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*Research Article*

# Existence of Homoclinic Orbits for a Class of Asymptotically $p$-Linear Difference Systems with $p$-Laplacian

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By applying a variant version of Mountain Pass Theorem in critical point theory, we prove the existence of homoclinic solutions for the following asymptotically $p$-linear difference system with $p$-Laplacian $\Delta(|\Delta u(n-1)|^{p-2}\Delta u(n-1)) + \nabla[-K(n,u(n)) + W(n,u(n))] = 0$, where $p \in (1,+\infty)$, $n \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $K, W : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$ are not periodic in $n$, and W is asymptotically $p$-linear at infinity.

## 1. Introduction

Consider the following $p$-Laplacian difference system:

$$\Delta\left(|\Delta u(n-1)|^{p-2}\Delta u(n-1)\right) + \nabla[-K(n,u(n)) + W(n,u(n))] = 0, \quad n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta$ is the forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$, $\Delta^2 u(n) = \Delta(\Delta u(n))$, $p \in (1,+\infty)$, $n \in \mathbb{Z}$, $u \in \mathbb{R}^N$, $K$, $W$: $\mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$ are not periodic in $n$, $W$ is asymptotically $p$-linear at infinity, and $K$ and $W$ are continuously differentiable in $x$. As usual, we say that a solution $u(n)$ of (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $n \to \pm\infty$. In addition, if $u(n) \not\equiv 0$, then $u(n)$ is called a nontrivial homoclinic solution.

When $p = 2$, (1.1) can be regarded as a discrete analogue of the following second-order Hamiltonian system:

$$\ddot{u}(t) + \nabla[-K(t,u(t)) + W(t,u(t))] = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

The existence of homoclinic orbits for Hamiltonian systems is a classical problem and its importance in the study of the behavior of dynamical systems has been recognized by Poincaré [1]. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation and its perturbed system probably produces chaotic phenomenon. For the existence of homoclinic solutions of problem (1.2), one can refer to the papers [2–5].

Difference equations usually describe evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$th generation $x(n+1)$ is a function of the $n$th generation $x(n)$. In fact, difference equations provide a natural description of many discrete models in real world. Since discrete models exist in various fields of science and technology such as statistics, computer science, electrical circuit analysis, biology, neural network, and optimal control, it is of practical importance to investigate the solutions of difference equations. For more details about difference equations, we refer the readers to the books [6–8].

In some recent papers [9–20], the authors studied the existence of periodic solutions and subharmonic solutions of difference equations by applying critical point theory. These papers show that the critical point theory is an effective method to the study of periodic solutions for difference equations. Along this direction, several authors [21–28] used critical point theory to study the existence of homoclinic orbits for difference equations. Motivated by the above papers, we consider the existence of homoclinic orbits for problem (1.1) by using the variant version of Mountain Pass Theorem. Our result is new, which seems not to have been considered in the literature. Here is our main result.

**Theorem 1.1.** *Suppose that $K$ and $W$ satisfy the following conditions.*

(K1) *There are two positive constants $b_1$ and $b_2$ such that*

$$b_1|x|^p \le K(n,x) \le b_2|x|^p, \quad \forall (n,x) \in \mathbb{Z}\times\mathbb{R}^N. \tag{1.3}$$

(K2) *There is a positive constant $b_3$ such that*

$$b_3|x|^p \le (\nabla K(n,x),x) \le |\nabla K(n,x)||x| \le pK(n,x), \quad \forall (n,x)\in\mathbb{Z}\times\mathbb{R}^N. \tag{1.4}$$

(W1) $W(n,0)=0$, $\nabla W(n,x)=o(|x|^{p-1})$ *as* $|x|\to 0$ *uniformly for* $n\in\mathbb{Z}$.

