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Research Article

Some Additions to the Fuzzy Convergent and Fuzzy Bounded Sequence Spaces of Fuzzy Numbers

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Some properties of the fuzzy convergence and fuzzy boundedness of a sequence of fuzzy numbers were studied in Choi (1996). In this paper, we have consider, some important problems on these spaces and shown that these spaces are fuzzy complete module spaces. Also, the fuzzy α-, fuzzy β-, and fuzzy γ-duals of the fuzzy module spaces of fuzzy numbers have been computeded, and some matrix transformations are given.

1. Introduction

As known, the ideas of fuzzy sets and fuzzy operations were first introduced by Zadeh [1], and after his innovation, many authors have studied different aspects of the fuzzy numbers theory and applications. One of them is the sequence spaces of the fuzzy numbers. A major direction in the study on sequence spaces of fuzzy numbers is the metric properties of these spaces (see [2–4]), but this direction has been altered by Talo and Başar [5].

Some important problems on sequence spaces of fuzzy numbers can be ordered as follows:

(1) to construct a sequence space of fuzzy numbers and compute α-, β- and γ-duals,

(2) to find some isomorphic spaces of it,

(3) to give some theorems about matrix transformation on sequence space of fuzzy numbers,

(4) to study some inclusion problems and other properties.

By using the metric, $\widehat{d}(u,v) = \sup_k \overline{d}(u_k, v_k)$, these problems have been nicely studied by Talo and Başar in [5]. But, as known, by defining different metrics on sequence spaces of fuzzy numbers, different metric spaces can be built up. By using $\widehat{d}$ metric, so many spaces of fuzzy sequences have been built and many metric properties have been investigated. In literature, one can easily reach many documents about sequence space of fuzzy numbers.

In this paper, using a *fuzzy metric*, we will deal with some problems about fuzzy convergent and fuzzy bounded sequence spaces of fuzzy numbers which did not come up in [6]. Thus, we believe that some gaps in this area will be filled.

The rest of this paper is structured as follows.

Some required definitions and consequences related with the fuzzy numbers given in Section 2. Besides, as a proposition, the complete fuzzy module space of fuzzy numbers is given, and the sequence spaces of fuzzy numbers with fuzzy metric are introduced in this section. In Section 3, we have stated and proved the theorems determining the fuzzy α-, fuzzy β-, and fuzzy γ-duals of the fuzzy sequence space of fuzzy numbers. Finally, in Section 4, the fuzzy classes $(\ell_\infty(E^1) : \ell_\infty(E^1))$ and $(c_0(E^1) : c_0(E^1))$ of infinite matrix of fuzzy numbers are characterized.

2. Preliminaries

Lets suppose that $\mathbb{N}$ is the set of all positive integer numbers, $\mathbb{R}$ is the set of all real numbers, is the E_i be the set of all bounded and closed intervals on the real line $\mathbb{R}$, that is, $E_i = \{a = [a^-, a^+] : a^- \leq x \leq a^+,\ a^- \text{ and } a^+ \in \mathbb{R}\}$. For $a, b \in E_i$ and define

$$d(a,b) = \max\{|a^- - b^-|, |a^+ - b^+|\}. \tag{2.1}$$

Then, it can be seen easily that d defines a metric on E_i and (E_i, d) is a complete metric space [7]. Let X be nonempty set. According to Zadeh, a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u : X \to [0,1]$, [8]. Consider a function $u : \mathbb{R} \to [0,1]$ as a subset of a nonempty base space $\mathbb{R}$ and denote the totality of all such functions or fuzzy sets by E. A fuzzy number (FN) is a function u from $\mathbb{R}$ to $[0,1]$, which satisfies the following properties:

FN1 u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,

FN2 u is fuzzy convex, that is, for any $x, y \in \mathbb{R}$ and $\mu \in [0,1]$, $u[\mu x + (1-\mu)y] \geq \min\{u(x), u(y)\}$,

FN3 u is upper semicontinuous,

FN4 the closure of $\{x \in \mathbb{R} : u(x) > 0\}$, denoted by u^0, is compact.

(FN1), (FN2), (FN3), and (FN4) imply that for each $\alpha \in [0,1]$, the α-level set defined by $[u]_\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$ is in E_i, as well as the support u^0, that is, $u_\alpha = [u_\alpha^-, u_\alpha^+]$ for each $\alpha \in [0,1]$. We denote the set of all fuzzy numbers by E^1.

Let us suppose that $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for $u \in E^1$ and for each $\lambda \in [0,1]$. Then the following statements are held:

(1) $u^-(\lambda)$ is a bounded and nondecreasing left continuous function on $[0,1)$,

(2) $u^+(\lambda)$ is a bounded and nonincreasing left continuous function on $(0,1]$,

(3) The functions $u^+(\lambda)$ and $u^-(\lambda)$ are right continuous at the point $\lambda = 0$,

(4) $u^+(\lambda) \geq u^-(\lambda)$.

Conversely, if the pair of functions α and β satisfy the conditions (1)–(4), then there exists a unique $u \in E^1$ such that $[u]_\lambda = [\alpha^-(\lambda), \beta^+(\lambda)]$ for each $\lambda \in [0,1]$, [9]. The fuzzy number u corresponding to the pair of functions α and β is defined by $u : \mathbb{R} \to [0,1]$, $u(x) = \sup\{\lambda : \alpha(x) \leq \lambda \leq \beta(x)\}$, [5].

