Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 421031, 7 pages
http://dx.doi.org/10.1155/2013/421031

Research Article

Some Geometric Properties of the Domain of the Double Sequential Band Matrix $B(\widetilde{r}, \widetilde{s})$ in the Sequence Space $\ell(p)$

Havva Nergiz and Feyzi Başar

Department of Mathematics, Faculty of Arts and Sciences, Fatih University, The Hadımköy Campus, Büyükçekmece, 34500 İstanbul, Turkey

Correspondence should be addressed to Feyzi Başar; feyzibasar@gmail.com

Received 2 January 2013; Accepted 12 February 2013

Academic Editor: Józef Banaś

The sequence space $\ell(p)$ was introduced by Maddox (1967). Quite recently, the sequence space $\ell(\widetilde{B}, p)$ of nonabsolute type has been introduced and studied which is the domain of the double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ in the sequence space $\ell(p)$ by Nergiz and Başar (2012). The main purpose of this paper is to investigate the geometric properties of the space $\ell(\widetilde{B}, p)$, like rotundity and Kadec-Klee and the uniform Opial properties. The last section of the paper is devoted to the conclusion.

1. Introduction

By ω, we denote the space of all real-valued sequences. Any vector subspace of ω is called a *sequence space*. We write ℓ_∞, c, and c_0 for the spaces of all bounded, convergent, and null sequences, respectively. Also by bs, cs, ℓ_1, and ℓ_p; we denote the spaces of all bounded, convergent, absolutely convergent, and p-absolutely convergent series, respectively, where $1 < p < \infty$.

Assume here and after that (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p) = \left\{x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty\right\} \quad (0 < p_k \le H < \infty) \tag{1}$$

which is complete paranormed space paranormed by

$$g(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M}. \tag{2}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 1 to ∞.

Quite recently, Nergiz and Başar [4] have introduced the space $\ell(\widetilde{B}, p)$ of nonabsolute type which consists of all sequences whose $B(\widetilde{r}, \widetilde{s})$-transforms are in the space $\ell(p)$, where $B(\widetilde{r}, \widetilde{s}) = \{b_{nk}(r_k, s_k)\}$ is defined by

$$b_{nk}(r_k, s_k) = \begin{cases} r_k, & k = n, \\ s_k, & k = n-1, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

for all $k, n \in \mathbb{N}$, where $\widetilde{r} = (r_k)$ and $\widetilde{s} = (s_k)$ are the convergent sequences. We should record that the double sequential band matrices were used for determining its fine spectrum over some sequence spaces by Kumar and Srivastava in [5, 6], Panigrahi and Srivastava in [7], and Akhmedov and El-Shabrawy in [8]. The reader may refer to Nergiz and Başar [4, 9] for relevant terminology and additional references on the space $\ell(\widetilde{B}, p)$, since the present paper is a natural continuation of them. Here and after, for short we write $\widetilde{B}$ instead of $B(\widetilde{r}, \widetilde{s})$. In the special case $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(\widetilde{B}, p)$ is reduced to the space $(\ell_p)_{\widetilde{B}}$; that is,

$$(\ell_p)_{\widetilde{B}} := \left\{(x_k) \in \omega : \sum_k |s_{k-1}x_{k-1} + r_k x_k|^p < \infty\right\}, \quad (0 < p < \infty). \tag{4}$$

2. The Rotundity of the Space $\ell(\widetilde{B},p)$

The rotundity of Banach spaces is one of the most important geometric property in functional analysis. For details, the reader may refer to [10–12]. In this section, we characterize the rotundity of the space $\ell(\widetilde{B}, p)$ and give some results related to this concept.

Definition 1. Let $S(X)$ be the unit sphere of a Banach space X. Then, a point $x \in S(X)$ is called an extreme point if $2x = y+z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.

Definition 2. A Banach space X is said to have Kadec-Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 3. A Banach space X is said to have

(i) the Opial property if every sequence (x_n) weakly convergent to $x_0 \in X$ satisfies

$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n + x\| \tag{5}$$

for every $x \in X$ with $x \neq x_0$;

(ii) the uniform Opial property if for each $\epsilon > 0$, there exists an $r > 0$ such that

$$1 + r \le \liminf_{n\to\infty} \|x_n + x\| \tag{6}$$

for each $x \in X$ with $\|x\| \ge \epsilon$ and each sequence (x_n) in X such that $x_n \to 0$ and $\liminf_{n\to\infty}\|x_n\| \ge 1$.

Definition 4. Let X be a real vector space. A functional $\sigma : X \to [0,\infty)$ is called a modular if

(i) $\sigma(x) = 0$ if and only if $x = \theta$;

(ii) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$;

(iii) $\sigma(\alpha x+\beta y) \le \sigma(x)+\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$;

(iv) the modular σ is called convex if $\sigma(\alpha x+\beta y) \le \alpha\sigma(x)+\beta\sigma(y)$ for all $x, y \in X$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$.

A modular σ on X is called

(a) right continuous if $\lim_{\alpha\to 1^+}\sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.