(W2) *There exists a constant $R>0$ such that*

$$\frac{|\nabla W(n,x)|}{|x|^{p-1}} \le R, \quad \forall n\in\mathbb{Z},\ x\in\mathbb{R}^N. \tag{1.5}$$

(W3) *There exists a function $V_\infty \in l^\infty(\mathbb{Z},\mathbb{R}^+)$ such that*

$$\lim_{|x|\to\infty}\frac{\left|\nabla W(n,x)-V_\infty(n)|x|^{p-2}x\right|}{|x|^{p-1}}=0 \quad \text{uniformly for } n\in\mathbb{Z}, \qquad \inf_{\mathbb{Z}} V_\infty(n) > \max\{1, pb_2\}. \tag{1.6}$$

(W4) $\widetilde{W}(n,x) = (\nabla W(n,x), x) - pW(n,x)$,

$$\lim_{|x|\to\infty} \widetilde{W}(n,x) = +\infty \quad \textit{uniformly for } n \in \mathbb{Z}, \tag{1.7}$$

*and for any fixed* $0 < c_1 < c_2 < +\infty$,

$$\inf_{n\in\mathbb{Z},\ c_1\le|x|\le c_2} \frac{\widetilde{W}(n,x)}{|x|^p} > 0. \tag{1.8}$$

*Then problem* (1.1) *has at least one nontrivial homoclinic solution.*

*Remark 1.2.* The function $W(n,x)$ in this paper is asymptotically $p$-linear at infinity. The behavior of the gradient of $W(n,x)$ at infinity is like that of a function $V_\infty(n)|x|^{p-2}x$, where $V_\infty(n)$ is a real function but not a matrix function. To the best of our knowledge, similar results of this kind of $p$-Laplacian difference systems with asymptotically $p$-linear $W(n,x)$ at infinity cannot be found in the literature. From this point, our result is new.

## 2. Preliminaries

Let

$$\begin{aligned} S &= \left\{ \{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^N, n \in \mathbb{Z} \right\}, \\ E &= \left\{ u \in S : \sum_{n\in\mathbb{Z}} \left[ |\Delta u(n-1)|^p + |u(n)|^p \right] < +\infty \right\}, \end{aligned} \tag{2.1}$$

and for $u \in E$, let

$$\|u\| = \left\{ \sum_{n\in\mathbb{Z}} \left[ |\Delta u(n-1)|^p + |u(n)|^p \right] < +\infty \right\}^{1/p}. \tag{2.2}$$

Then $E$ is a uniform convex Banach space with this norm. As usual, for $1 \le p < +\infty$, let

$$l^p\left(\mathbb{Z},\mathbb{R}^N\right) = \left\{ u \in S : \sum_{n\in\mathbb{Z}} |u(n)|^p < +\infty \right\}, \qquad l^\infty\left(\mathbb{Z},\mathbb{R}^N\right) = \left\{ u \in S : \sup_{n\in\mathbb{Z}} |u(n)| < +\infty \right\}, \tag{2.3}$$

and their norms are given by

$$\begin{aligned} \|u\|_{l^p} &= \left( \sum_{n\in\mathbb{Z}} |u(n)|^p \right)^{1/p}, \quad \forall u \in l^p\left(\mathbb{Z},\mathbb{R}^N\right), \\ \|u\|_\infty &= \sup\{|u(n)| : n \in \mathbb{Z}\}, \quad \forall u \in l^\infty\left(\mathbb{Z},\mathbb{R}^N\right), \end{aligned} \tag{2.4}$$

respectively.

For any $u \in E$, let

$$\varphi(u) = \frac{1}{p}\sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^p - \sum_{n\in\mathbb{Z}}[-K(n,u(n)) + W(n,u(n))]. \tag{2.5}$$

To prove our results, we need the following generalization of Lebesgue's dominated convergence theorem.

**Lemma 2.1** (see [29]). *Let $\{f_k(t)\}$ and $\{g_k(t)\}$ be two sequences of measurable functions on a measurable set $A$, and let*

$$|f_k(t)| \le g_k(t), \quad \forall a.e.\ t \in A. \tag{2.6}$$

*If*

$$\lim_{k\to\infty} f_k(t) = f(t), \quad \lim_{k\to\infty} g_k(t) = g(t), \quad \forall a.e.\ t \in A,$$
$$\lim_{k\to\infty}\int_A g_k(t)dt = \int_A g(t)dt < +\infty, \tag{2.7}$$

*then*

$$\lim_{k\to\infty}\int_A f_k(t)dt = \int_A f(t)dt. \tag{2.8}$$

**Lemma 2.2.** *For $u \in E$,*

$$\|u\|_\infty \le \|u\|_{l^p} \le 2\|u\|. \tag{2.9}$$

*Proof.* Since $u \in E$, it follows that $\lim_{|n|\to\infty}|u(n)| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u(n^*)| = \max_{n\in\mathbb{Z}}|u(n)|. \tag{2.10}$$