A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$, the set of all positive integers, into E^1, and fuzzy number u_k denotes the value of the function at k and is called the kth term of the sequence.

Let $u, v \in E^1$ and $\lambda \in \mathbb{R}$, then the operations addition and scalar multiplication are defined on E^1 in terms of α-level sets by

$$u + v = w \Longleftrightarrow [w]_\alpha = [u]_\alpha + [v]_\alpha, \quad [\lambda u]_\alpha = \lambda [u]_\alpha \quad \forall \alpha \in [0,1]. \tag{2.2}$$

Define a map $\overline{d} : E^1 \times E^1 \to \mathbb{R}$ by $\overline{d}(u,v) = \sup_{0\leq\alpha\leq1} d([u]_\alpha, [v]_\alpha)$. It is known that E^1 is a complete metric space with the metric $\overline{d}$ [3]. Let us suppose that $w(E^1)$, $c(E^1)$, and $\ell_\infty(E^1)$ are set of all sequences space of all fuzzy numbers, convergent and bounded sequences of fuzzy numbers, respectively.

Let us suppose that $u, v \in E^1$ and G is the set of all nonnegative fuzzy numbers. The function $d_f : E^1 \times E^1 \to G$ is called fuzzy metric [6] which satisfies the following properties:

(1) $d_f(u,v) \geq 0$,

(2) $d_f(u,v) = 0$ if and only if $u = v$,

(3) $d_f(u,v) = d_f(v,u)$,

(4) whenever $w \in E^1$, we have $d_f(u,v) \leq d_f(u,w) + d_f(w,v)$.

In [4], Nanda has studied the spaces of bounded and convergent sequences of fuzzy numbers and has shown that they are complete metric spaces with the metric $\widehat{d}(u,v) = \sup_k \overline{d}(u_k, v_k)$.

By using this metric, $\widehat{d}$, so many spaces of fuzzy sequences have been built and published in famous maths journals. By reviewing the literature, one can reach them easily. However, another important metric which is called as fuzzy metric is used for measuring *fuzzy distances* among fuzzy numbers.

If d_f is a fuzzy metric on E^1, then the pair of (E^1, d_f) is called as a fuzzy metric space. For any $u, v \in E^1$, the fuzzy metric of Zhang is [10–12] defined by

$$d_f(u,v) = \sup_{\lambda\in[0,1]} \lambda \left[d_1(u,v), \widehat{d}_\lambda(u,v) \right], \tag{2.3}$$

where $\widehat{d}_\lambda(u,v) = \sup_{\alpha\in[\lambda,1]} d([u]_\alpha, [v]_\alpha)$ and $d_1(u,v) = \sup_{\alpha=1} d([u]_\alpha, [v]_\alpha)$.

Also, fuzzy metric spaces have been studied in [13]. But the given metric space definition in [13] is different from our fuzzy metric space definition.

Theorem 2.1 (see [10–12]). *The metric d_f defined by equality* (2.3) *is a fuzzy distance of fuzzy numbers; thus, (E^1, d_f) is a fuzzy metric space.*

Theorem 2.2. *If $u, v \in E^1$, then $\overline{d}(u,v)$ is a point of the interval determined by the fuzzy metric $d_f(u,v)$.*

Proof. Clearly, if λ run from 0, then the second side of d_f, $\widehat{d}_\lambda(u,v)$, is equal to $\overline{d}(u,v)$ since

$$d_f(u,v) = \sup_{\lambda\in[0,1]} \lambda\Big[d_1(u,v), \widehat{d}_\lambda(u,v)\Big]. \tag{2.4}$$

□

Theorem 2.3 (see [12]). *The fuzzy metric space (E^1, d_f) is complete metric space.*

Definition 2.4. Let $\lambda(E^1)$ be the subset of all sequence spaces of fuzzy numbers and suppose that $\|\cdot\| : \lambda(E^1) \to G$ is a function. The function $\|\cdot\|$ is called fuzzy module or fuzzy norm if it has the following properties:

(N1) $\|u\| = \theta \Leftrightarrow u = \theta$,

(N2) $\|\alpha u\| = |\alpha|\|u\|_{E^1}$,

(N3) $\|u+v\| \leq \|u\| + \|v\|$

If the function $\|\cdot\| : \lambda(E^1) \to G$ satisfies $N1$, $N2$, and $N3$, then $\lambda(E^1)$ is called fuzzy module sequence space of the fuzzy numbers. And if $\lambda(E^1)$ is complete with respect to the fuzzy module, then $\lambda(E^1)$ is called complete fuzzy module sequence space of the fuzzy numbers.

Definition 2.5. The fuzzy module of the fuzzy number u is defined which corresponds to the fuzzy distance from u to $\overline{0}$, that is,

$$\|u\|_{E^1} := \sup_{\lambda\in[0,1]} \lambda\Big[d_1\Big(u,\overline{0}\Big), \widehat{d}_\lambda\Big(u,\overline{0}\Big)\Big]. \tag{2.5}$$

Proposition 2.6. *The set E^1 of the fuzzy numbers is fuzzy complete module space with the fuzzy module in* (2.5).