(b) left continuous if $\lim_{\alpha\to 1^-}\sigma(\alpha x) = \sigma(x)$ for all $x \in X_\sigma$.

(c) continuous if it is both right and left continuous, where

$$X_\sigma = \left\{x \in X : \lim_{\alpha\longrightarrow 0^+} \sigma(\alpha x) = 0\right\}. \tag{7}$$

We define σ_p on $\ell(\widetilde{B}, p)$ by $\sigma_p(x) = \sum_k |s_{k-1}x_{k-1}+r_kx_k|^{p_k}$. If $p_k \ge 1$ for all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, by the convexity of the function $t \mapsto |t|^{p_k}$ for each $k \in \mathbb{N}$, σ_p is a convex modular on $\ell(\widetilde{B}, p)$.

Proposition 5. *The modular σ_p on $\ell(\widetilde{B}, p)$ satisfies the following properties with $p_k \ge 1$ for all $k \in \mathbb{N}$:*

(i) *if $0 < \alpha \le 1$, then $\alpha^M\sigma_p(x/\alpha) \le \sigma_p(x)$ and $\sigma_p(\alpha x) \le \alpha\sigma_p(x)$.*

(ii) *If $\alpha \ge 1$, then $\sigma_p(x) \le \alpha^M\sigma_p(x/\alpha)$.*

(iii) *If $\alpha \ge 1$, then $\sigma_p(x) \le \alpha\sigma_p(x/\alpha)$.*

(iv) *The modular σ_p is continuous on the space $\ell(\widetilde{B}, p)$.*

Proof. Consider the modular σ_p on $\ell(\widetilde{B}, p)$.

(i) Let $0 < \alpha \le 1$, then $\alpha^M/\alpha^{p_k} \le 1$. So, we have

$$\begin{aligned} \alpha^M\sigma_p\left(\frac{x}{\alpha}\right) &= \alpha^M\sum_k \frac{1}{\alpha^{p_k}}\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} \\ &= \sum_k \frac{\alpha^M}{\alpha^{p_k}}\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} \\ &\le \sum_k\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} = \sigma_p(x), \\ \sigma_p(\alpha x) &= \sum_k \alpha^{p_k}\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} \\ &\le \alpha\sum_k\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} = \alpha\sigma_p(x). \end{aligned} \tag{8}$$

(ii) Let $\alpha \ge 1$. Then, $\alpha^M/\alpha^{p_k} \ge 1$ for all $p_k \ge 1$. So, we have

$$\sigma_p(x) \le \frac{\alpha^M}{\alpha^{p_k}}\sigma_p(x) = \alpha^M\sigma_p\left(\frac{x}{\alpha}\right). \tag{9}$$

(iii) Let $\alpha \ge 1$. Then, $\alpha/\alpha^{p_k} \ge 1$ for all $p_k \ge 1$. So, we have

$$\begin{aligned} \sigma_p(x) &= \sum_k\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} \\ &\le \sum_k \frac{\alpha}{\alpha^{p_k}}\left|s_{k-1}x_{k-1} + r_kx_k\right|^{p_k} = \alpha\sigma_p\left(\frac{x}{\alpha}\right). \end{aligned} \tag{10}$$

(iv) By (ii) and (iii), one can immediately see for $\alpha > 1$ that

$$\sigma_p(x) \le \alpha\sigma_p(x) \le \sigma_p(\alpha x) \le \alpha^M\sigma_p(x). \tag{11}$$

By passing to limit as $\alpha \to 1^+$ in (11), we have $\lim_{\alpha\to 1^+}\sigma_p(\alpha x) = \sigma_p(x)$. Hence, σ_p is right continuous. If $0 < \alpha < 1$, by (i) we have

$$\alpha^M\sigma_p(x) \le \sigma_p(\alpha x) \le \alpha\sigma_p(x). \tag{12}$$

By letting $\alpha \to 1^-$ in (12), we observe that $\lim_{\alpha\to 1^-}\sigma_p(\alpha x) = \sigma_p(x)$. Hence, σ_p is also left continuous, and so, it is continuous. □

Proposition 6. *For any $x \in \ell(\widetilde{B}, p)$, the following statements hold:*

(i) *if $\|x\| < 1$, then $\sigma_p(x) \leq \|x\|$.*

(ii) *If $\|x\| > 1$, then $\sigma_p(x) \geq \|x\|$.*

(iii) $\|x\| = 1$ *if and only if $\sigma_p(x) = 1$.*

(iv) $\|x\| < 1$ *if and only if $\sigma_p(x) < 1$.*

(v) $\|x\| > 1$ *if and only if $\sigma_p(x) > 1$.*

Proof. Let $x \in \ell(\widetilde{B}, p)$.

(i) Let $\epsilon > 0$ be such that $0 < \epsilon < 1 - \|x\|$. By the definition of $\|\cdot\|$, there exists an $\alpha > 0$ such that $\|x\| + \epsilon > \alpha$ and $\sigma_p(x) \leq 1$. From Parts (i) and (ii) of Proposition 5, we obtain

$$\sigma_p(x) \leq \sigma_p\left[(\|x\| + \epsilon)\frac{x}{\alpha}\right] \leq (\|x\| + \epsilon)\,\sigma_p\left(\frac{x}{\alpha}\right) \leq \|x\| + \epsilon. \tag{13}$$

Since ϵ is arbitrary, we have (i).