Hence, we have

$$\begin{aligned}\|u\|_\infty &\le \left(\sum_{n\in\mathbb{Z}}|u(n)|^p\right)^{1/p} = \|u\|_{l^p} = \left(\sum_{n\in\mathbb{Z}}|u(n)-u(n-1)+u(n-1)|^p\right)^{1/p}\\ &\le \left(\sum_{n\in\mathbb{Z}}(|u(n)-u(n-1)|+|u(n-1)|)^p\right)^{1/p}\\ &\le \left(2^p\sum_{n\in\mathbb{Z}}\left(|u(n)-u(n-1)|^p+|u(n-1)|^p\right)\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&= 2\left(\sum_{n\in\mathbb{Z}}\left(|\Delta u(n-1)|^p + |u(n-1)|^p\right)\right)^{1/p} \\
&= 2\left(\sum_{n\in\mathbb{Z}}\left(|\Delta u(n-1)|^p + |u(n)|^p\right)\right)^{1/p} \\
&= 2\|u\|.
\end{aligned}
\tag{2.11}
$$

□

**Lemma 2.3.** *Suppose that (K1), (K2), and (W2) hold. If $u_k \to u$ in $E$, then $\nabla K(n,u_k) \to \nabla K(n,u)$ and $\nabla W(n,u_k) \to \nabla W(n,u)$ in $l^{p'}(\mathbb{R},\mathbb{R}^N)$, where $p' > 1$ satisfies $1/p + 1/p' = 1$.*

*Proof.* From (K1) and (K2), we have

$$
|\nabla K(n,x)| \le pb_2|x|^{p-1}, \quad \forall (n,x) \in \mathbb{Z}\times\mathbb{R}^N. \tag{2.12}
$$

Hence, from (2.12), we have

$$
\begin{aligned}
|\nabla K(n,u_k(n)) - \nabla K(n,u(n))|^{p'} &\le \left[pb_2\left(|u_k(n)|^{p-1} + |u(n)|^{p-1}\right)\right]^{p'} \\
&\le \left[pb_2 2^{p-1}|u_k(n)-u(n)|^{p-1} + pb_2\left(1+2^{p-1}\right)|u(n)|^{p-1}\right]^{p'} \\
&\le 2^{pp'}(pb_2)^{p'}|u_k(n)-u(n)|^p + 2^{p'}(pb_2)^{p'}\left(1+2^{p-1}\right)^{p'}|u(n)|^p \\
&:= g_k(n).
\end{aligned}
\tag{2.13}
$$

Moreover, since $u_k \to u$ in $l^p(\mathbb{Z},\mathbb{R}^N)$ and $u_k(n) \to u(n)$ for almost every $n \in \mathbb{Z}$, hence,

$$
\begin{aligned}
\lim_{k\to\infty} g_k(n) &= 2^{p'}(pb_2)^{p'}\left(1+2^{p-1}\right)^{p'}|u(n)|^p := g(n), \quad \forall \text{a.e. } n\in\mathbb{Z}, \\
\lim_{k\to\infty}\sum_{n\in\mathbb{Z}} g_k(n) &= \lim_{k\to\infty}\sum_{n\in\mathbb{Z}}\left[2^{pp'}(pb_2)^{p'}|u_k(n)-u(n)|^p + 2^{p'}(pb_2)^{p'}\left(1+2^{p-1}\right)^{p'}|u(n)|^p\right] \\
&= 2^{pp'}(pb_2)^{p'}\lim_{k\to\infty}\sum_{n\in\mathbb{Z}}|u_k(n)-u(n)|^p + 2^{p'}(pb_2)^{p'}\left(1+2^{p-1}\right)^{p'}\sum_{n\in\mathbb{Z}}|u(n)|^p \\
&= 2^{p'}(pb_2)^{p'}\left(1+2^{p-1}\right)^{p'}\sum_{n\in\mathbb{Z}}|u(n)|^p \\
&= \sum_{n\in\mathbb{Z}} g(n) < +\infty.
\end{aligned}
\tag{2.14}
$$

It follows from Lemma 2.1, (2.13), and the previous equations that

$$\lim_{k\to\infty}\sum_{n\in\mathbb{Z}}|\nabla K(n,u_k(n))-\nabla K(n,u(n))|^{p'}=0. \tag{2.15}$$

