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Research Article

On Alzer and Qiu's Conjecture for Complete Elliptic Integral and Inverse Hyperbolic Tangent Function

Yu-Ming Chu, Miao-Kun Wang, and Ye-Fang Qiu

Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China

Correspondence should be addressed to Yu-Ming Chu, chuyuming2005@yahoo.com.cn

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We prove that the double inequality $(\pi/2)(\text{arth}r/r)^{3/4+\alpha^* r} < \mathcal{K}(r) < (\pi/2)(\text{arth}r/r)^{3/4+\beta^* r}$ holds for all $r \in (0,1)$ with the best possible constants $\alpha^* = 0$ and $\beta^* = 1/4$, which answer to an open problem proposed by Alzer and Qiu. Here, $\mathcal{K}(r)$ is the complete elliptic integrals of the first kind, and arth is the inverse hyperbolic tangent function.

1. Introduction

For $r \in [0,1]$, Lengedre's complete elliptic integrals of the first and second kind [1] are defined by

$$\begin{gathered}
\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2\sin^2\theta)^{-1/2} d\theta, \\
\mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\
\mathcal{K}(0) = \frac{\pi}{2}, \qquad \mathcal{K}(1) = \infty, \\
\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2\sin^2\theta)^{1/2} d\theta, \\
\mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\
\mathcal{E}(0) = \frac{\pi}{2}, \qquad \mathcal{E}(1) = 1,
\end{gathered} \tag{1.1}$$

respectively. Here and in what follows, we set $r' = \sqrt{1-r^2}$. These integrals are special cases of Guassian hypergeometric function

$$F_2(a,b;c;x) = F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!} \quad (-1 < x < 1), \tag{1.2}$$

where $(a,n) = \prod_{k=0}^{n-1}(a+k)$. Indeed, we have

$$\mathcal{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right), \qquad \mathcal{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \tag{1.3}$$

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory, and other related fields [2–13].

Recently, the complete elliptic integrals have been the subject of intensive research. In particular, many remarkable properties and inequalities can be found in the literature [3, 10–18].

In 1992, Anderson et al. [15] discovered that $\mathcal{K}$ can be approximated by the inverse hyperbolic tangent function, arth, and proved that

$$\frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{1/2} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right), \tag{1.4}$$

for $r \in (0,1)$.

In [16], Alzer and Qiu proved that the double inequality

$$\frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{\alpha} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{\beta}, \tag{1.5}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha = 3/4$ and $\beta = 1$ and proposed an open problem as follows.

Open Problem #

The double inequality

$$\frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{3/4+\alpha^* r} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{3/4+\beta^* r}, \tag{1.6}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha^* = 0$ and $\beta^* = 1/4$.

It is the aim of this paper to give a positive answer to the open problem #.

2. Lemmas and Theorem

In order to establish our main result, we need several formulas and lemmas, which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [4, Appendix E, pages 474-475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$
$$\frac{d\left(\mathcal{E} - r'^2\mathcal{K}\right)}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2}. \tag{2.1}$$

Lemma 2.1 (see [4, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and be differentiable on (a, b), let $g'(x) \neq 0$ be on (a, b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{2.2}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

The following Lemma 2.2 can be found in [9, Lemma 3(1)] and [4, Theorem 3.21(1) and Exercise 3.43(30) and (46)].

Lemma 2.2. (1) *$[(r')^c \operatorname{arth} r]/r$ is strictly decreasing in $(0, 1)$ if and only if $c \geq 2/3$;*

(2) *$(\mathcal{E} - r'^2\mathcal{K})/r^2$ is strictly increasing from $(0, 1)$ onto $(\pi/4, 1)$;*

(3) *$(\mathcal{E} - r'^2\mathcal{K})/(r^2\mathcal{K})$ is strictly decreasing from $(0, 1)$ onto $(0, 1/2)$;*

(4) *$r\mathcal{K}/\operatorname{arth} r$ is strictly decreasing from $(0, 1)$ onto $(1, \pi/2)$.*

Lemma 2.3. (1) *$f_1(r) = [r - r'^2 \operatorname{arth} r]/r^3$ is strictly increasing from $(0, 1)$ onto $(2/3, 1)$;*

(2) *$f_2(r) = (\log[\operatorname{arth}(r)/r])/r^2$ is strictly increasing from $(0, 1)$ onto $(1/3, \infty)$;*

(3) *$f_3(r) = [\mathcal{E} \operatorname{arth} r - r'^2\mathcal{K} \operatorname{arth}(r)/4 - 3r\mathcal{K}/4]/r^5$ is strictly increasing from $(0, 1)$ onto $(\pi/480, \infty)$;*

