Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 634675, 11 pages
doi:10.1155/2011/634675

Review Article

Some Identities on the q-Integral Representation of the Product of Several q-Bernstein-Type Polynomials

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Received 28 August 2011; Accepted 5 November 2011

Academic Editor: Pavel Drábek

The purpose of this paper is to give some properties of several q-Bernstein-type polynomials to express the q-integral on [0, 1] in terms of q-beta and q-gamma functions. Finally, we derive some identities on the q-integral of the product of several q-Bernstein-type polynomials.

1. Introduction

Let $q \in \mathbb{R}$ with $0 \leq q < 1$. We assume that q-number is defined by $[x]_q = (1 - q^x)/(1 - q)$ and $[0]_q = 0$. Note that $\lim_{q\to 1}[x]_q = x$. The q-derivative of a map $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R} \setminus \{0\}$ is given by

$$D_q(f) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x} \tag{1.1}$$

(see [1–6]). For $n \in \mathbb{N}$, by (1.1), we get $D_q^n(x^n) = [n]_q[n-1]_q \cdots [2]_q[1]_q = [n]_1!$. The q-binomial formula is given by

$$(a+b)_q^n = \prod_{i=0}^{n-1}\left(a + bq^i\right) = \sum_{l=0}^{n} \binom{n}{l}_q q^{\binom{l}{2}} a^{n-l} b^l \tag{1.2}$$

(see [2, 5, 7–11]), where $\binom{n}{k}_q = [n]_q!/[k]_q![n-k]_q! = [n]_q[n-1]_q \cdots [n-k+1]_q/[k]_q!$.

For $a, b \in \mathbb{R}$, the Jackson q-integral of $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} q^n \left(b f(bq^n) - a f(aq^n) \right) \tag{1.3}$$

(see [1, 2, 5, 6, 9, 12, 13]). From (1.2), we note that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k \binom{n}{k}_q = q^{n-k} \binom{n}{k-1}_q + \binom{n}{k}_q. \tag{1.4}$$

By (1.2) and (1.4), we get

$$\begin{aligned} (1-b)_q^n &= (b:q)_n = \prod_{i=0}^{n-1} \left(1 - q^i b\right) = \sum_{i=0}^{n} \binom{n}{i}_q q^{\binom{i}{2}} (-b)^i, \\ \frac{1}{(1-b)_q^n} &= \frac{1}{(b:q)_n} = \frac{1}{\prod_{i=0}^{n-1}(1-q^i b)} = \sum_{i=0}^{\infty} \binom{n+i-1}{i}_q b^i. \end{aligned} \tag{1.5}$$

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear operators (see [1, 4, 9, 11, 14]):

$$\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x). \tag{1.6}$$

Here $\mathbb{B}_n(f \mid x)$ is called Bernstein operator of order n for f. For $k, n \in \mathbb{Z}_+ (= \mathbb{N} \cup \{0\})$, the Bernstein polynomials of degree n are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{1.7}$$

(see [1, 3, 4, 11–14]). By the definition of Bernstein polynomials (see (1.6) and (1.7)), we can see that Bernstein basis is the probability mass function of binomial distribution. A Bernoulli trial involves performing an experiment once and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be "success" if A occurs and a "failure" otherwise. Let k be the number of successes in n independent Bernoulli trials, the probabilities of k are given by the binomial probability law:

$$p_n(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k = 0, 1, \ldots, n, \tag{1.8}$$

where $p_n(k)$ is the probability of k successes in n trials. For example, a communication system transmits binary information over channel that introduces random bit errors with probability $\xi = 10^{-3}$. The transmitter transmits each information bit three times, an a decoder takes a majority vote of the received bits to decide on what the transmitted bit was. The receiver can correct a single error, but it will make the wrong decision if the channel introduces two or more errors. If we view each transmission as a Bernoulli trial in which a "success" corresponds to the introduction of an error, then the probability of two or more errors in three Bernoulli trials is

$$p(k \geq 2) = \binom{3}{2}(0.001)^2(0.999) + \binom{3}{3}(0.001)^3 \approx 3\left(10^{-6}\right), \tag{1.9}$$

see [9]. Based on the q-integers Phillips introduced the q-analogue of well-known Bernstein polynomials (see [4, 5, 9, 11, 15]). For $f \in C([0,1])$, Phillips introduced the q-extension of (1.6) as follows:

$$\begin{aligned} \mathbb{B}_{n,q}(f \mid x) &= \sum_{n=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\binom{n}{k}_q (1-x)_q^{n-k} \\ &= \sum_{n=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) B_{k,n}(x,q), \quad \text{for } k, n \in \mathbb{Z}_+ \end{aligned} \tag{1.10}$$