This shows that $\nabla K(n,u_k)\to\nabla K(n,u)$ in $l^{p'}(\mathbb{Z},\mathbb{R}^N)$. By a similar proof, we can prove that $\nabla W(n,u_k)\to\nabla W(n,u)$ in $l^{p'}(\mathbb{Z},\mathbb{R}^N)$. The proof is complete. □

**Lemma 2.4.** *Under the conditions of Theorem 1.1, one has*

$$\begin{aligned}\langle\varphi'(u),v\rangle=\sum_{n\in\mathbb{Z}}\Big[&|\Delta u(n-1)|^{p-2}(\Delta u(n-1),\Delta v(n-1))\\&+(\nabla K(n,u(n))-\nabla W(n,u(n)),v(n))\Big]\end{aligned} \tag{2.16}$$

*for $u,v\in E$, which yields that*

$$\langle\varphi'(u),u\rangle=\sum_{n\in\mathbb{Z}}\big[|\Delta u(n-1)|^{p}+(\nabla K(n,u(n)),u(n))-(\nabla W(n,u(n)),u(n))\big]. \tag{2.17}$$

*Moreover, $\varphi$ is continuously Fréchet-differential defined on $E$; that is, $\varphi\in C^1(E,\mathbb{R})$ and any critical point $u$ of $\varphi$ on $E$ is classical solution of* (1.1) *with $u(\pm\infty)=0$.*

*Proof.* Firstly, we show that $\varphi:E\to\mathbb{R}$. Let $u\in E$, by (2.9) and (K1), we have

$$\sum_{n\in\mathbb{Z}}K(n,u(n))\le\sum_{n\in\mathbb{Z}}b_2|u(n)|^p\le b_2 2^p\|u\|^p. \tag{2.18}$$

By (W2), we get

$$|W(n,x)|=\left|\int_0^1(\nabla W(n,sx),x)ds\right|\le R|x|^p,\quad\forall(n,x)\in\mathbb{Z}\times\mathbb{R}^N. \tag{2.19}$$

Hence, from (2.9) and (2.19), we have

$$\left|\sum_{n\in\mathbb{Z}}W(n,u(n))\right|\le\sum_{n\in\mathbb{Z}}|W(n,u(n))|\le\sum_{n\in\mathbb{Z}}R|u(n)|^p\le R2^p\|u\|^p. \tag{2.20}$$

It follows from (2.5), (2.18), and (2.20) that $\varphi:E\to\mathbb{R}$. Next we prove that $\varphi\in C^1(E,\mathbb{R})$. Rewrite $\varphi$ as follows:

$$\varphi(u)=\varphi_1(u)+\varphi_2(u)-\varphi_3(u), \tag{2.21}$$

where

$$\varphi_1(u) := \frac{1}{p}\sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^p, \quad \varphi_2(u) := \sum_{n\in\mathbb{Z}}K(n,u(n)), \quad \varphi_3(u) := \sum_{n\in\mathbb{Z}}W(n,u(n)). \tag{2.22}$$

It is easy to check that $\varphi_1 \in C^1(E,\mathbb{R})$ and

$$\langle \varphi_1'(u), v\rangle = \sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^{p-2}(\Delta u(n-1), \Delta v(n-1)), \quad \forall u, v \in E. \tag{2.23}$$

Next, we prove that $\varphi_i \in C^1(E,\mathbb{R})$, $i = 2, 3$, and

$$\langle \varphi_2'(u), v\rangle = \sum_{n\in\mathbb{Z}}(\nabla K(n,u(n)), v(n)), \quad \forall u, v \in E, \tag{2.24}$$

$$\langle \varphi_3'(u), v\rangle = \sum_{n\in\mathbb{Z}}(\nabla W(n,u(n)), v(n)), \quad \forall u, v \in E. \tag{2.25}$$