(ii) If we choose $\epsilon > 0$ such that $0 < \epsilon < 1 - (1/\|x\|)$, then $1 < (1-\epsilon)\|x\| < \|x\|$. By the definition of $\|\cdot\|$ and Part (i) of Proposition 5, we have

$$1 < \sigma_p\left[\frac{x}{(1-\epsilon)\|x\|}\right] \leq \frac{1}{(1-\epsilon)\|x\|}\sigma_p(x). \tag{14}$$

So, $(1-\epsilon)\|x\| < \sigma_p(x)$ for all $\epsilon \in (0, 1 - (1/\|x\|))$. This implies that $\|x\| < \sigma_p(x)$.

(iii) Since σ_p is continuous, by Theorem 1.4 of [12] we directly have (iii).

(iv) This follows from Parts (i) and (iii).

(v) This follows from Parts (ii) and (iii). □

Now, we consider the space $\ell(\widetilde{B}, p)$ equipped with the Luxemburg norm given by

$$\|x\| = \inf\left\{\alpha > 0 : \sigma_p\left(\frac{x}{\alpha}\right) \leq 1\right\}. \tag{15}$$

Theorem 7. *$\ell(\widetilde{B}, p)$ is a Banach space with Luxemburg norm.*

Proof. Let $S_x = \{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}$ and $\|x\| = \inf S_x$ for all $x \in \ell(\widetilde{B}, p)$. Then, $S_x \subset (0, \infty)$. Therefore, $\|x\| \geq 0$ for all $x \in \ell(\widetilde{B}, p)$.

For $x = \theta$, $\sigma_p(\theta) = 0$ for all $\alpha > 0$. Hence, $S_0 = (0, \infty)$ and $\|\theta\| = \inf S_0 = \inf(0, \infty) = 0$.

Let $x \neq \theta$ and $Y = \{kx : k \in \mathbb{C}$ and $x \in \ell(\widetilde{B}, p)\}$ be a nonempty subset of $\ell(\widetilde{B}, p)$. Since $Y \subsetneq S[\ell(\widetilde{B}, p)]$, there exists $k_1 \in \mathbb{C}$ such that $k_1 x \notin S[\ell(\widetilde{B}, p)]$. Obviously, $k_1 \neq 0$. We assume that $0 < \alpha < 1/k_1$ and $\alpha \in S_x$. Then, $(x/\alpha) \in S[\ell(\widetilde{B}, p)]$. Since $|k_1\alpha| < 1$, we get

$$k_1 x = k_1 \alpha \frac{x}{\alpha} \in S\left[\ell\left(\widetilde{B}, p\right)\right] \tag{16}$$

which contradicts the assumption. Hence, we obtain that if $\alpha \in S_x$, then $\alpha > 1/|k_1|$. This means that $\|x\| \geq 1/|k_1| > 0$. Thus, we conclude that $\|x\| = 0$ if and only if $x = \theta$.

Now, let $k \neq 0$ and $\alpha \in S_{kx}$. Then, we have

$$\sigma_p\left(\frac{kx}{\alpha}\right) \leq 1, \quad \frac{kx}{\alpha} \in S\left[\ell\left(\widetilde{B}, p\right)\right]. \tag{17}$$

Therefore, we obtain

$$\frac{|k|\,x}{\alpha} = \frac{|k|}{k} \times \frac{kx}{\alpha} \in S\left[\ell\left(\widetilde{B}, p\right)\right], \quad \frac{\alpha}{|k|} \in S_x. \tag{18}$$

That is, $\|x\| \leq \alpha/|k|$ and $|k|\|x\| \leq \alpha$ for all $\alpha \in S_{kx}$. So, $|k|\|x\| \leq \|kx\|$.

If we take $1/k$ and kx instead of k and x, respectively, then we obtain that

$$\left|\frac{1}{kx}\right| \|kx\| \leq \left\|\frac{1}{k}kx\right\| = \|x\|, \qquad \|kx\| \leq |k|\,\|x\|. \tag{19}$$

Hence, we get $\|kx\| = |k|\|x\|$. This also holds when $k = 0$.

To prove the triangle inequality, let $x, y \in \ell(\widetilde{B}, p)$ and $\epsilon > 0$ be given. Then, there exist $\alpha \in S_x$ and $\beta \in S_y$ such that $\alpha < \|x\| + \epsilon$ and $\beta < \|y\| + \epsilon$. Since $S[\ell(\widetilde{B}, p)]$ is convex,

$$\begin{gathered} \frac{x}{\alpha} \in S\left[\ell\left(\widetilde{B}, p\right)\right], \qquad \frac{y}{\beta} \in S\left[\ell\left(\widetilde{B}, p\right)\right], \\ \frac{(x+y)}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}\left(\frac{x}{\alpha}\right) + \frac{\beta}{\alpha+\beta}\left(\frac{y}{\beta}\right) \in S\left[\ell\left(\widetilde{B}, p\right)\right]. \end{gathered} \tag{20}$$

Therefore, $\alpha + \beta \in S_{x+y}$. Then, we have $\|x + y\| \leq \alpha + \beta < \|x\| + \|y\| + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we obtain $\|x + y\| \leq \|x\| + \|y\|$. Hence, $\|x\| = \inf\{\alpha > 0 : \sigma_p(x/\alpha) \leq 1\}$ is a norm on $\ell(\widetilde{B}, p)$.

