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Research Article

Tight Representations of 0-E-Unitary Inverse Semigroups

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We study the tight representation of a semilattice in $\{0,1\}$ by some examples. Then we introduce the concept of the complex tight representation of an inverse semigroup S by the concept of the tight representation of the semilattice of idempotents E of S in $\{0,1\}$. Specifically we describe the tight representation of a 0-E-unitary inverse semigroup and prove that if σ is a tight semilattice representation of the 0-E-unitary inverse semigroup S in $\{0,1\}$, then σ is a complex tight representation.

1. Introduction

A *semigroup* is a set equipped with an associative binary operation. A *monoid* is a semigroup with an identity. A semigroup S is said to be an *inverse semigroup*, provided there exists, for each s in S, a unique element s^* in S such that

$$s = ss^*s, \qquad s^* = s^*ss^* \tag{1.1}$$

Good references for inverse semigroups are [1–3].

For a given set X, let $I(X)$ be the set of all bijective functions $f : A \to B$, where A and B are subsets of X. The multiplication on $I(X)$ is by composition of functions, defined on the largest possible domain. More precisely, for $f, g \in I(X)$, let fog be the function with $\mathrm{dom}(fog) = g^{-1}(\mathrm{ran}(g) \cap \mathrm{dom}(f))$, and $fog(x) = f(g(x))$. The involution on $I(X)$ sends a function to its inverse. $I(X)$ is called the inverse semigroup of partial bijections on X.

By the Wagner-Preston representation theorem, (see [1, 1.5.1]) every inverse semigroup is an inverse semigroup of partial bijection.

Let S be an inverse semigroup. An *idempotent* is an element $e \in S$ such that $e^2 = e$. The set of idempotents of S is usually denoted by $E(S)$, or just E. A partial bijection is idempotent if and only if it is the identity function on its domain.

The *natural partial order* $\leq$ on S is defined by

$$s \leq t \quad \text{iff } s = te \text{ for some idempotent } e. \tag{1.2}$$

The natural partial order induces a semilattice structure on the set $E(S)$ of idempotents by the order

$$e \leq f \quad \text{iff } e = ef. \tag{1.3}$$

So, one often refers to $E(S)$ as the semilattices of idempotents of S. For f, g in $I(X)$, $f \leq g$ if and only if g restricted to $\operatorname{dom}(f)$ is f.

Let $B_n = \{(i, j) : 1 \leq i, j \leq n\} \cup \{0\}$. Define a multiplication on B_n by

$$(i, j)(k, l) = \begin{cases} (i, l), & i = j, \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

and $(i, j)0 = 0(i, j) = 0$. Define the involution on B_n by $(i, j)^* = (j, i)$. The inverse semigroup B_n in called a Brandt semigroup.

2. Tight Representations of Semilattices

In this section we define the tight representation of a semilattice E on $\{0, 1\}$ and introduce two characteristic functions on E that are tight representations. One can see more about representations and semilattices in [4–7].

Definition 2.1. Let E be a partially ordered set. A subset $F \subseteq E$ is said to be *connected* if, for every f_1 and f_2 in F, there exists an element f in F such that

$$f \leq f_1, \qquad f \leq f_2. \tag{2.1}$$

A *component* of E is a maximal connected subset of E. For a partially ordered set E with the minimum element 0, we denote by $E_{\min}$ the set of all minimal elements of $E^* = E \setminus \{0\}$.

Definition 2.2. Given a partially ordered set E with smallest element 0, we say that two elements s and t in E are *disjoint*, in symbols $s \perp t$, if there is no nonzero $u \in E$ such that $u \leq s, t$. Otherwise we say that s and t *intersect*, in symbols $s \cap t \neq \emptyset$.

For any subset U of E, we say that a subset $V \subseteq U$ is a *cover* for U if, for every nonzero $u \in U$, there exists $v \in V$ such that $u \cap v \neq \emptyset$.

A *semilattice* is a partially ordered set E such that for every $s, t \in E$, the set $\{u \in E : u \leq s, t\}$ contains a maximum element.

From now on we will fix a semilattice E.

Definition 2.3. For a finite subset $F \subseteq E$, define $[0, F]$ to be the subset of E given by

$$[0, F] = \{e \in E : e \leq f, \quad \forall f \in F\}, \tag{2.2}$$

and denote by $F^{\perp}$ the subset of E given by

$$F^{\perp} = \{e \in E : e \perp f, \quad \forall f \in F\}. \tag{2.3}$$

It is obvious that $0 \in [0, F]$ and if F is not contained in a component of E^*, then $[0, F] = \{0\}$. If F and G are finite subsets of E, we denote by $E^{F,G}$ the subset $[0, F] \cap G^{\perp}$ of E.