For any $u, v \in E$ and for any function $\theta : \mathbb{R} \to (0,1)$, by (K2), we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\max_{h\in[0,1]}|(\nabla K(n,u(n)+\theta(t)hv(n)), v(n))| &\le pb_2\sum_{n\in\mathbb{Z}}\max_{h\in[0,1]}|u(n)+\theta(t)hv(n)|^{p-1}|v(n)| \\ &\le 2^{p-1}pb_2\sum_{n\in\mathbb{Z}}\left(|u(n)|^{p-1}+|v(n)|^{p-1}\right)|v(n)| \\ &\le 2^{p-1}pb_2\left[\|u\|_{l^p}^{p-1}\|v\|_{l^p}+\|v\|_{l^p}^{p}\right] \\ &< +\infty.\end{aligned} \tag{2.26}$$

Then by the previous equations and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\langle \varphi_2'(u), v\rangle &= \lim_{h\to 0^+}\frac{\varphi_2(u+hv)-\varphi_2(u)}{h} \\ &= \lim_{h\to 0^+}\frac{1}{h}\sum_{n\in\mathbb{Z}}[K(n,u(n)+hv(n))-K(n,u(n))] \\ &= \lim_{h\to 0^+}\sum_{n\in\mathbb{Z}}(\nabla K(t,u(n)+\theta(t)hv(n)), v(n)) \\ &= \sum_{n\in\mathbb{Z}}(\nabla K(n,u(n)), v(n)), \quad \forall u, v \in E.\end{aligned} \tag{2.27}$$

Similarly, we can prove that (2.25) holds by using (W2) instead of (K2). Finally, we prove that $\varphi_i \in C^1(E,\mathbb{R})$, $i = 2,3$. Let $u_k \longrightarrow u$ in $E$; then by Lemma 2.3, we have

$$\begin{aligned}
&\left|\langle \varphi_2'(u_k) - \varphi_2'(u), v\rangle\right| \\
&\quad = \left|\sum_{n\in\mathbb{Z}} (\nabla K(n,u_k(n)) - \nabla K(n,u(n)), v(n))\right| \\
&\quad \le \sum_{n\in\mathbb{Z}} |\nabla K(n,u_k(n)) - \nabla K(n,u(n))\|v(n)| \\
&\quad \le \|v\| \left[\sum_{n\in\mathbb{Z}} |\nabla K(n,u_k(n)) - \nabla K(n,u(n))|^{p'}\right]^{1/p'} \longrightarrow 0, \quad k \longrightarrow \infty,\ \forall v \in E.
\end{aligned} \tag{2.28}$$

This shows that $\varphi_2 \in C^1(E,\mathbb{R})$. Similarly, we can prove that $\varphi_3 \in C^1(E,\mathbb{R})$. Furthermore, by a standard argument, it is easy to show that the critical points of $\varphi$ in $E$ are classical solutions of (1.1) with $u(\pm\infty) = 0$. The proof is complete. □

**Lemma 2.5** (see [30]). *Let $E$ be a real Banach space with its dual space $E^*$ and suppose that $\varphi \in C^1(E,\mathbb{R})$ satisfies*

$$\max\{\varphi(0), \varphi(e)\} \le \eta_0 < \eta \le \inf_{\|u\|=\rho} \varphi(u), \tag{2.29}$$

*for some $\eta_0 < \eta$, $\rho > 0$, and $e \in E$ with $\|e\| > \rho$. Let $c \ge \eta$ be characterized by*

$$c = \inf_{\Upsilon\in\Gamma} \max_{0\le\tau\le1} \varphi(\Upsilon(\tau)), \tag{2.30}$$

*where $\Gamma = \{\Upsilon \in C([0,1],E) : \Upsilon(0) = 0, \Upsilon(1) = e\}$ is the set of continuous paths joining 0 to $e$; then there exists $\{u_k\}_{k\in\mathbb{N}} \subset E$ such that*

$$\varphi(u_k) \longrightarrow c, \quad (1 + \|u_k\|)\left\|\varphi'(u_k)\right\|_{E^*} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{2.31}$$

## 3. Proof of Theorem 1.1

*Proof of Theorem 1.1.* We divide the proof of Theorem 1.1 into three steps.