Now, we need to show that every Cauchy sequence in $\ell(\widetilde{B}, p)$ is convergent according to the Luxemburg norm. Let $\{x_k^{(n)}\}$ be a Cauchy sequence in $\ell(\widetilde{B}, p)$ and $\epsilon \in (0, 1)$. Thus, there exists n_0 such that $\|x^{(n)} - x^{(m)}\| < \epsilon$ for all $n, m \geq n_0$. By Part (i) of Proposition 6, we have

$$\sigma_p\left(x^{(n)} - x^{(m)}\right) \leq \left\|x^{(n)} - x^{(m)}\right\| < \epsilon \tag{21}$$

for all $n, m \geq n_0$. This implies that

$$\sum_k \left|\left[\widetilde{B}\left(x^{(n)} - x^{(m)}\right)\right]_k\right|^{p_k} < \epsilon. \tag{22}$$

Then, for each fixed k and for all $n, m \geq n_0$,

$$\left|\left[\widetilde{B}\left(x^{(n)} - x^{(m)}\right)\right]_k\right| = \left|\left(\widetilde{B}x^{(n)}\right)_k - \left(\widetilde{B}x^{(m)}\right)_k\right| < \epsilon. \tag{23}$$

Hence, the sequence $\{(\widetilde{B}x^{(n)})_k\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there is a $(\widetilde{B}x)_k \in \mathbb{R}$ such that $(\widetilde{B}x^{(m)})_k \to (\widetilde{B}x)_k$ as $m \to \infty$. Therefore, as $m \to \infty$ by (22), we have

$$\sum_k \left|\left[\widetilde{B}\left(x^{(n)} - x\right)\right]_k\right|^{p_k} < \epsilon \tag{24}$$

for all $n \geq n_0$.

Now, we have to show that (x_k) is an element of $\ell(\widetilde{B}, p)$. Since $(\widetilde{B}x^{(m)})_k \to (\widetilde{B}x)_k$ as $m \to \infty$, we have

$$\lim_{m\to\infty} \sigma_p\left(x^{(n)} - x^{(m)}\right) = \sigma_p\left(x^{(n)} - x\right). \tag{25}$$

Then, we see by (21) that $\sigma_p(x^{(n)} - x) \leq \|x^{(n)} - x\| < \epsilon$ for all $n \geq n_0$. This implies that $x^n \to x$ as $n \to \infty$. So, we have $x = x^{(n)} - (x^{(n)} - x) \in \ell(\widetilde{B}, p)$. Therefore, the sequence space $\ell(\widetilde{B}, p)$ is complete with respect to Luxemburg norm. This completes the proof. □

Theorem 8. *The space $\ell(\widetilde{B}, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Let $\ell(\widetilde{B}, p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$ for $k < 3$. Consider the following sequences given by

$$\begin{aligned} x &= \left(0, \frac{1}{r_1}, \frac{-s_1}{r_1 r_2}, \frac{s_1 s_2}{r_1 r_2 r_3}, \ldots\right), \\ y &= \left(0, 0, \frac{1}{r_2}, \frac{-s_2}{r_2 r_3}, \frac{s_2 s_3}{r_2 r_3 r_4}, \ldots\right). \end{aligned} \tag{26}$$

Then, obviously $x \neq y$ and

$$\sigma_p(x) = \sigma_p(y) = \sigma_p\left(\frac{x+y}{2}\right) = 1. \tag{27}$$

By Part (iii) of Proposition 6, $x, y, (x+y)/2 \in S[\ell(\widetilde{B}, p)]$ which leads us to the contradiction that the sequence space $\ell(\widetilde{B}, p)$ is not rotund. Hence, $p_k > 1$ for all $k \in \mathbb{N}$.

Conversely, let $x \in S[\ell(\widetilde{B}, p)]$ and $v, z \in S[\ell(\widetilde{B}, p)]$ with $x = (v+z)/2$. By convexity of σ_p and Part (iii) of Proposition 6, we have

$$1 = \sigma_p(x) \leq \frac{\sigma_p(v) + \sigma_p(z)}{2} \leq \frac{1}{2} + \frac{1}{2} = 1, \tag{28}$$

which gives that $\sigma_p(v) = \sigma_p(z) = 1$, and

$$\sigma_p(x) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \tag{29}$$

Also, we obtain from (29) that

$$\sum_k \left|s_{k-1}x_{k-1} + r_k x_k\right|^{p_k} = \frac{1}{2}\left(\sum_k \left|s_{k-1}v_{k-1} + r_k v_k\right|^{p_k} + \sum_k \left|s_{k-1}z_{k-1} + r_k z_k\right|^{p_k}\right). \tag{30}$$