Let $u = (u_k)$ be a sequence of fuzzy numbers, and let $\|\cdot\|$ be a fuzzy module, then the sequence (u_k) is said to converge fuzzy to $u_0 \in E^1$ with the fuzzy module $\|\cdot\|$ if for any given $\epsilon > 0$, there exists an integer n_0 such that $\|u_k - u_0\| < [\epsilon, \epsilon] = \epsilon$ for $k \geq n_0$. The sequence (u_k) is said to be fuzzy bounded in fuzzy module $\|\cdot\|$ if $\sup_k \|u_k\| < \infty$ for all $k \in \mathbb{N}$.

We will write $\mathfrak{L}_\infty(E^1)$, $\mathfrak{C}(E^1)$, and $\mathfrak{C}_0(E^1)$ for the fuzzy sets of all fuzzy bounded, fuzzy convergent, fuzzy null sequences, respectively, that is,

$$\begin{aligned}
\mathfrak{L}_\infty\Big(E^1\Big) &:= \left\{u = (u_k) \in w\Big(E^1\Big) : \sup_k \sup_{\lambda\in[0,1]} \lambda\Big[d_1\Big(u_k,\overline{0}\Big), \widehat{d}_\lambda\Big(u_k,\overline{0}\Big)\Big] < \infty\right\},\\
\mathfrak{C}\Big(E^1\Big) &:= \left\{u = (u_k) \in w\Big(E^1\Big) : \exists u_0 \in E^1 \ni \lim_k \sup_{\lambda\in[0,1]} \lambda\Big[d_1(u_k,u_0), \widehat{d}_\lambda(u_k,u_0)\Big] = \theta\right\},\\
\mathfrak{C}_0\Big(E^1\Big) &:= \left\{u = (u_k) \in w\Big(E^1\Big) : \lim_k \sup_{\lambda\in[0,1]} \lambda\Big[d_1\Big(u_k,\overline{0}\Big), \widehat{d}_\lambda\Big(u_k,\overline{0}\Big)\Big] = \theta\right\}.
\end{aligned} \tag{2.6}$$

In [6], Kong and Cho has proved that the fuzzy convergent sequence spaces of fuzzy numbers $\mathfrak{C}(E^1)$ and fuzzy bounded sequence spaces of fuzzy numbers $\mathfrak{L}_\infty(E^1)$ are fuzzy complete metric spaces with fuzzy metric $\widehat{d}_f$ defined by

$$\widehat{d}_f(u,v) = \sup_k \sup_{\lambda\in[0,1]} \lambda\left[d_1(u_k,v_k), \widehat{d}_\lambda(u_k,v_k)\right]. \tag{2.7}$$

Now, let us give some definitions that will facilitate our work.

A sequence $u = (u_n)$ in E^1 is said to be a fuzzy fundamental sequence if forever $\varepsilon > 0$, there exists an integer n_0 such that $d_f(u_n,u_m) < \varepsilon$ for $n,m > n_0$. A fuzzy metric space (E^1, d_f) is called the fuzzy complete metric space if every fundamental sequence converges in E^1.

Theorem 2.7 (see [10–12]). *The sequence u in E^1 is fuzzy convergent in the metric d_f if and only if u is a fuzzy fundamental sequence.*

Theorem 2.8. *The fuzzy metric spaces $(\mathfrak{C}(E^1), \widehat{d}_f)$ and $(\mathfrak{L}_\infty(E^1), \widehat{d}_f)$ are fuzzy complete metric spaces.*

Theorem 2.9 (see [14]). *Let $\mathcal{U} \subset E^1$, and let (u_k) be a sequence of fuzzy functions from $\mathcal{U}$ to E^1. If for each k, there exists a real number M_k such that $\max\{|(u_k(U))^-_\lambda|, |(u_k(U))^+_\lambda|\} \le M_k$ for all $U \in \mathcal{U}$ and $\lambda \in (0,1]$, and if the series $\sum_k M_k$ converges, then there is a fuzzy function $u : \mathcal{U} \to E^1$ such that $\sum_k u_k$ converges uniformly to u.*

Now, let us define the sequence sets $\mathfrak{C}s(E^1)$, $\mathfrak{B}s(E^1)$, and $\mathfrak{L}_p(E^1)$ as the set of all fuzzy convergent series of FNs, the set of all fuzzy bounded series of FNs, and the set of p-absolutely fuzzy convergent series of the FNs, respectively, that is,

$$\begin{aligned}
\mathfrak{C}s\left(E^1\right) &= \left\{u = (u_k) \in w\left(E^1\right) : \lim_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_{k=1}^n u_k, u_0\right), \widehat{d}_\lambda\left(\sum_{k=1}^n u_k, u_0\right)\right] = 0\right\},\\
\mathfrak{B}s\left(E^1\right) &= \left\{u = (u_k) \in w\left(E^1\right) : \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_{k=1}^n u_k, \overline{0}\right), \widehat{d}_\lambda\left(\sum_{k=1}^n u_k, \overline{0}\right)\right] < \infty\right\},\\
\mathfrak{L}_p\left(E^1\right) &= \left\{u = (u_k) \in w\left(E^1\right) : \left(\sum_k \left(\sup_{\lambda\in[0,1]} \lambda[d_1(u_k,\overline{0}), \widehat{d}_\lambda(u_k,\overline{0})]\right)^p\right)^{1/p} < \infty,\ 1 \le p < \infty\right\}.
\end{aligned} \tag{2.8}$$

Let us see the following theorems which about the sets $\mathfrak{L}_\infty(E^1)$, $\mathfrak{C}(E^1)$, and $\mathfrak{C}_0(E^1)$.