*Step 1.* From (W1), there exists $\rho_0 > 0$ such that

$$\nabla W(n,x) \le \frac{C_1}{2^p}|x|^{p-1}, \qquad \forall n \in \mathbb{Z},\ |x| \le \rho_0, \tag{3.1}$$

where $C_1 = \min\{1/p, b_1\}$. From (3.1), we have

$$\begin{aligned} W(n,x) &= \int_0^1 (\nabla W(n,sx),x)ds \\ &\le \int_0^1 \frac{C_1}{2^p}|x|^p s^{p-1}ds = \frac{C_1}{p2^p}|x|^p, \quad \forall n \in \mathbb{Z},\ |x| \le \rho_0. \end{aligned} \tag{3.2}$$

Let $\rho = \rho_0/2$ and $S = \{u \in E \mid \|u\| = \rho\}$; then from (2.9), we obtain

$$\|u\|_\infty \le \rho_0, \quad \|u\|_{l^p} \le 2\rho, \quad \forall u \in S, \tag{3.3}$$

which together with (2.9), (3.2), and (K1) implies that

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^p - \sum_{n\in\mathbb{Z}}[-K(n,u(n)) + W(n,u(n))] \\ &\ge \frac{1}{p}\sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^p + b_1\sum_{n\in\mathbb{Z}}|u(n)|^p - \sum_{n\in\mathbb{Z}}\frac{C_1}{p2^p}|u(n)|^p \\ &\ge \min\left\{\frac{1}{p}, b_1\right\}\|u\|^p - \frac{C_1}{p2^p}\|u\|_{l^p}^p \\ &\ge C_1\|u\|^p - \frac{C_1}{p}\|u\|^p = \frac{(p-1)C_1}{p}\|u\|^p = \alpha_1 > 0, \quad u \in S. \end{aligned} \tag{3.4}$$

*Step 2.* From (K1), we have

$$\begin{aligned} \varphi(u) &= \frac{1}{p}\sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^p - \sum_{n\in\mathbb{Z}}[-K(n,u(n)) + W(n,u(n))] \\ &\le \frac{1}{p}\sum_{n\in\mathbb{Z}}|\Delta u(n-1)|^p + b_2\sum_{n\in\mathbb{Z}}|u(n)|^p - \sum_{n\in\mathbb{Z}}W(n,u(n)) \\ &\le \max\left\{\frac{1}{p}, b_2\right\}\|u\|^p - \sum_{n\in\mathbb{Z}}W(n,u(n)) \\ &\equiv C_2\|u\|^p - \sum_{n\in\mathbb{Z}}W(n,u(n)). \end{aligned} \tag{3.5}$$

By (W2) and (W3), we get

$$\lim_{|x|\to\infty}\frac{pW(n,x)}{|x|^p} = V_\infty(n) \quad \text{uniformly for } n \in \mathbb{Z}. \tag{3.6}$$

Let $\overline{W}(n,x) = pW(n,x) - V_\infty(n)|x|^p$; it follows from (W2), (W3), (2.19), and (3.6) that

$$\overline{W}(n,x) \le \left(pR + \sup_{\mathbb{Z}} V_\infty(n)\right)|x|^p, \quad \forall x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} \frac{\overline{W}(n,x)}{|x|^p} = 0. \tag{3.7}$$

Define $E_1 := \{u(n) = xe^{-|n|} : x \in \mathbb{R}^N,\ n \in \mathbb{Z}\} \subset E$ with

$$\inf_{\mathbb{Z}} V_\infty(n) > \max\{1, pb_2\}\left(1 + \left|1 - e^{|n|-|n-1|}\right|^p\right). \tag{3.8}$$

By an easy calculation, we have

$$\|u\|^p = \left(1 + \left|1 - e^{|n|-|n-1|}\right|^p\right)\|u\|_{l^p}^p. \tag{3.9}$$

In what follows, we prove that for some $u \in E_1$ with $\|u\| = 1$, $\varphi(su) \to -\infty$ as $s \to \infty$. Otherwise, there exist a sequence $\{s_k\}$ with $s_k \to \infty$ as $k \to \infty$ and a positive constant $C_3$ such that $\varphi(s_k u) \ge -C_3$ for all $k$. From (3.5), we obtain

$$\begin{aligned} -\frac{C_3}{s_k^p} \le \frac{\varphi(s_k u)}{s_k^p} &\le C_2 - \frac{1}{p}\sum_{n\in\mathbb{Z}} \frac{\overline{W}(n, s_k u(n))}{s_k^p} - \frac{1}{p}\sum_{n\in\mathbb{Z}} V_\infty(n)|u(n)|^p \\ &\le C_2 - \frac{1}{p}\sum_{n\in\mathbb{Z}} \frac{\overline{W}(n, s_k u(n))}{s_k^p} - \frac{1}{p}\inf_{\mathbb{Z}} V_\infty(n)\|u\|_{l^p}^p. \end{aligned} \tag{3.10}$$