(see [4, 5, 9, 11, 15]). Here $\mathbb{B}_{n,q}(f \mid x)$ is called the q-Bernstein operator of order n for f. For $k, n \in \mathbb{Z}_+$, the q-Bernstein polynomial of degree n is defined by

$$B_{k,n}(x,q) = \binom{n}{k}_q x^k (1-x)_q^{n-k}, \quad \text{where } x \in [0,1]. \tag{1.11}$$

Note that (1.11) is the q-extension of (1.7). That is, $\lim_{q\to 1} B_{k,n}(x,q) = B_{k,n}(x)$. For example, $B_{0,1}(x,q) = 1-x$, $B_{1,1}(x,q) = x$, and $B_{0,2}(x,q) = 1-[2]_q x + qx^2, \ldots$. Also $B_{k,n}(x,q) = 0$ for $k > n$, because $\binom{n}{k}_q = 0$. For $n, k \in \mathbb{Z}_+$, its probabilities are given by

$$p(x = k) = \binom{n}{k}_q x^k (1-x)_q^{n-k}, \quad \text{where } x \in [0,1]. \tag{1.12}$$

This distributions are studied by several authors and they have applications in physics as well as in approximation theory due to the q-Bernstein polynomials and the q-Bernstein operators (see [1–16]). By the definition of the q-Bernstein polynomials, we easily see that the q-Bernstein basis is the probability mass function of q-binomial distribution. In this paper we use the two q-analogues of exponential function as follows:

$$E_q(x) = ((1-q)x : q)_\infty = (1+(1-q)x)_q^\infty = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_q!}, \tag{1.13}$$

$$e_q(x) = \frac{1}{((1-q)x : q)_\infty} = \frac{1}{(1+(1-q)x)_q^\infty} = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{1.14}$$

(see [2–4, 6, 10]). From (1.3), the improper q-integral is given by

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n\in\mathbb{Z}} \frac{q^n}{A} f\left(\frac{q^n}{A}\right) \tag{1.15}$$

(see [6]), where the improper q-integral depends on A. The purpose of this paper is to give some properties of several q-Bernstein type polynomials to express the q-integral on $[0,1]$ in terms of q-beta and q-gamma functions. Finally, we derive some identities on the q-integral of the product of several q-Bernstein type polynomials.

2. q-Integral Representation of q-Bernstein Polynomials

The gamma and beta functions are defined as the following definite integrals ($\alpha > 0, \beta > 0$):

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt, \tag{2.1}$$

(see [1–11, 14–16])

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt. \tag{2.2}$$

From (2.1) and (2.2), we can derive the following equations:

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2.3}$$

As the q-extensions of (2.1) and (2.2), the q-gamma and q-beta functions are defined as the following q-integrals ($\alpha > 0, \beta > 0$):

$$\Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(-qx) d_q x \tag{2.4}$$

(see [2–6, 10]),

$$B_q(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-qx)_q^{\beta-1} d_qx \tag{2.5}$$

(see [2, 4, 6, 10]).

By (2.4) and (2.5), we obtain the following lemma.

Lemma 2.1 (see [2, 6]). (a) Γ_q *can be equivalently expressed as*

$$\Gamma_q(\alpha) = \frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}, \quad \textit{where } \alpha > 0. \tag{2.6}$$

In particular, one has

$$\Gamma_q(\alpha+1) = [\alpha]_q\Gamma_q(\alpha), \quad \textit{for } \alpha > 0,\ \Gamma_q(1) = 1. \tag{2.7}$$

(b) *The q-gamma and q-beta functions are related to each other by the following two equations:*

$$\Gamma_q(\alpha) = \frac{B_q(\alpha,\infty)}{(1-q)^{\alpha}}, \quad B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}, \quad \textit{where } \alpha > 0,\ \beta > 0. \tag{2.8}$$