Theorem 2.10. *The sets $\mathfrak{L}_\infty(E^1)$, $\mathfrak{C}(E^1)$, and $\mathfrak{C}_0(E^1)$ are fuzzy complete module spaces defined by fuzzy module*

$$\|u\| = \sup_k \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k,\overline{0}\right), \widehat{d}_\lambda\left(u_k,\overline{0}\right)\right]. \tag{2.9}$$

Proof. Since the proof for $\mathfrak{C}(E^1)$ and $\mathfrak{C}_0(E^1)$ can be proved in a similar way, we will consider only $\mathfrak{L}_\infty(E^1)$. Clearly, it is straightforward to see that $\|u\|$ is a fuzzy module on $\mathfrak{L}_\infty(E^1)$. To show that $\mathfrak{L}_\infty(E^1)$ is fuzzy complete in this fuzzy module, let us suppose that (u_k) is a fuzzy fundamental sequence in $\mathfrak{L}_\infty(E^1)$ where $(u_k) = (u_0^{(i)}, u_1^{(i)}, u_2^{(i)}, \ldots)$, then, for any $\epsilon > 0$, there exists an integer n_0 such that

$$\left\|u_k^i - u_k^j\right\| = \sup_k \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k^i(1) - u_k^j(1), \overline{0}\right), \widehat{d}_\lambda\left(u_k^i(\alpha) - u_k^j(\alpha), \overline{0}\right)\right] \le [\epsilon, \epsilon] = \epsilon, \tag{2.10}$$

for $i, j \ge n_0$. Hence, we obtain $|u_k^i(1)^- - u_k^j(1)^-| < \epsilon/\lambda$ and $\sup_{\alpha\in[\lambda,1]} \max\{|u_k^{i-}(\alpha) - u_k^{j-}(\alpha)|, |u_k^{i+}(\alpha) - u_k^{j+}(\alpha)|\} < \epsilon/\lambda$. This shows that $(u_k^i(1)^-)$ is fundamental sequence of real numbers in $\mathbb{R}$, and $(u_k^i(\alpha))$ is fundamental sequence of fuzzy numbers in E^1. Since $\mathbb{R}$ and E^1 are complete, so $(u_k^i(1))$ converges in $\mathbb{R}$ and $(u_k^i(\alpha))$ converges in E^1 for all $k \in \mathbb{N}$.

Let us suppose that $\lim_i u_k^i(1)^- = u_k(1)^-$ and $\lim_i u_k^i(\alpha) = u_k(\alpha)$ for each $k \in \mathbb{N}$. Put $u(1)^- = (u_k(1)^-)$ and $u(\alpha) = (u_k(\alpha))$. Now, we shall show that $\lim_i u^i(1)^- = u(1)^-$ and $u(1)^- \in \mathbb{R}$, $\lim_i u^i(\alpha) = u(\alpha)$ and $u(\alpha) \in E^1$. Since $(u^i(1))$ is a fundamental sequence in $\mathbb{R}$, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $i, j > n_0$, $|u^i(1) - u^j(1)| < \epsilon/\lambda$, and if we take limit over j, we get $|u^i(1) - u(1)| < \epsilon/\lambda$. Therefore, $\lim_i u^i(1) = u(1)$. Similarly, since $(u_k^i(\alpha))$ is a fundamental sequence in E^1, given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $i, j > n_0$, $\sup_{\alpha\in[\lambda,1]} \max\{|u_k^{i-}(\alpha) - u_k^{j-}(\alpha)|, |u_k^{i+}(\alpha) - u_k^{j+}(\alpha)|\} < \epsilon/\lambda$, and if we take limit over j we get $\sup_{\alpha\in[\lambda,1]} \max\{|u_k^{i-}(\alpha) - u_k^{-}(\alpha)|, |u_k^{i+}(\alpha) - u_k^{+}(\alpha)|\} < \epsilon/\lambda < \epsilon$. Therefore, $\lim_i u^i(\alpha) = u_k(\alpha)$. Let us show that $u_k(1)^- \in \mathbb{R}$ and $u_k(\alpha) \in E^1$. Also, since $|u(1)^-| \le |u(1)^- - u_k^i(1)^-| + |u_k^i(1)^-| < \infty$ and $\sup_{\alpha\in[\lambda,1]} \max\{|u_k^-(\alpha)|, |u_k^+(\alpha)|\} \le \sup_{\alpha\in[\lambda,1]} \max\{|u_k^-(\alpha) - u_k^{i-}(\alpha)|, |u_k^+(\alpha) - u_k^{i+}(\alpha)|\} + \sup_{\alpha\in[\lambda,1]} \max\{|u_k^{i-}(\alpha)|, |u_k^{i+}(\alpha)|\} < \infty$, this shows that $u \in \mathfrak{L}_\infty(E^1)$. □