It follows from (3.7) that

$$\frac{\overline{W}(n, s_k u(n))}{s_k^p} \le \left(pR + \sup_{\mathbb{Z}} V_\infty(n)\right)|u(n)|^p, \quad \frac{\overline{W}(n, s_k u(n))}{|s_k|^p} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.11}$$

Hence, from Lebesgue's dominated theorem and (3.11), we have

$$\sum_{n\in\mathbb{Z}} \frac{\overline{W}(n, s_n u(n))}{|s_k|^p} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{3.12}$$

It follows from (3.8), (3.9), (3.10), and (3.12) that

$$0 \longleftarrow -\frac{C_3}{s_k^p} \le C_2 - \frac{1}{p\left(1 + \left|1 - e^{|n|-|n-1|}\right|^p\right)} \inf_{\mathbb{Z}} V_\infty(n) < 0 \quad \text{as } k \longrightarrow \infty, \tag{3.13}$$

which is a contradiction. Hence, there exists $e \in E$ with $\|e\| > \rho$ such that $\varphi(e) \le 0$.

*Step 3.* From Step 1, Step 2, and Lemma 2.5, we know that there is a sequence $\{u_k\}_{k\in\mathbb{N}} \subset E$ such that

$$\varphi(u_k) \longrightarrow c, \quad (1+\|u_k\|)\left\|\varphi'(u_k)\right\|_{E^*} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.14}$$

where $E^*$ is the dual space of $E$. In the following, we will prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $E$. Otherwise, assume that $\|u_k\| \to \infty$ as $k \to \infty$. Let $z_k = u_k/\|u_k\|$; we have $\|z_k\| = 1$. It follows from (2.5), (2.16), (3.14), and (K2) that

$$\begin{aligned}
C_4 &\geq p\varphi(u_k) - \langle \varphi'(u_k), u_k \rangle \\
&= \sum_{n\in\mathbb{Z}} \left[(\nabla W(n,u_k(n)), u_k(n)) - pW(n,u_k(n))\right] \\
&\quad + \sum_{n\in\mathbb{Z}} \left[pK(n,u_k(n)) - (\nabla K(n,u_k(n)), u_k(n))\right] \\
&\geq \sum_{n\in\mathbb{Z}} \left[(\nabla W(n,u_k(n)), u_k(n)) - pW(t,u_k(n))\right] := \sum_{n\in\mathbb{Z}} \widetilde{W}(n,u_k(n)).
\end{aligned} \tag{3.15}$$

Set $\Omega_k(\alpha,\beta) = \{n \in \mathbb{Z} : \alpha \leq |u_k(n)| \leq \beta\}$ for $0 < \alpha < \beta$. Then from (3.15), we have

$$C_4 \geq \sum_{n\in\Omega_k(0,\alpha)} \widetilde{W}(n,u_k(n)) + \sum_{n\in\Omega_k(\alpha,\beta)} \widetilde{W}(n,u_k(n)) + \sum_{n\in\Omega_k(\beta,+\infty)} \widetilde{W}(n,u_k(n)). \tag{3.16}$$

From (K1), (K2), and (3.14), we get

$$\begin{aligned}
o(1) &= \langle \varphi'(u_k), u_k \rangle \\
&= \sum_{n\in\mathbb{Z}} \left[|\Delta u_k(n-1)|^p + (\nabla K(n,u_k(n)) - \nabla W(n,u_k(n)), u_k(n))\right] \\
&\geq \sum_{n\in\mathbb{Z}} \left[|\Delta u_k(n-1)|^p + b_3|u_k(n)|^p - (\nabla W(n,u_k(n)), u_k(n))\right] \\
&\geq \min\{1,b_3\}\|u_k\|^p - \sum_{n\in\mathbb{Z}} (\nabla W(n,u_k(n)), u_k(n)) \\
&:= C_5\|u_k\|^p - \sum_{n\in\mathbb{Z}} (\nabla W(n,u_k(n)), u_k(n)) \\
&= \|u_k\|^p \left( C_5 - \sum_{n\in\mathbb{Z}} \frac{(\nabla W(n,u_k(n)), u_k(n))}{\|u_k\|^p} \right),
\end{aligned} \tag{3.17}$$

which implies that

$$\limsup_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{(\nabla W(n,u_k(n)), u_k(n))}{|u_k(n)|^p} |z_k(n)|^p = \limsup_{k\to\infty} \sum_{n\in\mathbb{Z}} \frac{(\nabla W(n,u_k(n)), u_k(n))}{\|u_k\|^p} \geq C_5. \tag{3.18}$$