Since $x = (v+z)/2$, we have

$$\sum_k \left|s_{k-1}(v_{k-1} + z_{k-1}) + r_k(v_k + z_k)\right|^{p_k} = \frac{1}{2}\left(\sum_k \left|s_{k-1}v_{k-1} + r_k v_k\right|^{p_k} + \sum_k \left|s_{k-1}z_{k-1} + r_k z_k\right|^{p_k}\right). \tag{31}$$

This implies that

$$\begin{aligned} &\left|s_{k-1}(v_{k-1} + z_{k-1}) + r_k(v_k + z_k)\right|^{p_k} \\ &\quad = \frac{1}{2}\left|s_{k-1}v_{k-1} + r_k v_k\right|^{p_k} + \frac{1}{2}\left|s_{k-1}z_{k-1} + r_k z_k\right|^{p_k} \end{aligned} \tag{32}$$

for all $k \in \mathbb{N}$. Since the function $t \mapsto |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (32) that $v_k = z_k$ for all $k \in \mathbb{N}$. Hence, $v = z$. That is, the sequence space $\ell(\widetilde{B}, p)$ is rotund. □

Theorem 9. *Let $x \in \ell(\widetilde{B}, p)$. Then, the following statements hold:*

(i) $0 < \alpha < 1$ *and* $\|x\| > \alpha$ *imply* $\sigma_p(x) > \alpha^M$.

(ii) $\alpha \geq 1$ *and* $\|x\| < \alpha$ *imply* $\sigma_p(x) < \alpha^M$.

Proof. Let $x \in \ell(\widetilde{B}, p)$.

(i) Suppose that $\|x\| > \alpha$ with $0 < \alpha < 1$. Then, $\|x/\alpha\| > 1$. By Part (ii) of Proposition 6, $\|x/\alpha\| > 1$ implies $\sigma_p(x/\alpha) \geq \|x/\alpha\| > 1$. That is, $\sigma_p(x/\alpha) > 1$. Since $0 < \alpha < 1$, by Part (i) of Proposition 5, we get $\alpha^M \sigma_p(x/\alpha) \leq \sigma_p(x)$. Thus, we have $\alpha^M < \sigma_p(x)$.

(ii) Let $\|x\| < \alpha$ and $\alpha \geq 1$. Then, $\|x/\alpha\| < 1$. By Part (i) of Proposition 6, $\|x/\alpha\| < 1$ implies $\sigma_p(x/\alpha) \leq \|x/\alpha\| < 1$. That is, $\sigma_p(x/\alpha) < 1$. If $\alpha = 1$, then $\sigma_p(x/\alpha) = \sigma_p(x) < 1 = \alpha^M$. If $\alpha > 1$, then by Part (ii) of Proposition 5, we have $\sigma_p(x) \leq \alpha^M \sigma_p(x/\alpha)$. This means that $\sigma_p(x) < \alpha^M$. □

Theorem 10. *Let (x_n) be a sequence in $\ell(\widetilde{B}, p)$. Then, the following statements hold:*

(i) $\lim_{n\to\infty}\|x_n\| = 1$ *implies* $\lim_{n\to\infty}\sigma_p(x_n) = 1$.

(ii) $\lim_{n\to\infty}\sigma_p(x_n) = 0$ *implies* $\lim_{n\to\infty}\|x_n\| = 0$.

Proof. Let (x_n) be a sequence in $\ell(\widetilde{B}, p)$.

(i) Let $\lim_{n\to\infty}\|x_n\| = 1$ and $\epsilon \in (0,1)$. Then, there exists $n_0 \in \mathbb{N}$ such that $1-\epsilon < \|x_n\| < \epsilon + 1$ for all $n \geq n_0$. By Parts (i) and (ii) of Theorem 9, $1-\epsilon < \|x_n\|$ implies $\sigma_p(x_n) > (1-\epsilon)^M$ and $\|x_n\| < \epsilon + 1$ implies $\sigma_p(x_n) < (1+\epsilon)^M$ for all $n \geq n_0$. This means $\epsilon \in (0,1)$ and for all $n \geq n_0$ there exists $n_0 \in \mathbb{N}$ such that $(1-\epsilon)^M < \sigma_p(x_n) < (1+\epsilon)^M$. That is, $\lim_{n\to\infty}\sigma_p(x_n) = 1$.