Theorem 2.11. *The space $\mathfrak{L}_p(E^1)$ is fuzzy complete module space defined by module*

$$\|u\|_{\mathfrak{L}_p(E^1)} = \left(\sum_k \left(\sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k, \overline{0}\right), \widehat{d}_\lambda\left(u_k, \overline{0}\right)\right]\right)^p\right)^{1/p} < \infty, \quad 1 \le p < \infty. \tag{2.11}$$

Proof. Since the proof is similar to the proof of the Theorem 2.10, we omit it. □

3. Construction of the Fuzzy Duals of the Fuzzy Module Sequence Spaces

For the fuzzy sequence spaces $\lambda(E^1)$ and $\mu(E^1)$, define the set $S(\lambda(E^1), \mu(E^1))$ by

$$S\left(\lambda\left(E^1\right), \mu\left(E^1\right)\right) = \left\{u \in w\left(E^1\right) : uv = (u_k v_k) \in \mu\left(E^1\right)\ \forall u = (u_k) \in \lambda\left(E^1\right)\right\}. \tag{3.1}$$

With the notation of (3.1), the fuzzy α-, fuzzy β-, and fuzzy γ-duals of a fuzzy sequence space $\lambda(E^1)$, which are, respectively, denoted by $\lambda^\alpha(E^1)$, $\lambda^\beta(E^1)$, and $\lambda^\gamma(E^1)$, are defined by

$$\begin{aligned}
\lambda^\alpha\left(E^1\right) &= S\left(\lambda\left(E^1\right), \mathfrak{L}_1\left(E^1\right)\right),\\
\lambda^\beta\left(E^1\right) &= S\left(\lambda\left(E^1\right), \mathfrak{C}s\left(E^1\right)\right),\\
\lambda^\gamma\left(E^1\right) &= S\left(\lambda\left(E^1\right), \mathfrak{B}s\left(E^1\right)\right).
\end{aligned}\tag{3.2}$$

Definition 3.1. Let us suppose that $\lambda(E^1)$, $\mu(E^1)$ are sets of the fuzzy sequences of FNs and $\lambda(E^1) \subset \mu(E^1)$, then $\lambda(E^1)$ is called fuzzy cofinal in $\mu(E^1)$ if for $(u_k) \in \lambda(E^1)$ there is $(v_k) \in \mu(E^1)$ such that $\|u_k\| \leq \|v_k\|$ for all $k \in \mathbb{N}$.

If $\lambda(E^1)$ is fuzzy cofinal in $\mu(E^1)$, then $\lambda^\alpha(E^1) = \mu^\alpha(E^1)$; the converse of this assertion is not true.

Now, we may give results concerning the fuzzy α-dual, fuzzy β-dual, and fuzzy γ-dual of the sets $\mathfrak{C}_0(E^1)$, $\mathfrak{C}(E^1)$, $\mathfrak{L}_\infty(E^1)$, and $\mathfrak{L}_p(E^1)$.

Theorem 3.2. *The fuzzy α-dual of the set $\mathfrak{L}_\infty(E^1)$ of sequence spaces of FNs is the set $\mathfrak{L}_1(E^1)$.*

Proof. Let $(u_k) \in \mathfrak{L}_\infty^\alpha(E^1)$. If we consider $(v_k) = ([\overline{1}, \overline{1}]) \in \mathfrak{L}_\infty(E^1)$, then the series

$$\begin{aligned}
&\sum_k \left(\sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k v_k, \overline{0}\right), \widehat{d}_\lambda\left(u_k v_k, \overline{0}\right)\right]\right)\\
&\quad= \sum_k \left(\sup_{\lambda\in[0,1]} \lambda\left[\left|u_k^-(1)v_k^-(1)\right|, \sup_{\alpha\in[\lambda,1]} \max\left\{\left|u_k^-(\alpha)v_k^-(\alpha)\right|, \left|u_k^+(\alpha)v_k^+(\alpha)\right|\right\}\right]\right)\\
&\quad\leq \sum_k \left(\sup_{\lambda\in[0,1]} \lambda\left[\left|u_k^-(1)\right|, \sup_{\alpha\in[\lambda,1]} \max\left\{\left|u_k^-(\alpha)\right|, \left|u_k^+(\alpha)\right|\right\}\right]\right.\\
&\qquad\qquad \left.\sup_{\lambda\in[0,1]} \lambda\left[\left|v_k^-(1)\right|, \sup_{\alpha\in[\lambda,1]} \max\left\{\left|v_k^-(\alpha)\right|, \left|v_k^+(\alpha)\right|\right\}\right]\right)\\
&\quad= \sum_k \left(\sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k, \overline{0}\right), \widehat{d}_\lambda\left(u_k, \overline{0}\right)\right]\right)
\end{aligned}\tag{3.3}$$

converges, that is to say, $(u_k) \in \mathfrak{L}_1(E^1)$. Therefore, we have

$$\mathfrak{L}_\infty^\alpha\left(E^1\right) \subseteq \mathfrak{L}_1\left(E^1\right).\tag{3.4}$$