Notice that if $F = G = \emptyset$, than $E^{F,G} = E$, if $F = \emptyset$, $E^{F,G} = G^{\perp}$ and if $G = \emptyset$, $E^{F,G} = [0, F]$. If $e \leq f$, then $E^{\{e\},\{f\}} = \{0\}$ and $E^{*\{e\},\{f\}} = \emptyset$. However $E^{\{f\},\{e\}}$ is not necessarily zero. Note that if e and f belong to different components of E^*, then $E^{\{e\},\{f\}} = (0, e]$. For elements e and f in E such that $e \leq f$, e is said to be *dense* in f if $E^{\{f\},\{e\}} = \{0\}$.

Definition 2.4. A map $\sigma : E \to \{0, 1\}$ is said to be a *representation* of E in $\{0, 1\}$, if $\sigma(0) = 0$ and $\sigma(x \wedge y) = \sigma(x)\sigma(y)$, for all x, y in E. We say that σ is *tight* if for all finite subsets $F, G \subseteq E$, and for all finite cover H for $E^{F,G}$, one has that

$$\operatorname{sgn}\left(\sum_{h \in H} \sigma(h)\right) = \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)). \tag{2.4}$$

Proposition 2.5. *Let e and f be in E with e being dense in f. Then $\sigma(e) = \sigma(f)$ for every tight representation σ of E in $\{0, 1\}$.*

Proof. Suppose that σ is a tight representation of E in $\{0, 1\}$ and choose e, f in E such that $E^{\{f\},\{e\}} = \{0\}$. Then $\emptyset$ is a cover for $E^{\{f\},\{e\}}$. So by the definition of tight representation we have $\sigma(f)(1 - \sigma(e)) = 0$. Therefore $\sigma(f) \leq \sigma(e)$. On the other hand, since $e \leq f$, then $\sigma(e) \leq \sigma(f)$. ☐

Theorem 2.6. *Let E be a semilattice with minimum element 0. If $e \in E_{\min}$, then $\chi_{[e,\infty)}$ is a tight representation of E in $\{0, 1\}$.*

Proof. Set $\sigma = \chi_{[e,\infty)}$. If $x, y \in E$ are such that $x \leq y$, then $\sigma(x) \leq \sigma(y)$. On the other hand if x and y are disjoint, then $\sigma(x)$ and $\sigma(y)$ are disjoint too. So $\sigma(x) \leq 1 - \sigma(y)$. More generally, if F and G are finite subsets of E, and $h \in E$ is such that $h \leq f$ for every $f \in F$, and $h \perp g$, for every $g \in G$, then

$$\sigma(h) \leq \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)). \tag{2.5}$$

Conversely, let F, G be finite subsets of E, and let H be a cover for $E^{F,G}$. To prove the inequality

$$\operatorname{sgn}\left(\sum_{h \in H} \sigma(h)\right) \geq \prod_{f \in F} \sigma(f) \prod_{g \in G} (1 - \sigma(g)), \tag{2.6}$$

we see that if the right-hand side is 0, then the inequality holds obviously. So suppose that the right-hand side is 1. Then we show that the left-hand side is 1 too. Since $\sigma = \chi_{[e,\infty)}$, we have $F \subseteq [e,\infty)$ and $G \cap [0,\infty) = \emptyset$. Also $e \in E^{F,G}$. Then there exists $h \in H$ such that $h \cap e \neq \emptyset$. This means that there exists a nonzero $t \in E$ such that $t \leq h, e$. Since $e \in E_{\min}$, then $e \leq h$ and so $h \in [0,\infty)$ and $\sigma(h) = 1$. Therefore the left-hand side is 1 too. □

By the definition of $E_{\min}$, one can show that every element of $E_{\min}$ is the minimum element of some component of E^*. But it may happen that some component of E^* does not have a minimum element. So the following theorem holds.

Theorem 2.7. *If F is a component of E^*, then χ_F is a tight representation of E in $\{0,1\}$.*

3. Complex Tight Representations of 0-*E*-Unitary Inverse Semigroups

The class of E-unitary inverse semigroups is one of the most important in inverse semigroup theory. When an inverse semigroup contains a zero, then every element of E must be idempotent. Thus motivated by Szendrei [8], we define the class of 0-E-unitary inverse semigroups (although she called them E^*-unitary). The term 0-E-unitary appears to be due to Meakin and Sapir [9]. More references for 0-E-unitary inverse semigroups are [10–12].

Throughout this section we define complex tight representations of inverse semigroups and prove that every semilattice tight representation on a 0-E-unitary inverse semigroup is a complex tight representation.

Definition 3.1. An inverse semigroup S with semilattice of idempotent E is E-unitary if, for every $e \in E$, $e \leq s$ for some $s \in S$ implies that s is idempotent.

Proposition 3.2 (see [1]). *Let S be an inverse semigroup. For $s, t \in S$, the following are equivalent:*

(i) $s \leq t$,

(ii) *there exists $f \in E$ such that $s = ft$,*

(iii) $s = ts^*s$,

(iv) $s = ss^*t$,

(v) $s^* \leq t^*$.