(ii) We assume that $\lim_{n\to\infty}\|x_n\| \neq 0$ and $\epsilon \in (0,1)$. Then, there exists a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \epsilon$ for all $k \in \mathbb{N}$. By Part (i) of Theorem 9, $0 < \epsilon < 1$ and $\|x_{n_k}\| > \epsilon$ imply $\sigma_p(x_{n_k}) > \epsilon^M$. Thus, $\lim_{n\to\infty}\sigma_p(x_n) \neq 0$ for all $k \in \mathbb{N}$. Hence, we obtain that $\lim_{n\to\infty}\sigma_p(x_n) = 0$ implies $\lim_{n\to\infty}\|x_n\| = 0$. □

Theorem 11. *Let $x \in \ell(\widetilde{B}, p)$ and $(x^{(n)}) \subset \ell(\widetilde{B}, p)$. If $\sigma_p(x^{(n)}) \to \sigma_p(x)$ as $n \to \infty$ and $x_k^{(n)} \to x_k$ as $n \to \infty$ for all $k \in \mathbb{N}$, then $x^{(n)} \to x$ as $n \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Since $\sigma_p(x) = \sum_k |(\widetilde{B}x)_k|^{p_k} < \infty$, there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left|\left(\widetilde{B}x\right)_k\right|^{p_k} < \frac{\epsilon}{3\left(2^{M+1}\right)}. \tag{33}$$

It follows from the fact

$$\lim_{n\longrightarrow\infty}\left[\sigma_p\left(x^{(n)}\right) - \sum_{k=1}^{k_0}\left|\left(\widetilde{B}x^{(n)}\right)_k\right|^{p_k}\right] = \sigma_p(x) - \sum_{k=1}^{k_0}\left|\left(\widetilde{B}x\right)_k\right|^{p_k} \tag{34}$$

that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}\sigma_p\left(x_{n_k}\right) &- \sum_{k=1}^{k_0}\left|\left(\widetilde{B}x^{(n)}\right)_k\right|^{p_k}\\ &< \sigma_p(x) - \sum_{k=1}^{k_0}\left|\left(\widetilde{B}x\right)_k\right|^{p_k} + \frac{\epsilon}{3\left(2^{M}\right)},\end{aligned} \tag{35}$$

and for all $n \geq n_0$,

$$\sum_{k=1}^{k_0}\left|\left\{\widetilde{B}\left(x^{(n)} - x\right)\right\}_k\right|^{p_k} < \frac{\epsilon}{3}. \tag{36}$$

Therefore, we obtain from (33), (35), and (36) that

$$\begin{aligned}\sigma_p(x_n - x) &= \sum_{k=1}^{\infty}\left|\left\{\widetilde{B}\left(x^{(n)} - x\right)\right\}_k\right|^{p_k}\\
&< \sum_{k=1}^{k_0}\left|\left\{\widetilde{B}\left(x^{(n)} - x\right)\right\}_k\right|^{p_k} + \sum_{k=k_0+1}^{\infty}\left|\left\{\widetilde{B}\left(x^{(n)} - x\right)\right\}_k\right|^{p_k}\\
&< \frac{\epsilon}{3} + 2^M\left[\sum_{k=k_0+1}^{\infty}\left|\left(\widetilde{B}x^{(n)}\right)_k\right|^{p_k} + \sum_{k=k_0+1}^{\infty}\left|\left(\widetilde{B}x\right)_k\right|^{p_k}\right]\\
&< \frac{\epsilon}{3} + 2^M\left[\sigma_p(x_n) - \sum_{k=1}^{k_0}\left|\left(\widetilde{B}x^{(n)}\right)_k\right|^{p_k} + \sum_{k_0+1}^{\infty}\left|\left(\widetilde{B}x\right)_k\right|^{p_k}\right]\\
&< \frac{\epsilon}{3} + 2^M\left[\sigma_p(x) - \sum_{k=1}^{k_0}\left|\left(\widetilde{B}x\right)_k\right|^{p_k} + \frac{\epsilon}{3\left(2^M\right)} + \sum_{k=k_0+1}^{\infty}\left|\left(\widetilde{B}x\right)_k\right|^{p_k}\right]\\
&< \frac{\epsilon}{3} + 2^M\left[2\sum_{k=k_0+1}^{\infty}\left|\left(\widetilde{B}x\right)_k\right|^{p_k} + \frac{\epsilon}{3\left(2^M\right)}\right]\\
&< \frac{\epsilon}{3} + 2^M\left[2\frac{\epsilon}{3\left(2^{M+1}\right)} + \frac{\epsilon}{3\left(2^M\right)}\right] = \epsilon.\end{aligned} \tag{37}$$

This means that $\sigma_p(x^{(n)} - x) \to 0$ as $n \to \infty$. By Part (ii) of Theorem 10, $\sigma_p(x^{(n)} - x) \to 0$ as $n \to \infty$ implies $\|x_n - x\| \to 0$ as $n \to \infty$. Hence, $x_n \to x$ as $n \to \infty$. □

Theorem 12. *The sequence space $\ell(\widetilde{B}, p)$ has the Kadec-Klee property.*

Proof. Let $x \in S[\ell(\widetilde{B}, p)]$ and $(x^{(n)}) \subset \ell(\widetilde{B}, p)$ such that $\|x^{(n)}\| \to 1$ and $x^{(n)} \xrightarrow{w} x$ are given. By Part (ii) of Theorem 10, we have $\sigma_p(x^{(n)}) \to 1$ as $n \to \infty$. Also $x \in S[\ell(\widetilde{B}, p)]$ implies $\|x\| = 1$. By Part (iii) of Proposition 6, we obtain $\sigma_p(x) = 1$. Therefore, we have $\sigma_p(x^{(n)}) \to \sigma_p(x)$ as $n \to \infty$.