Now one takes the q-integral for one q-Bernstein polynomial as follows: for $n,k \in \mathbb{Z}_+$,

$$\begin{aligned} q^{-k}\int_0^1 B_{k,n}(qx,q)d_qx &= \binom{n}{k}_q \int_0^1 x^k(1-qx)_q^{n-k}d_qx \\ &= \binom{n}{k}_q \sum_{l=0}^{n-k}\binom{n-k}{l}_q (-1)^l q^{\binom{l+1}{2}} \int_0^1 x^{l+k}d_qx \\ &= \binom{n}{k}_q \sum_{l=0}^{n-k}\binom{n-k}{l}_q (-1)^{n-k-l} q^{\binom{n-k-l+1}{2}} \frac{1}{[n-l+1]_q}. \end{aligned} \tag{2.9}$$

Therefore, by (2.9), one obtains the following proposition.

Proposition 2.2. *For $n,k \in \mathbb{Z}_+$, one has*

$$\int_0^1 B_{k,n}(qx,q)d_qx = q^k\binom{n}{k}_q \sum_{l=0}^{n-k}\binom{n-k}{l}_q (-1)^{n-k-l} q^{\binom{n-k-l+1}{2}} \frac{1}{[n-l+1]_q}. \tag{2.10}$$

The Proposition 2.2 is closely related to the q-beta function which is given by

$$B_q(n,m) = \int_0^1 x^{n-1}(1-qx)_q^{m-1} d_q x, \tag{2.11}$$

$$\Gamma_q(m) = \int_0^{1/(1-q)} x^{n-1} E_q(-qx) d_q x, \tag{2.12}$$

(see (2.5)). From Lemma 2.1, one has

$$B_q(n,m) = \frac{\Gamma_q(m)\Gamma_q(n)}{\Gamma_q(n+m)}, \quad \text{where } m,n \in \mathbb{N}. \tag{2.13}$$

By (2.9) and (2.13), one gets

$$\begin{aligned} q^{-k}\int_0^1 B_{k,n}(qx,q) d_q x &= \binom{n}{k}_q B_q(k+1, n-k+1) \\ &= \binom{n}{k}_q \frac{\Gamma_q(k+1)\Gamma_q(n-k+1)}{\Gamma_q(n+2)}, \quad \text{where } k > -1,\ n > k-1. \end{aligned} \tag{2.14}$$

Therefore, by (2.14), one obtains the following theorem.

Theorem 2.3. *For $n,k \in \mathbb{Z}_+$ with $k > -1$ and $n > k-1$, one has*

$$\int_0^1 B_{k,n}(qx,q) d_q x = \binom{n}{k}_q [k]_q [n-k]_q \Big((q-1)[k]_q + 1\Big) \frac{\Gamma_q(k)\Gamma_q(n-k)}{\Gamma_q(n+2)}. \tag{2.15}$$

By comparing the coefficients on the both sides of Proposition 2.2 and Theorem 2.3, one obtains the following corollary.

Corollary 2.4. *For $n,k \in \mathbb{Z}_+$ with $k > -1$ and $n > k-1$, one has*

$$\sum_{l=0}^{n-k} \binom{n-k}{l}_q (-1)^{n-k-l} \frac{q^{\binom{n-k-l+1}{2}}}{[n-l+1]_q} = \frac{\Gamma_q(k+1)\Gamma_q(n-k+1)}{\Gamma_q(n+2)}. \tag{2.16}$$

According to this result one can say that the q-integral of q-Bernstein polynomials from 0 to 1 is symmetric. Now one considers the q-integral for the multiplication of two q-Bernstein polynomials which is given by the following relation:

$$\begin{aligned}\frac{\int_0^1 B_{k,n}(qx,q)B_{k,m(q^{n-k+1}x,q)d_qx}}{q^{nk-k^2+2k}} &= \binom{n}{k}_q\binom{m}{k}_q\int_0^1 x^2k(1-qx)_q^{n+m-2k}d_qx \\ &= \binom{n}{k}_q\binom{m}{k}_q\int_0^1 u^{n+m-2k}(1-qu)_q^{2k}d_qu.\end{aligned} \tag{2.17}$$