(4) *$f_4(r) = (3/4 + r/4)(r - r'^2 \operatorname{arth} r)\mathcal{K} - (\mathcal{E} - r'^2\mathcal{K}) \operatorname{arth} r$ is positive and strictly increasing in $(\sqrt{2}/2, 1)$;*

(5) *$f_5(r) = (3/4 + r^2)\log[\operatorname{arth}(r)/r] - \log(2\mathcal{K}/\pi)$ is positive and strictly increasing on $(0, 1/4)$.*

Proof. For part (1), let $h_1(r) = r - r'^2 \operatorname{arth} r$ and $h_2(r) = r^3$. Then $f_1(r) = h_1(r)/h_2(r)$, $h_1(0) = h_2(0) = 0$ and

$$\frac{h_1'(r)}{h_2'(r)} = \frac{2}{3}\frac{\operatorname{arth} r}{r}. \tag{2.3}$$

It is well known that the function $r \mapsto \operatorname{arth}(r)/r$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$. Therefore, from (2.3) and Lemma 2.1 together with l'Hôpital's rule, we know that $f_1(r)$ is strictly increasing in $(0, 1)$, $f_1(0^+) = 2/3$ and $f_1(1^-) = 1$.

For part (2), clearly $f_2(1^-) = +\infty$. Let $h_3(r) = \log[\operatorname{arth}(r)/r]$ and $h_4(r) = r^2$, then $f_2(r) = h_3(r)/h_4(r)$, $h_3(0) = h_4(0) = 0$, and

$$\frac{h_3'(r)}{h_4'(r)} = \frac{r - r'^2 \operatorname{arth} r}{2r^2 r'^2 \operatorname{arth} r} = \frac{1}{2}\frac{r - r'^2 \operatorname{arth} r}{r^3}\frac{r}{r'^2 \operatorname{arth} r}. \tag{2.4}$$

It follows from Lemma 2.1, Lemma 2.2(1), part (1), (2.4), and l'Hôpital's rule that $f_2(r)$ is strictly increasing in $(0,1)$ and $f_2(0^+) = 1/3$.

For part (3), from Lemma 2.2(4), we clearly see that $f_3(1^-) = +\infty$. Let $h_5(r) = \mathcal{E} \operatorname{arth} r - r'^2\mathcal{K} \operatorname{arth}(r)/4 - 3r\mathcal{K}(r)/4$, $h_6(r) = r^5$, $h_7(r) = (\mathcal{E} - r'^2\mathcal{K})/(4r'^2) - r\mathcal{K} \operatorname{arth}(r)/2 + 3\operatorname{arth}(r)(\mathcal{E} - r'^2\mathcal{K})/(4r)$, and $h_8(r) = r^4$, then $f_3(r) = h_5(r)/h_6(r)$, $h_5(0) = h_6(0) = h_7(0) = h_8(0) = 0$,

$$\begin{gathered}\frac{h_5'(r)}{h_6'(r)} = \frac{1}{5}\frac{h_7(r)}{h_8(r)},\\ \frac{h_7'(r)}{h_8'(r)} = \frac{1}{4r'^4}\frac{r - r'^2 \operatorname{arth} r}{r^3}\left[\frac{3}{4}\frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r^2} - \frac{1}{4}\mathcal{E}(r)\right].\end{gathered} \tag{2.5}$$

From Lemma 2.2(2) and part (1), we clearly see that $h_7'(r)/h_8'(r)$ is strictly increasing in $(0,1)$. Thus, the monotonicity of $f_3(r)$ can be obtained from (2.5) and Lemma 2.1. Moreover, making use of l'Hôpital's rule, we have $f_3(0^+) = \pi/480$.

For part (4), let $h_9(r) = 2(1 + r) - \mathcal{E}/(r\mathcal{K}) - 3(\mathcal{E} - r'^2\mathcal{K})/(r^2\mathcal{K})$. Then, Lemma 2.2(3) leads to the conclusion that $h_9(r)$ is strictly increasing in $(0,1)$. Note that

$$h_9\left(\frac{\sqrt{2}}{2}\right) = 1.013\cdots > 0\ , \tag{2.6}$$

$$f_4\left(\frac{\sqrt{2}}{2}\right) = 0.084\cdots > 0, \tag{2.7}$$

$$f_4'(r) = \frac{(\mathcal{K} - \mathcal{E}) + r\mathcal{K}(r)}{4(1 + r)} + \frac{r\mathcal{K} \operatorname{arth} r}{4}h_9(r) > \frac{r\mathcal{K} \operatorname{arth} r}{4}h_9\left(\frac{\sqrt{2}}{2}\right) > 0 \tag{2.8}$$

for $r \in (\sqrt{2}/2, 1)$.

Therefore, part (4) follows from (2.7) and (2.8).