Since $x^{(n)} \xrightarrow{w} x$ and $q_k : \ell(\widetilde{B}, p) \to \mathbb{R}$ defined by $q_k(x) = x_k$ is continuous, $x_k^{(n)} \to x_k$ as $n \to \infty$ for all $k \in \mathbb{N}$. Therefore, $x^{(n)} \to x$ as $n \to \infty$.

Since any weakly convergent sequence in $\ell(\widetilde{B}, p)$ is convergent, the sequence space $\ell(\widetilde{B}, p)$ has the Kadec-Klee property. □

Theorem 13. *For any $1 < p < \infty$, the space $(\ell_p)_{\widetilde{B}}$ has the uniform Opial property.*

Proof. Let $\epsilon > 0$ and $\epsilon_0 \in (0, \epsilon)$ be given such that $1 + (\epsilon^p/2) > (1 + \epsilon_0)^p$. Also let $x \in (\ell_p)_{\widetilde{B}}$ and $\|x\| \geq \epsilon$. There exists $k_1 \in \mathbb{N}$ such that

$$\sum_{k=k_1+1}^{\infty}\left|\left(\widetilde{B}x\right)_k\right|^p < \left(\frac{\epsilon_0}{4}\right)^p. \tag{38}$$

Hence, we have

$$\left\|\sum_{k=k_1+1}^{\infty} x_k e_k\right\| < \frac{\epsilon_0}{4}. \tag{39}$$

Furthermore, we have

$$\begin{aligned}\epsilon^p &\leq \sum_{k=1}^{k_1}\left|\left(\widetilde{B}x\right)_k\right|^p + \sum_{k=k_1+1}^{\infty}\left|\left(\widetilde{B}x\right)_k\right|^p \\ &< \sum_{k=1}^{k_1}\left|\left(\widetilde{B}x\right)_k\right|^p + \left(\frac{\epsilon_0}{4}\right)^p \\ &< \sum_{k=1}^{k_1}\left|\left(\widetilde{B}x\right)_k\right|^p + \frac{\epsilon^p}{4},\end{aligned} \tag{40}$$

which yields that

$$\frac{3\epsilon^p}{4} < \sum_{k=1}^{k_1}\left|\left(\widetilde{B}x\right)_k\right|^p. \tag{41}$$

For any weakly null sequence $(x^{(m)}) \subset S[(\ell_p)_{\widetilde{B}}]$, since $x_k^{(m)} \to 0$ as $m \to \infty$ for each $k \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that for all $m > m_0$,

$$\left\|\sum_{k=1}^{k_1} x_k^{(m)} e_k\right\| < \frac{\epsilon^p}{4}. \tag{42}$$

Therefore, for all $m > m_0$,

$$\begin{aligned}\left\|x^{(m)} + x\right\| &= \left\|\sum_{k=1}^{k_1}\left(x_k^{(m)} + x_k\right)e_k + \sum_{k=k_1+1}^{\infty}\left(x_k^{(m)} + x_k\right)e_k\right\| \\ &\geq \left\|\sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k\right\| \\ &\quad - \left\|\sum_{k=1}^{k_1} x_k^{(m)} e_k\right\| - \left\|\sum_{k=k_1+1}^{\infty} x_k e_k\right\| \\ &\geq \left\|\sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k\right\| - \frac{\epsilon^p}{4} - \frac{\epsilon^p}{4}.\end{aligned} \tag{43}$$

Moreover,

$$\begin{aligned}&\left\|\sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k\right\|^p \\ &\quad= \sum_{k=1}^{k_1}\left|\left(\widetilde{B}x\right)_k e_k\right|^p + \sum_{k=k_1+1}^{\infty}\left|\left(\widetilde{B}x^{(m)}\right)_k e_k\right|^p \\ &\quad\geq \frac{3\epsilon^p}{4} + \left(1 - \frac{\epsilon^p}{4}\right) \\ &\quad= 1 + \frac{\epsilon^p}{2} \\ &\quad> \left(1 + \epsilon_0\right)^p.\end{aligned} \tag{44}$$

Then, we have

$$\begin{aligned}\left\|x^{(m)} + x\right\| &\geq \left\|\sum_{k=1}^{k_1} x_k e_k + \sum_{k=k_1+1}^{\infty} x_k^{(m)} e_k\right\| - \frac{\epsilon^p}{2} \\ &\geq 1 + \epsilon_0 - \frac{\epsilon^p}{2} \\ &> 1 + \frac{\epsilon_0^p}{2}.\end{aligned} \tag{45}$$

This means that $(\ell_p)_{\widetilde{B}}$ has the uniform Opial property. □

3. Conclusion

The sequence spaces $bv(u, p)$ and $bv_\infty(u, p)$ of nonabsolute type consisting of all sequences $x = (x_k)$ such that $\{u_k(x_k - x_{k-1})\}$ is in the Maddox' spaces $\ell(p)$ and $\ell_\infty(p)$ were introduced by Başar et al. [13], where $u = (u_k)$ is a sequence such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and the rotundity of the space $bv(u, p)$ was examined.