For $n,k,m\in\mathbb{Z}_+$, one can derive the following equation (2.20) from (2.17):

$$\begin{aligned}\frac{\int_0^1 B_{k,n}(qx,q)B_{k,m}(q^{n-k+1}x,q)d_qx}{q^{nk-k^2+2k}} &= \binom{n}{k}_q\binom{m}{k}_q\sum_{l=0}^{2k}\frac{\binom{2k}{l}_q(-1)^l q^{\binom{l+1}{2}}}{[n+m+l-2k+1]_q} \\ &= \binom{n}{k}_q\binom{m}{k}_q\sum_{l=0}^{2k}\frac{\binom{2k}{l}_q(-1)^{2k-l} q^{\binom{2k-l+1}{2}}}{[n+m-l+1]_q}.\end{aligned} \tag{2.18}$$

Therefore, one obtains the following theorem.

Theorem 2.5. *For $m,n,k\in\mathbb{Z}_+$, one has*

$$\int_0^1 B_{k,n}(qx,q)B_{k,m}\left(q^{n-k+1}x,q\right)d_qx = q^{nk-k^2+2k}\binom{n}{k}_q\binom{m}{k}_q\sum_{l=0}^{2k}\frac{\binom{2k}{l}_q(-1)^{2k-l} q^{\binom{2k-l+1}{2}}}{[n+m-l+1]_q}. \tag{2.19}$$

For $m,n,k\in\mathbb{Z}_+$, by (2.5) and (2.9), one gets

$$\frac{\int_0^1 B_{k,n}(qx,q)B_{k,m}(q^{n-k+1}x,q)d_qx}{q^{nk-k^2+2k}} = \binom{n}{k}_q\binom{m}{k}_q B_q(n+m-2k+1,2k+1). \tag{2.20}$$

Therefore, by Theorem 2.5 and (2.20), one obtains the following corollary.

Corollary 2.6. *For $k>-1$ and $n+m-2k>-1$, one has*

$$\sum_{l=0}^{2k}\frac{\binom{2k}{l}_q(-1)^{2k-l} q^{\binom{2k-l+1}{2}}}{[n+m-l+1]_q} = \frac{\Gamma_q(n+m-2k+1)\Gamma_q(2k+1)}{\Gamma_q(n+m+2)}. \tag{2.21}$$

By the same method, the multiplication of three q-Bernstein polynomials is given by the following relation: for $k, n, m, s \in \mathbb{Z}_+$,

$$\begin{aligned}
&\frac{\int_0^1 B_{k,n}(qx,q)B_{k,m}(q^{n-k+1}x,q)B_{k,s}(q^{n+m-2k+1}x,q)d_qx}{q^{3k+2nk-3k^2+mk}}\\
&\quad=\binom{n}{k}_q\binom{m}{k}_q\binom{s}{k}_q\int_0^1 x^{3k}(1-qx)_q^{n+m+s-3k}d_qx\\
&\quad=\binom{n}{k}_q\binom{m}{k}_q\binom{s}{k}_q\int_0^1 u^{n+m+s-3k}(1-qu)_q^{3k}d_qu\\
&\quad=\binom{n}{k}_q\binom{m}{k}_q\binom{s}{k}_q\sum_{l=0}^{3k}\binom{3k}{l}_q q^{\binom{l+1}{2}}(-1)^l\int_0^1 u^{n+m+s-3k+l}d_qu\\
&\quad=\binom{n}{k}_q\binom{m}{k}_q\binom{s}{k}_q\sum_{l=0}^{3k}\binom{3k}{l}_q q^{\binom{3k-l+1}{2}}(-1)^{l+3k}\frac{1}{[n+m+s-l+1]_q}.
\end{aligned}\tag{2.22}$$

Therefore, by (2.22), one obtains the following theorem.