For part (5), simple computations lead to

$$\lim_{r\to 0^+} f_5(r) = 0, \tag{2.9}$$

$$f_5'(r) = 2r\log\left(\frac{\operatorname{arth} r}{r}\right) + \left(\frac{3}{4} + r^2\right)\frac{r - r'^2 \operatorname{arth} r}{rr'^2 \operatorname{arth} r} - \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2\mathcal{K}}. \tag{2.10}$$

Making use of parts (1)–(4), one has

$$\frac{r'^2\mathcal{K}\,\mathrm{arth}\,r}{r^4}f_5'(r)=\frac{2r'^2\mathcal{K}\,\mathrm{arth}\,r}{r}f_2(r)+\mathcal{K}f_1(r)-f_3(r)$$
$$>Kf_1(r)-f_3(r)>\frac{\pi}{3}-f_3\left(\frac{1}{4}\right)=1.040\cdots>0 \tag{2.11}$$

for $r\in(0,1/4)$.

Therefore, part (5) follows from (2.9) and (2.11). □

Lemma 2.4. *Let*

$$g_c(r)=\left(\frac{3}{4}+cr\right)\log\left[\frac{\mathrm{arth}(r)}{r}\right]-\log\left(\frac{2\mathcal{K}}{\pi}\right)\quad(c\in\mathbb{R}), \tag{2.12}$$

then the following statements are true:

(1) $g_c(r)>0$ *for all* $r\in(0,1)$ *if and only if* $c\in[1/4,\infty)$;

(2) $g_c(r)<0$ *for all* $r\in(0,1)$ *if and only if* $c\in(-\infty,0]$.

Proof. Firstly, we prove that $g_c(r)>0$ for $c\in[1/4,\infty)$. Since $g_c(r)$ is continuous and strictly increasing with respect to $c\in\mathbb{R}$ for fixed $r\in(0,1)$, it suffices to prove that $g_{1/4}(r)>0$ for all $r\in(0,1)$. Note that

$$\lim_{r\to 0^+}g_{1/4}(r)=0, \tag{2.13}$$

$$g_{1/4}'(r)=\frac{1}{4}\log\left(\frac{\mathrm{arth}\,r}{r}\right)+\left(\frac{3}{4}+\frac{1}{4}r\right)\frac{r-r'^2\,\mathrm{arth}\,r}{rr'^2\,\mathrm{arth}\,r}-\frac{\mathcal{E}-r'^2\mathcal{K}}{rr'^2\mathcal{K}}. \tag{2.14}$$

We divide the proof into two cases.

Case 1 ($r\in(0,\sqrt{2}/2]$). Then, making use of Lemma 2.3(1)–(3) and (2.14), we have

$$\frac{r'^2\mathcal{K}\,\mathrm{arth}\,r}{r^3}g_{1/4}'(r)=\frac{r'^2\mathcal{K}\,\mathrm{arth}\,r}{4r}f_2(r)+\frac{1}{4}\mathcal{K}(r)f_1(r)-rf_3(r)$$
$$>\frac{1}{4}\mathcal{K}(r)f_1(r)-rf_3(r)>\frac{\pi}{12}-\frac{\sqrt{2}}{2}f_3\left(\frac{\sqrt{2}}{2}\right) \tag{2.15}$$
$$=0.250\cdots>0.$$

Case 2 ($r\in(\sqrt{2}/2,1)$). Then, making use of Lemma 2.3(4) and (2.14), we get

$$\frac{g_{1/4}'(r)}{\log[\mathrm{arth}(r)/r]}=\frac{1}{4}+\frac{f_4(r)}{rr'^2\mathcal{K}\,\mathrm{arth}\,r\log[\mathrm{arth}(r)/r]}>0. \tag{2.16}$$

Inequalities (2.15) and (2.16) imply that $g_{1/4}(r)$ is strictly increasing in $(0,1)$. Therefore, $g_{1/4}(r)>0$ follows from (2.13) and the monotonicity of $g_{1/4}(r)$.

On the other hand, inequality (1.5) leads to the conclusion that $g_c(r) < 0$ for all $r \in (0,1)$ and $c \in (-\infty, 0]$.

Next, we prove that the parameters $1/4$ and 0 are the best possible parameters in Lemma 2.4(1) and (2), respectively.