Conversely, let us suppose that $(u_k) \in \mathfrak{L}_\infty(E^1)$ and $(v_k) \in \mathfrak{L}_1(E^1)$, then there exists a $\overline{0} < K \in E^1$ such that $K = \sup_k \sup_{\lambda \in [0,1]} \lambda[|d_1(u_k(1), \overline{0}), \widehat{d}_\lambda(u_k(\alpha), \overline{0})] < \infty$. From here, we have

$$\begin{aligned}
&\sum_k \left(\sup_{\lambda \in [0,1]} \lambda \left[d_1\left(u_k(1)v_k(1), \overline{0}\right), \widehat{d}_\lambda\left(u_k(\alpha)v_k(\alpha), \overline{0}\right) \right] \right) \\
&\quad = \sum_k \left(\sup_{\lambda \in [0,1]} \lambda \left[d_1\left(v_k(1), \overline{0}\right), \widehat{d}_\lambda\left(v_k(\alpha), \overline{0}\right) \right] \right) \|u\| \\
&\quad \leq K \sum_k \left(\sup_{\lambda \in [0,1]} \lambda \left[d_1\left(v_k(1), \overline{0}\right), \widehat{d}_\lambda\left(v_k(\alpha), \overline{0}\right) \right] \right) = K \|v\|_{\mathfrak{L}_1(E^1)} < \infty,
\end{aligned} \tag{3.5}$$

which gives that

$$\mathfrak{L}_1\left(E^1\right) \subseteq \mathfrak{L}_\infty^\alpha\left(E^1\right). \tag{3.6}$$

From (3.4) and (3.6), we see that $\mathfrak{L}_1(E^1) = \mathfrak{L}_\infty^\alpha(E^1)$. □

Theorem 3.3. *The fuzzy sequence spaces $\mathfrak{C}_0(E^1)$, $\mathfrak{C}(E^1)$ are cofinal in $\mathfrak{L}_1(E^1)$.*

Proof. Denote any of the spaces $\mathfrak{C}_0(E^1)$ and $\mathfrak{C}(E^1)$ by $\lambda(E^1)$, and suppose that $\|u_k\| \leq \|v_k\|$ holds for some $(v_k) \in \mathfrak{L}_1(E^1)$, then we can easily see that $\sup_k \|u_k\| \leq \sup_k \|v_k\|$, $\lim_k \|u_k\| \leq \sup_k \|v_k\|$ which lead us to the desired results. □

Theorem 3.4. *The fuzzy α-dual of the sets $\mathfrak{C}_0(E^1)$ and $\mathfrak{C}(E^1)$ of sequence spaces of FNs is the set $\mathfrak{L}_1(E^1)$.*

Proof. Since the sets $\mathcal{C}_0(E^1)$ and $\mathfrak{C}(E^1)$ are cofinal in $\mathfrak{L}_\infty(E^1)$ (see Theorem 3.3), the proof is clear. □

Theorem 3.5. *The fuzzy β-dual of the sets $\mathfrak{C}(E^1)$ and $\mathfrak{L}_\infty(E^1)$ of sequence spaces of FNs is the set $\mathfrak{L}_1(E^1)$.*

Proof. We give the proof only for the set $\mathfrak{L}_\infty(E^1)$. Let us suppose that $(u_k) \in \mathfrak{L}_\infty(E^1)$ and $(v_k) \in w(E^1)$, then there is a $K > 0$ such that $K = [K_1, K_2] = \sup_k \sup_{\lambda \in [0,1]} \lambda[d_1(u_k(1), \overline{0}), \widehat{d}_\lambda(u_k(\alpha), \overline{0})] < \infty$, also since

$$\begin{aligned}
\left|u_k^-(1)v_k^-(1)\right| &= d_1\left(u_k v_k, \overline{0}\right) \leq d_1\left(u_k, \overline{0}\right) d_1\left(v_k, \overline{0}\right) \leq K_1 d_1\left(v_k, \overline{0}\right), \\
\left|u_k^-(\alpha)v_k^-(\alpha)\right| &\leq d_\lambda\left(u_k v_k, \overline{0}\right) \leq d_\lambda\left(u_k, \overline{0}\right) d_\lambda\left(v_k, \overline{0}\right) \leq K_1 d_\lambda\left(v_k, \overline{0}\right), \\
\left|u_k^+(\alpha)v_k^+(\alpha)\right| &\leq d_\lambda\left(u_k v_k, \overline{0}\right) \leq d_\lambda\left(u_k, \overline{0}\right) d_\lambda\left(v_k, \overline{0}\right) \leq K_1 d_\lambda\left(v_k, \overline{0}\right).
\end{aligned} \tag{3.7}$$

The series $\sum_k (u_k^-(\alpha)v_k^-(\alpha))$, $\sum_k (u_k^+(\alpha)v_k^+(\alpha))$, and $\sum_k (u_k^-(1)v_k^-(1))$ are convergent uniformly from Theorem 2.9; therefore, $\sum_k u_k v_k$ converges whenever $\sum_k d_1(v_k, \overline{0})$ and $\sum_k d_\lambda(v_k, \overline{0})$ converge. From here, we can write $\mathfrak{L}_1(E^1) \subseteq \mathfrak{L}_\infty^\beta(E^1)$.