Theorem 2.7. *For $n, m, s, k \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\int_0^1 B_{k,n}(qx,q)B_{k,m}\left(q^{n-k+1}x,q\right)B_{k,s}\left(q^{n+m-2k+1}x,q\right)d_qx\\
&\quad=q^{3k+2nk-3k^2+mk}\binom{n}{k}_q\binom{m}{k}_q\binom{s}{k}_q\sum_{l=0}^{3k}\binom{3k}{l}_q q^{\binom{3k-l+1}{2}}\frac{(-1)^{l+3k}}{[n+m+s-l+1]_q}.
\end{aligned}\tag{2.23}$$

From (2.5) and (2.22), one has

$$\begin{aligned}
&\frac{\int_0^1 B_{k,n}(qx,q)B_{k,m}(q^{n-k+1}x,q)B_{k,s}(q^{n+m-2k+1}x,q)d_qx}{q^{3k+2nk-3k^2+mk}}\\
&\quad=\binom{n}{k}_q\binom{m}{k}_q\binom{s}{k}_q B_q(n+m+s-3k+1,3k+1).
\end{aligned}\tag{2.24}$$

Therefore, by Theorem 2.7 and (2.24), one obtains the following corollary.

Corollary 2.8. *For $k > -1/3$ and $n + m + s - 3k > -1$, one has*

$$\sum_{k=0}^{3k}\binom{3k}{l}_q\frac{(-1)^{l+3k}q^{\binom{3k-l+1}{2}}}{[n+m+s-l+1]_q}=\frac{\Gamma_q(n+m+s-3k+1)\Gamma_q(3k+1)}{\Gamma_q(n+m+s+2)}.\tag{2.25}$$

For $s \in \mathbb{N}$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$. Then one has

$$\begin{aligned}
&\frac{\int_0^1 B_{k,n_1}(qx,q)\left(\prod_{i=1}^{s-1} B_{k,n_{i+1}}\left(q^{\sum_{l=1}^{i} n_l - ik+1}x, q\right) d_q x\right)}{q^{sk+k\sum_{i=1}^{s-1} i n_{s-i} - k^2\binom{s}{2}}} \\
&\quad = \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \int_0^1 x^{sk}(1-qx)_q^{n_1+\cdots+n_s-sk} d_q x \\
&\quad = \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \sum_{l=0}^{sk} \binom{sk}{l}_q (-1)^l q^{\binom{l+1}{2}} \int_0^1 x^{n_1+\cdots+n_s-sk+l} d_q x \\
&\quad = \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \sum_{l=0}^{sk} \binom{sk}{l}_q \frac{(-1)^{l+sk} q^{\binom{sk-l+1}{2}}}{[n_1+\cdots+n_s-l+1]_q}.
\end{aligned} \tag{2.26}$$

Therefore, by (2.26), one obtains the following theorem.

Theorem 2.9. *For $s \in \mathbb{N}$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$. Then one has*

$$\begin{aligned}
&\int_0^1 B_{k,n_1}(qx,q)\left(\prod_{i=1}^{s-1} B_{k,n_{i+1}}\left(q^{\sum_{l=1}^{i} n_l - ik+1}x, q\right)\right) d_q x \\
&\quad = q^{sk+k\sum_{i=1}^{s-1} i n_{s-i} - k^2\binom{s}{2}} \binom{n_1}{k}_q \cdots \binom{n_s}{k}_q \sum_{l=0}^{sk} \frac{\binom{sk}{l}_q (-1)^{l+sk} q^{\binom{sk-l+1}{2}}}{[n_1+\cdots+n_s-l+1]_q}.
\end{aligned} \tag{2.27}$$

By (2.5) and (2.26), we get

$$\begin{aligned}
&\frac{\int_0^1 B_{k,n_1}(qx,q)\left(\prod_{i=1}^{s-1} B_{k,n_{i+1}}\left(q^{\sum_{l=1}^{i} n_l - ik+1}x, q\right) d_q x\right)}{q^{sk+k\sum_{i=1}^{s-1} i n_{s-i} - k^2\binom{s}{2}}} \\
&\quad = \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q B_q(sk+1, n_1+\cdots+n_s-sk+1) \\
&\quad = \binom{n_1}{k}_q \binom{n_2}{k}_q \cdots \binom{n_s}{k}_q \frac{\Gamma_q(sk+1)\Gamma_q(n_1+\cdots+n_s-sk+1)}{\Gamma_q(n_1+\cdots+n_s+2)}.
\end{aligned} \tag{2.28}$$

By comparing the coefficients on the both sides of Theorem 2.9 and (2.28), one obtains the following corollary.