If $c \in (0, 1/4)$, then $g_c(c) = f_5(c) > 0$ follows from Lemma 2.3(5). Moreover, let

$$F(r) = \frac{g_c(r)}{\log[\operatorname{arth}(r)/r]} = \frac{3}{4} + cr - \frac{\log(2\mathcal{K}/\pi)}{\log[\operatorname{arth}(r)/r]}, \tag{2.17}$$

then, using l'Hôpital's rule and Lemma 2.2(4), we get

$$\lim_{r \to 1^+} F(r) = c - \frac{1}{4} < 0. \tag{2.18}$$

Inequality (2.18) implies that there exists $\delta = \delta(c) > 0$ such that $F(r) < 0$ for all $r \in (1-\delta, 1)$. Therefore, $g_c(r) < 0$ for $r \in (1-\delta, 1)$ follows from (2.17). □

From Lemma 2.4, we clearly see that the following Theorem 2.5 holds, which give a positive answer to the open problem #.

Theorem 2.5. *The double inequality*

$$\frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{3/4+\alpha^* r} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{arth} r}{r}\right)^{3/4+\beta^* r} \tag{2.19}$$

holds for all $r \in (0,1)$ *with the best possible constants* $\alpha^* = 0$ *and* $\beta^* = 1/4$.

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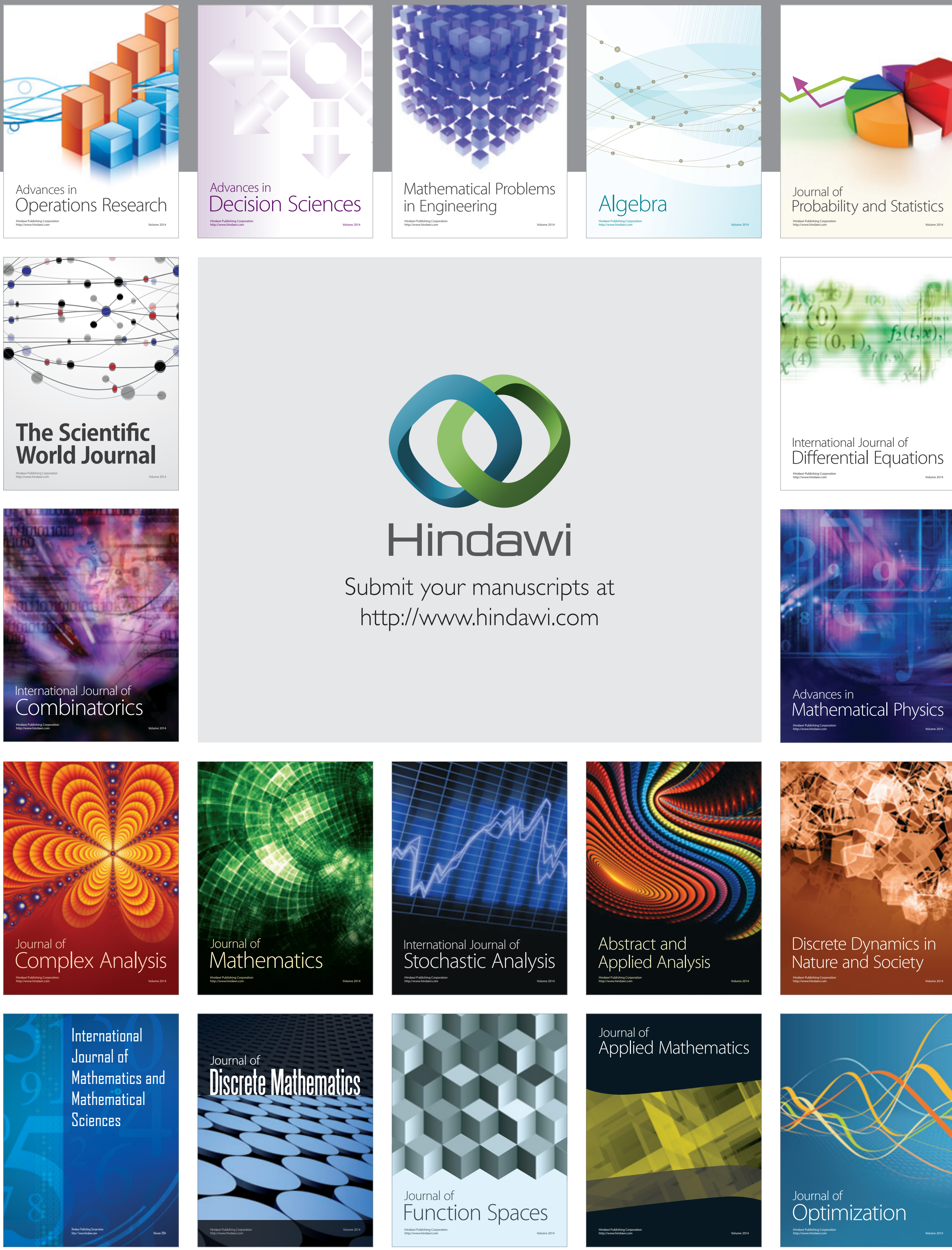
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