Finally, we will show that the inclusion $\mathfrak{L}_{\infty}^{\beta}(E^1) \subseteq \mathfrak{L}_1(E^1)$ is held. Let us suppose that $(v_k) \in \mathfrak{L}_{\infty}^{\beta}(E^1)$, then we have $\sum_k(\sup_{\lambda\in[0,1]}\lambda[d_1(u_k(1)v_k(1),0), \widehat{d}_\lambda(u_k(\alpha)v_k(\alpha),\overline{0})]) < \infty$ for all $(u_k) \in \mathfrak{L}_{\infty}(E^1)$. This holds for the sequence $(u_k) = ([\overline{1},\overline{1}]) \in \mathfrak{L}_{\infty}(E^1)$, then we can write

$$\begin{aligned}
&\sum_k\left(\sup_{\lambda\in[0,1]}\lambda\left[d_1\left(u_k(1)v_k(1),\overline{0}\right),\widehat{d}_\lambda\left(u_k(\alpha)v_k(\alpha),\overline{0}\right)\right]\right) \\
&\quad = K\sum_k\left(\sup_{\lambda\in[0,1]}\lambda\left[d_1\left(v_k(1),\overline{0}\right),\widehat{d}_\lambda\left(v_k(\alpha),\overline{0}\right)\right]\right) = \|v\|_{\mathfrak{L}_1(E^1)}.
\end{aligned} \tag{3.8}$$

This shows that $v \in \mathfrak{L}_1(E^1)$. □

Proposition 3.6. *The γ-dual of the set $\mathfrak{L}_{\infty}(E^1)$ of sequence spaces fuzzy numbers is the set $\mathfrak{L}_1(E^1)$.*

4. Fuzzy Matrix Transformations

Let $\lambda(E^1)$ and $\mu(E^1)$ be two sequence spaces of fuzzy numbers, and let $A = (a_{nk})$ be an infinite matrix of fuzzy numbers a_{nk} and $u = (u_k) \in \lambda(E^1)$, where $n, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, then we can say that A defines a matrix mapping from $\lambda(E^1)$ to $\mu(E^1)$, and we denote it by writing $A : \lambda(E^1) \rightarrow \mu(E^1)$ if for every sequence $u = (u_k) \in \lambda(E^1)$, the sequence $Au = \{(Au)_n\}$, the A-transform of u, is in $\mu(E^1)$, where

$$Au = \sum_k a_{nk}u_k. \tag{4.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $(\lambda(E^1) : \mu(E^1))$, we denote the class of matrices A such that $A : \lambda(E^1) \rightarrow \mu(E^1)$. Thus, $A \in (\lambda(E^1) : \mu(E^1))$ if and only if the series on the right side of (4.1) converges for each $n \in \mathbb{N}$.

When does $A \in (\mathfrak{L}_{\infty}(E^1) : \mathfrak{L}_{\infty}(E^1))$? It is obvious that sufficient and efficient conditions for this are in following theorem.

Theorem 4.1. *$A \in (\mathfrak{L}_{\infty}(E^1) : \mathfrak{L}_{\infty}(E^1))$ if and only if*

$$\|A\| = \sup_n\sum_k\sup_{\lambda\in[0,1]}\lambda\left[d_1\left(a_{nk}(1),\overline{0}\right),\widehat{d}_\lambda\left(a_{nk}(\alpha),\overline{0}\right)\right] < \infty. \tag{4.2}$$

Proof. Let us suppose that (4.2) holds and $u \in \mathfrak{L}_{\infty}(E^1)$, then

$$\begin{aligned}
\|Au\|_{\mathfrak{L}_{\infty}(E^1)} &= \sup_n\sup_{\lambda\in[0,1]}\lambda\left[d_1\left(\sum_k a_{nk}u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(\sum_k a_{nk}u_k(\alpha),\overline{0}\right)\right] \\
&\le \sup_n\sum_k\sup_{\lambda\in[0,1]}\lambda\left[d_1\left(a_{nk}u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(a_{nk}u_k(\alpha),\overline{0}\right)\right] \\
&\le M\|u\|_{\mathfrak{L}_{\infty}(E^1)} < \infty,
\end{aligned} \tag{4.3}$$

that is, $Au \in \ell_{\infty}(E^1)$.

Conversely, let us suppose that $A \in (\mathfrak{L}_\infty(E^1) : \mathfrak{L}_\infty(E^1))$ and $u \in \mathfrak{L}_\infty(E^1)$, then, since $Au \in \mathfrak{L}_\infty(E^1)$ exists, the series $\sum_k a_{nk}u_k$ converges for each fixed $n \in \mathbb{N}$, and hence $A \in \mathfrak{L}_\infty^\beta(E^1)$. This holds for the sequence $(u_k) = ([\overline{1}, \overline{1}]) \in \mathfrak{L}_\infty(E^1)$, then, we can write

$$\begin{aligned}
\|Au\|_{\mathfrak{L}_\infty(E^1)} &= \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_k a_{nk}u_k(1), \overline{0}\right), \widehat{d}_\lambda\left(\sum_k a_{nk}u_k(\alpha), \overline{0}\right)\right] \\
&\leq \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_k a_{nk}(1), \overline{0}\right), \widehat{d}_\lambda\left(\sum_k a_{nk}(\alpha), \overline{0}\right)\right] \\
&\quad \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_n(1), \overline{0}\right), \widehat{d}_\lambda\left(u_n(\alpha), \overline{0}\right)\right] \\
&\leq M \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_k a_{nk}(1), \overline{0}\right), \widehat{d}_\lambda\left(\sum_k a_{nk}(\alpha), \overline{0}\right)\right] < \infty
\end{aligned} \tag{4.4}$$

which means that (4.2) holds. □

Theorem 4.2. *Let* $\lim_n \sup_{\lambda\in[0,1]}\lambda[d_1(a_{nk}(1), \overline{0}), \widehat{d}_\lambda(a_{nk}(\alpha), \overline{0})] = 0$ *(k fixed), and suppose that* (4.2) *is held, then* $A = (a_{nk})$ *defines a bounded operator on* $\mathfrak{C}_0(E^1)$ *into itself, where* $a_{nk} \in E^1$ *for all* $n, k \in \mathbb{N}$.