Corollary 2.10. *For $s \in \mathbb{N}$, let $k > -1/s$ and $n_1 + \cdots + n_s - sk > -1$. Then one has*

$$\sum_{l=0}^{sk} \frac{\binom{sk}{l}_q (-1)^{l+sk} q^{\binom{sk-l+1}{2}}}{[n_1+\cdots+n_s-l+1]_q} = \frac{\Gamma_q(sk+1)\Gamma_q(n_1+\cdots+n_s-sk+1)}{\Gamma_q(n_1+\cdots+n_s+2)}. \tag{2.29}$$

For $n \in \mathbb{Z}_+$, one gets

$$\begin{aligned}
&\frac{\int_0^1 B_{0,n}(qx,q)\left(\prod_{l=1}^n B_{l,n}\left(q^{nl-\binom{l}{2}+1}x,q\right)\right)d_qx}{q^{\sum_{l=1}^n (nl-\binom{l}{2}+1)l}} \\
&= \left(\prod_{i=0}^n \binom{n}{i}_q\right)\int_0^1 x^{\binom{n+1}{2}}(1-qx)_q^{\binom{n+1}{2}}d_qx \\
&= \left(\prod_{i=0}^n \binom{n}{i}_q\right)B_q\left(\binom{n+1}{2}+1,\binom{n+1}{2}+1\right) \\
&= \left(\frac{(\Gamma_q(n+1))^{n+1}}{\left(\prod_{i=1}^n \Gamma_q(i+1)\right)^2}\right)\left(\frac{(\Gamma_q(n(n+1)/2+1))^2}{\Gamma_q(n(n+1)+2)}\right).
\end{aligned} \tag{2.30}$$

Therefore, by (2.30), one obtains the following theorem.

Theorem 2.11. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\int_0^1 B_{0,n}(qx,q)\left(\prod_{l=1}^n B_{l,n}\left(q^{nl-\binom{l}{2}+1}x,q\right)\right)d_qx \\
&= q^{\sum_{l=1}^n (nl-\binom{l}{2}+1)l}\left(\frac{(\Gamma_q(n+1))^{n+1}}{\left(\prod_{i=1}^n \Gamma_q(i+1)\right)^2}\right)\left(\frac{(\Gamma_q(n(n+1)/2+1))^2}{\Gamma_q(n(n+1)+2)}\right).
\end{aligned} \tag{2.31}$$

From (2.30), one can also derive the following equation:

$$\begin{aligned}
&\frac{\int_0^1 B_{0,n}(qx,q)\left(\prod_{l=1}^n B_{l,n}\left(q^{nl-\binom{l}{2}+1}x,q\right)\right)d_qx}{q^{\sum_{l=1}^n (nl-\binom{l}{2}+1)l}} \\
&= \left(\prod_{i=0}^n \binom{n}{i}_q\right)\sum_{l=0}^{\binom{n+1}{2}}\left(\begin{pmatrix} n+1 \\ 2 \end{pmatrix} \atop l\right)_q (-1)^l q^{\binom{l+1}{2}}\int_0^1 x^{\binom{n+1}{2}+l}d_qx \\
&= \left(\prod_{i=0}^n \binom{n}{i}_q\right)\sum_{l=0}^{\binom{n+1}{2}}\left(\begin{pmatrix} n+1 \\ 2 \end{pmatrix} \atop l\right)_q (-1)^l q^{\binom{l+1}{2}}\frac{1}{[n(n+1)/2+l+1]_q}.
\end{aligned} \tag{2.32}$$

By comparing the coefficients on the both sides of Theorem 2.11 and (2.30), one can see that

$$\sum_{l=0}^{n(n+1)/2} \frac{\binom{n(n+1)/2}{l}_q (-1)^l q^{\binom{l+1}{2}}}{[n(n+1)/2+l+1]_q} = B_q\left(\frac{n(n+1)}{2}+1, \frac{n(n+1)}{2}+1\right). \tag{2.33}$$

Therefore, by (2.33), one obtains the following corollary.