Proposition 3.3. *Let S be an inverse semigroup and e is an idempotent in E. If $s \in S$ such that $s \leq e$, then s is also an idempotent.*

Proof. If $s \leq e$, then by the previous proposition there exists an idempotent $f \in E$ such that $s = ef$. Since the semilattice of idempotents is closed under multiplication, we have $s \in E$. □

Definition 3.4. An inverse semigroup S is said to be a 0-E-unitary if, for every nonzero idempotent e, $e \leq s$ for some $s \in S$ implies s is idempotent. The components of E^* are in the form $[s,\infty)$ or (s,∞) for some nonzero element $s \in S$. By Proposition 3.3, if F is any component of $S^* = S \setminus \{0\}$, then $F \subseteq E$ or $F \cap E = \emptyset$.

Lemma 3.5 (see [4]). *If S is a 0-E-unitary inverse semigroup and $s, t \in S$ are such that $s^*s = t^*t$ and $se = te$ for some nonzero idempotent $e \leq s^*s$, then $s = t$.*

Proposition 3.6. *If S is a 0-E-unitary inverse semigroup with zero, then S is a semilattice with respect to natural order.*

Proof. Let $s, t \in S$. If there is no nonzero $u \in S$ such that $u \leq s, t$, then $st = 0$. So 0 is the infimum of s, t. Now suppose that there exists a nonzero element u such that $u \leq s, t$. By [1], $u^*u \leq s^*s$ and $u^*u \leq t^*t$. Let $f = s^*st^*t$. Then $u^*u \leq f$. Setting $s_1 = sf$ and $t_1 = tf$, we have

$$s_1^* s_1 = fs^*sf = f = ft^*tf = t_1^* t_1. \tag{3.1}$$

Since

$$s_1 u^* u = sfu^*u = su^*u = u = tu^*u = tfu^*u = t_1 u^* u, \tag{3.2}$$

by Lemma 3.5 we have $s_1 = t_1$. So

$$st^*t = ss^*st^*t = sf = s_1 = t_1 = tf = ts^*s. \tag{3.3}$$

Since $0 \neq u_1 \leq s_1, t_1$ we may apply the above argument to s_1, u_1, t_1 in order to prove that $s^*tt^* = t^*ss^*$, which implies that $tt^*s = ss^*t$.

The fact that $u \leq s, t$ implies that $su^*u = u = tu^*u$. So

$$t^*su^*u = t^*tu^*u = u^*u. \tag{3.4}$$

Since S is 0-E-unitary, t^*s is an idempotent. Also we can prove similarly that ts^* is an idempotent. Thus $st^*t = ts^*t = tt^*s$. Therefore

$$st^*t = ts^*s = tt^*s = ss^*t. \tag{3.5}$$

We claim that st^*t is the infimum of s, t. It is obvious that $st^*t \leq s, t$. Since

$$u = su^*u = sfu^*u = ss^*st^*tu^*u = st^*tu^*u, \tag{3.6}$$

then $u \leq st^*t$. □

Note that if σ is a representation of an inverse semigroup S in the complex plane (as a Hilbert space), then $\sigma(e) = 0$ or 1, for every idempotent element $e \in E(S)$. Such representations are called *complex representations*.

Now we will fix an inverse semigroup S with 0.

Definition 3.7. A complex representation σ of S on the complex plane is said to be *tight* if the restriction of σ to $E(S)$ is a tight representation of $E(S)$ in $\{0, 1\}$.

From the definition one can show that if s_0 is a minimum element of $S^* = S \setminus \{0\}$, then $\chi_{[s_0,\infty)}$ is a complex tight representation on S. Also if T is a component of S^*, then χ_T is a complex tight representation on S.

Since every 0-E-unitary inverse semigroup is a semilattice with zero, a representation of S in $\{0,1\}$ is both a representation of the semilattice S in $\{0,1\}$ and a complex representation of the inverse semigroup S.

Theorem 3.8. *Let S be a 0-E-unitary inverse semigroup and let σ be a representation of S in $\{0,1\}$. If σ is tight as a semilattice representation, then it is tight as a complex representation.*

Proof. Suppose that σ is a semilattice tight representation of S in $\{0,1\}$. Let F and G be finite subsets of E and H a cover for $E^{F,G}$. Since $E \subseteq S$, then $E^{F,G} \subseteq S^{F,G}$. Since H is a cover of $E^{F,G}$, then there is a cover K of $S^{F,G}$ such that $H \subseteq K$. Therefore

$$\sum_{h\in H}\sigma(h) \leq \sum_{k\in K}\sigma(k), \tag{3.7}$$

and hence

$$\operatorname{sgn}\left(\sum_{k\in K}\sigma(k)\right) \geq \prod_{f\in F}\sigma(f)\prod_{g\in G}(1-\sigma(g)). \tag{3.8}$$

Then $\sigma|_E$ is a tight representation of E in $\{0,1\}$ and therefore σ is a complex tight representation of S in $\{0,1\}$. □

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