Proof. Let $u = (u_k) \in \mathfrak{C}_0(E^1)$. If $u = (u_k) = \overline{0}$, then $A_n(u) = \sum_k a_{nk}u_k = \overline{0}$, for all $n \in \mathbb{N}$. Hence, $A(u) \in \mathfrak{C}_0(E^1)$. Now, we suppose that $u \neq \overline{0}$. Then, under conditions of hypothesis, since $u \in \mathfrak{C}_0(E^1)$ and $\sum_k \sup_{\lambda\in[0,1]}\lambda[d_1(a_{nk}(1), \overline{0}), \widehat{d}_\lambda(a_{nk}(\alpha), \overline{0})] < \infty$, for all $n \in \mathbb{N}$, the series $\sum_k a_{nk}u_k$ is fuzzy absolute convergent for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|A_n(u)\|_{\mathfrak{C}_0(E^1)} &= \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_k a_{nk}(1)u_k(1), \overline{0}\right), \widehat{d}_\lambda\left(\sum_k a_{nk}(\alpha)u_k(\alpha), \overline{0}\right)\right] \\
&\leq \sup_n \sum_k \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1)u_k(1), \overline{0}\right), \widehat{d}_\lambda\left(a_{nk}(\alpha)u_k(\alpha), \overline{0}\right)\right] \\
&= \sup_n \sum_{k=0}^m \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1)u_k(1), \overline{0}\right), \widehat{d}_\lambda\left(a_{nk}(\alpha)u_k(\alpha), \overline{0}\right)\right] \\
&\quad + \sup_n \sum_{k\geq m+1} \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1)u_k(1), \overline{0}\right), \widehat{d}_\lambda\left(a_{nk}(\alpha)u_k(\alpha), \overline{0}\right)\right] \\
&\leq \sup_n \sum_{k=0}^m \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1), \overline{0}\right), \widehat{d}_\lambda\left(a_{nk}(\alpha), \overline{0}\right)\right] \\
&\quad \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k(1), \overline{0}\right), \widehat{d}_\lambda\left(u_k(\alpha), \overline{0}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_n \sum_{k\geq m+1} \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1),\overline{0}\right),\widehat{d}_\lambda\left(a_{nk}(\alpha),\overline{0}\right)\right] \\
&\quad \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(u_k(\alpha),\overline{0}\right)\right] \\
&\leq \|u\| \sup_n \sum_{k=0}^{m} \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1),\overline{0}\right),\widehat{d}_\lambda\left(a_{nk}(\alpha),\overline{0}\right)\right] \\
&+ M \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(u_k(\alpha),\overline{0}\right)\right].
\end{aligned} \tag{4.5}
$$

Since $u \in \mathfrak{C}_0(E^1)$, for enough big $k > n_0$, we can write

$$
\sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(u_k(\alpha),\overline{0}\right)\right] < \frac{\epsilon}{2M}, \tag{4.6}
$$

and from $\lim_n \sup_{\lambda\in[0,1]}\lambda[d_1(a_{nk}(1),\overline{0}),\widehat{d}_\lambda(a_{nk}(\alpha),\overline{0})] = 0$ (k fixed), we can choose n so large that

$$
\sum_{k=0}^{m} \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk}(1),\overline{0}\right),\widehat{d}_\lambda\left(a_{nk}(\alpha),\overline{0}\right)\right] \leq \frac{\epsilon}{2\sup_k \sup_{\lambda\in[0,1]}\lambda\left[d_1\left(u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(u_k(\alpha),\overline{0}\right)\right]}. \tag{4.7}
$$

From (4.6) and (4.7), we see that $Au \in \mathfrak{C}_0(E^1)$. Finally, we will show that A is bounded as follows:

$$
\begin{aligned}
\|Au\|_{\mathfrak{C}_0(E^1)} &= \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(\sum_k a_{nk}(1)u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(\sum_k a_{nk}(\alpha)u_k(\alpha),\overline{0}\right)\right] \\
&\leq \sup_n \sum_k \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(a_{nk},\overline{0}\right),\widehat{d}_\lambda\left(a_{nk}(\alpha),\overline{0}\right)\right] \sup_n \sup_{\lambda\in[0,1]} \lambda\left[d_1\left(u_k(1),\overline{0}\right),\widehat{d}_\lambda\left(u_k(\alpha),\overline{0}\right)\right] \\
&\leq M\|u\|_{\mathfrak{C}_0(E^1)}.
\end{aligned} \tag{4.8}
$$

□

The above theorem shows that a certain type of matrix defines a linear operator on $\mathfrak{C}_0(E^1)$ into itself.

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