Corollary 2.12. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n(n+1)/2} \frac{\binom{n(n+1)/2}{l}_q (-1)^l q^{\binom{l+1}{2}}}{[n(n+1)/2+l+1]_q} = \left(\frac{(\Gamma_q(n(n+1)/2+1))^2}{\Gamma_q(n(n+1)+2)}\right). \tag{2.34}$$

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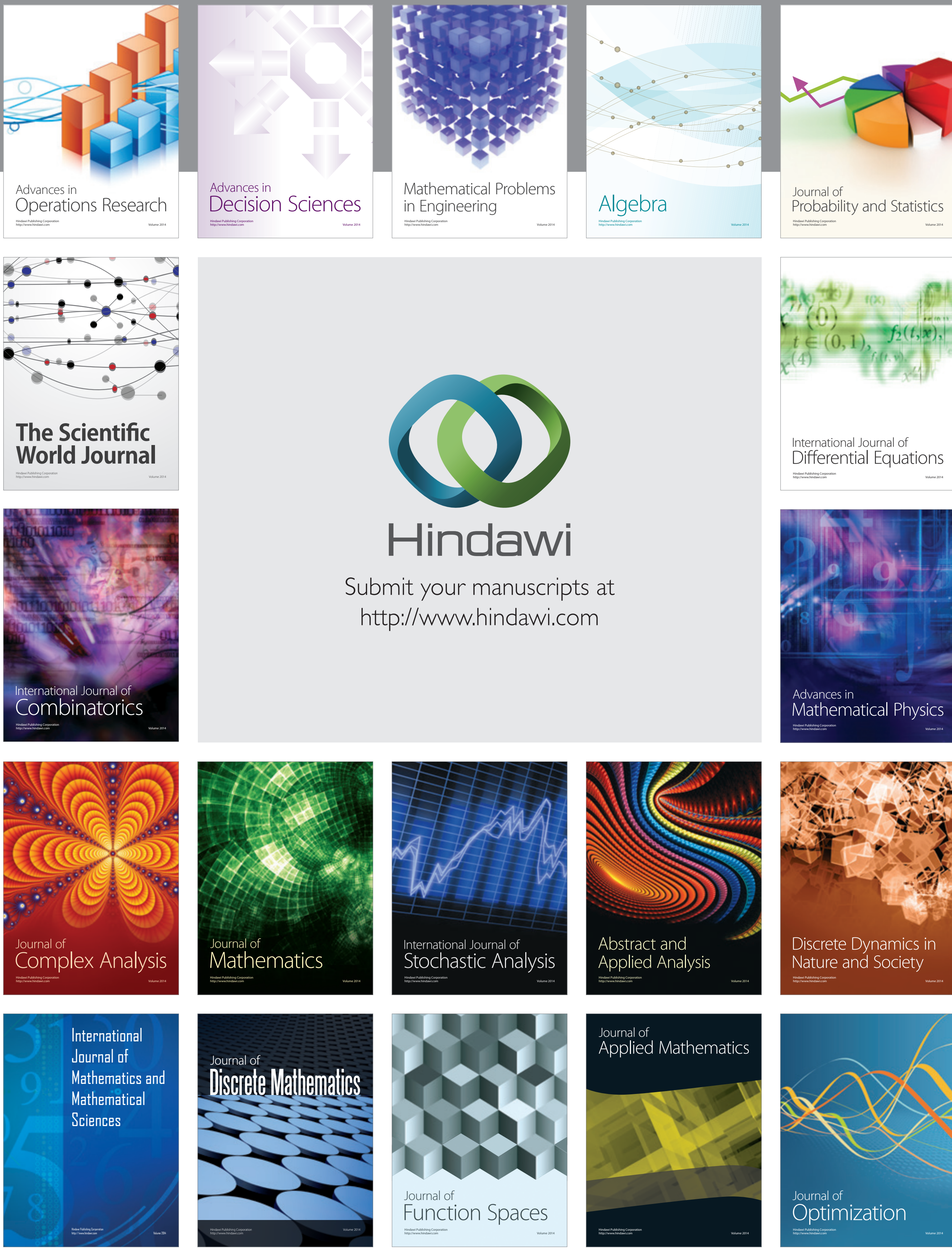

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