in M_2$ .

Finally, if  $\max_{i=1, \dots, n} \tau_i \leq t$ , we have

$$\begin{aligned} x(t) &= \mathcal{X}(t) \varphi(0) + \mathcal{Y}(t) \dot{\varphi}(0) \\ &\quad - \sum_{i=1}^n B_i^2 \int_{-\tau_i}^0 \mathcal{Y}(t - \tau_i - s) \varphi(s) ds \\ &\quad + \int_0^t \mathcal{Y}(t - s) f(s) ds. \end{aligned} \tag{83}$$

So, differentiating this formula twice and applying (4) of Lemma 4 result in (11). Hence, one can see that function  $x(t)$  given by (70) really solves (11) and satisfies initial condition (8) and, moreover, that  $x \in C^2((0, \infty), \mathbb{R}^N)$ . To see the last one, one has to put  $\tau_1, \dots, \tau_n$  into the computed derivatives, for example, if  $\tau_k := \min_{i=1, \dots, n} \tau_i < \tau_i$  for each  $i = 1, \dots, k - 1, k + 1, \dots, n$ , then by (75) and (82) we get

$$\begin{aligned} \lim_{t \rightarrow \tau_k^-} \ddot{x}(t) &= - \sum_{\substack{i=1 \\ i \neq k}}^n B_i^2 \varphi(\tau_k - \tau_i) - B_k^2 \varphi(0) + f(\tau_k) \\ &= - \sum_{j \in M_2} B_j^2 \varphi(\tau_k - \tau_j) \\ &\quad - B_k^2 x(0) + f(\tau_k) = \lim_{t \rightarrow \tau_k^+} \ddot{x}(t), \end{aligned} \tag{84}$$

where  $M_2 = \{1, \dots, n\} \setminus \{k\}$ . □

It is easy to see that defining functions

$$\begin{aligned} \widetilde{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1, \dots, B_n}(t) &:= \mathcal{X}_{\tau_1, \dots, \tau_n}^{-B_1, \dots, -B_n}(t), \\ \widetilde{\mathcal{Y}}_{\tau_1, \dots, \tau_n}^{B_1, \dots, B_n}(t) &:= \mathcal{Y}_{\tau_1, \dots, \tau_n}^{-B_1, \dots, -B_n}(t) \end{aligned} \tag{85}$$

leads to the solution of

$$\ddot{x}(t) = B_1 x(t - \tau_1) + \dots + B_n x(t - \tau_n) + f(t) \tag{86}$$

with pairwise permutable matrices  $B_1, \dots, B_n$  and initial condition (8). More precisely, we have the following corollary of Theorem 7.

**Corollary 8.** *Let  $n \geq 3$ ,  $\tau_1, \dots, \tau_n > 0$ ,  $\tau := \max_{i=1, \dots, n} \tau_i$ ,  $\varphi \in C^1([- \tau, 0], \mathbb{R}^N)$ , and let  $B_1, \dots, B_n$  be  $N \times N$  pairwise permutable matrices; that is,  $B_i B_j = B_j B_i$  for each  $i, j \in \{1, \dots, n\}$ , and let  $f : [0, \infty) \rightarrow \mathbb{R}^N$  be a given function. Solution  $x(t)$  of (86) satisfying initial condition (8) has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau \leq t < 0, \\ \mathcal{X}(t) \varphi(0) + \mathcal{Y}(t) \dot{\varphi}(0) \\ \quad + \sum_{i=1}^n B_i \int_{-\tau_i}^0 \mathcal{Y}(t - \tau_i - s) \varphi(s) ds \\ \quad + \int_0^t \mathcal{Y}(t - s) f(s) ds, & 0 \leq t, \end{cases} \tag{87}$$

where  $\mathcal{X}(t) = \widetilde{\mathcal{X}}_{\tau_1, \dots, \tau_n}^{B_1, \dots, B_n}(t)$  and  $\mathcal{Y}(t) = \widetilde{\mathcal{Y}}_{\tau_1, \dots, \tau_n}^{B_1, \dots, B_n}(t)$ .

*Proof.* The corollary can be proved exactly in the same way as Theorem 7. □

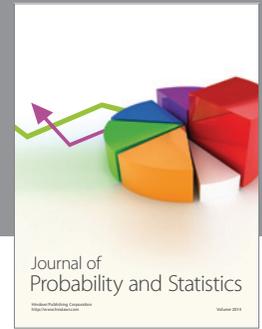
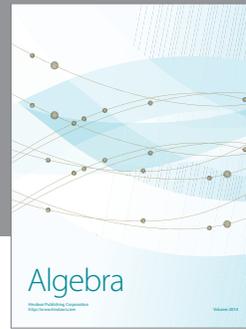
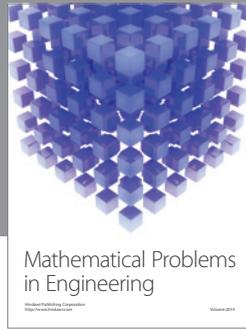
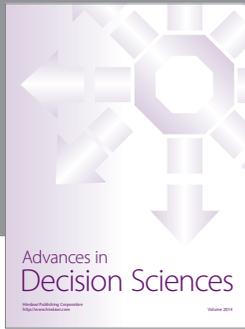
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