The sequence space $a^r(u, p)$ of nonabsolute type consisting of all sequences $x = (x_k)$ such that $A^r x = \{\sum_{k=0}^{n}(1 + r^k)x_k/(n + 1)\} \in \ell(p)$ was studied by Aydın and Başar [14], and some results related to the rotundity of the space $a^r(u, p)$ were given.

Quite recently, the sequence space $\widehat{\ell}(p)$ of nonabsolute type consisting of all sequences $x = (x_k)$ such that $B(r, s)x = (sx_{k-1} + rx_k) \in \ell(p)$ was defined by Aydın and Başar [15], and emphasized the rotundity of the space $\widehat{\ell}(p)$ together with some related results.

Although the sequence spaces $a^r(u, p)$ and $\ell(\widetilde{B}, p)$ are not comparable, since the double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ reduces to the generalized difference matrix $B(r, s)$ in the special case $\widetilde{r} = re$ and $\widetilde{s} = se$, the new space $\ell(\widetilde{B}, p)$ is more general than the space $\widehat{\ell}(p)$. Similarly, the sequence space $\ell(\widetilde{B}, p)$ is also reduced to the space $bv(u, p)$ in the case $\widetilde{r} = (u_k)$ and $\widetilde{s} = (-u_k)$. So, the results on the space $\ell(\widetilde{B}, p)$ are much more comprehensive than the results on the space $bv(u, p)$. Additionally, the corresponding theorems on the Kadec-Klee property of the space $\ell(\widetilde{B}, p)$ and the uniform Opial property of the space $(\ell_p)_{\widetilde{B}}$ were not given by Başar et al. [13] and Aydın and Başar [15] which make the present paper significant.

Acknowledgments

The main results of this paper were presented in part at the conference First International Conference on Analysis and Applied Mathematics (ICAAM 2012) held on October 18–21, 2012 in Gümüşhane, Turkey, at the University of Gümüşhane.

References

[1] I. J. Maddox, "Spaces of strongly summable sequences," *The Quarterly Journal of Mathematics. Oxford*, vol. 18, no. 2, pp. 345–355, 1967.

[2] S. Simons, "The sequence spaces $\ell(p_v)$ and $m(p_v)$," *Proceedings of the London Mathematical Society*, vol. 15, no. 3, pp. 422–436, 1965.

[3] H. Nakano, "Modulared sequence spaces," *Proceedings of the Japan Academy*, vol. 27, no. 2, pp. 508–512, 1951.

[4] H. Nergiz and F. Başar, "Some topological and geometric properties of the domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$ in the sequence space $\ell(p)$," *AIP Conference Proceedings*, vol. 1470, no. 1, pp. 163–168, 2012.

[5] P. D. Srivastava and S. Kumar, "Fine spectrum of the generalized difference operator Δ_v on sequence space l_1," *Thai Journal of Mathematics*, vol. 8, no. 2, pp. 221–233, 2010.

[6] P. D. Srivastava and S. Kumar, "Fine spectrum of the generalized difference operator Δ_{uv} on sequence space l_1," *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6407–6414, 2012.

[7] B. L. Panigrahi and P. D. Srivastava, "Spectrum and fine spectrum of generalized second order difference operator Δ^2_{uv} on sequence space c_0," *Thai Journal of Mathematics*, vol. 9, no. 1, pp. 57–74, 2011.

[8] A. M. Akhmedov and S. R. El-Shabrawy, "On the fine spectrum of the operator $\Delta_{a,b}$ over the sequence space c," *Computers & Mathematics with Applications*, vol. 61, no. 10, pp. 2994–3002, 2011.

[9] H. Nergiz and F. Başar, "Domain of the double sequential band matrix $B(\tilde{r}, \tilde{s})$ in the sequence space $\ell(p)$," *Abstract and Applied Analysis*. In press.

[10] S. Chen, "Geometry of Orlicz spaces," *Dissertationes Mathematicae*, vol. 356, pp. 1–224, 1996.

[11] J. Diestel, *Geometry of Banach Spaces—Selected Topics*, Springer, Berlin, Germany, 1984.

[12] L. Maligranda, *Orlicz Spaces and Interpolation*, Institute of Mathematics, Polish Academy of Sciences, Poznan, Poland, 1985.

[13] F. Başar, B. Altay, and M. Mursaleen, "Some generalizations of the space bv_p of p-bounded variation sequences," *Nonlinear Analysis: Theory, Methods &Applications*, vol. 68, no. 2, pp. 273–287, 2008.

[14] C. Aydın and F. Başar, "Some generalizations of the sequence space a^r_p," *Iranian Journal of Science and Technology. Transaction A*, vol. 30, no. A2, pp. 175–190, 2006.

[15] C. Aydın and F. Başar, "Some topological and geometric properties of the domain of the generalized difference matrix $B(r, s)$ in the sequence space $\ell(p)$," *Thai Journal of Mathematics*. In press.