Let $0 < \varepsilon < C_5/3$. From (W1), there exists $\alpha_\varepsilon > 0$ such that

$$|\nabla W(n,x)| \le \frac{\varepsilon}{2^p}|x|^{p-1} \quad \text{for } |x| \le \alpha_\varepsilon \text{ uniformly for } n \in \mathbb{Z}. \tag{3.19}$$

Since $\|z_k\| = 1$, it follows from (2.9) and (3.19) that

$$\sum_{n\in\Omega_k(0,\alpha_\varepsilon)} \frac{|\nabla W(n,u_k(n))|}{|u_k(n)|^{p-1}}|z_k(n)|^p \le \sum_{n\in\Omega_k(0,\alpha_\varepsilon)} \frac{\varepsilon}{2^p}|z_k(n)|^p \le \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.20}$$

For $s > 0$, let

$$h(s) := \inf\left\{\widetilde{W}(n,x) \mid n \in \mathbb{Z}, \quad x \in \mathbb{R}^N \text{ with } |x| \ge s\right\}. \tag{3.21}$$

Thus, from (W4), we have $h(s) \to +\infty$ as $s \to +\infty$, which together with (3.16) implies that

$$\operatorname{meas}(\Omega_k(\beta,+\infty)) \le \frac{C_6}{h(\beta)} \longrightarrow 0, \quad \text{as } \beta \longrightarrow +\infty. \tag{3.22}$$

Hence, we can take $\beta_\varepsilon$ sufficiently large such that

$$\sum_{n\in\Omega_k(\beta_\varepsilon,+\infty)} |z_k(n)|^p < \frac{\varepsilon}{R}. \tag{3.23}$$

The previous inequality and (W2) imply that

$$\sum_{n\in\Omega_k(\beta_\varepsilon,+\infty)} \frac{|\nabla W(n,u_k(n))|}{|u_k(n)|^{p-1}}|z_k(n)|^p \le R \sum_{n\in\Omega_k(\beta_\varepsilon,+\infty)} |z_k(n)|^p < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.24}$$

Next, for the previous $0 < \alpha_\varepsilon < \beta_\varepsilon$, let

$$\begin{aligned} c_\varepsilon &:= \inf\left\{\frac{\widetilde{W}(n,x)}{|x|^p} : n \in \mathbb{Z},\ x \in \mathbb{R}^N \text{ with } \alpha_\varepsilon \le |x| \le \beta_\varepsilon\right\}, \\ d_\varepsilon &:= \max\left\{\frac{|\nabla W(n,x)|}{|x|^{p-1}} : n \in \mathbb{Z},\ x \in \mathbb{R}^N \text{ with } \alpha_\varepsilon \le |x| \le \beta_\varepsilon\right\}. \end{aligned} \tag{3.25}$$

From (W4), we have $c_\varepsilon > 0$ and

$$\widetilde{W}(n,u_k(n)) \ge c_\varepsilon |u_k(n)|^p, \quad \forall n \in \Omega_k(\alpha_\varepsilon,\beta_\varepsilon). \tag{3.26}$$

From (3.15) and (3.26), we get

$$\begin{aligned}\sum_{n\in\Omega_k(\alpha_\varepsilon,\beta_\varepsilon)}|z_k(n)|^p &= \frac{1}{\|u_k\|^p}\sum_{n\in\Omega_k(\alpha_\varepsilon,\beta_\varepsilon)}|u_k(n)|^p \\ &\le \frac{1}{\|u_k\|^p}\sum_{n\in\Omega_k(\alpha_\varepsilon,\beta_\varepsilon)}\frac{1}{c_\varepsilon}\widetilde{W}(n,u_k(n)) \\ &\le \frac{C_4}{c_\varepsilon\|u_k\|^p}\longrightarrow 0 \quad \text{as } k\longrightarrow\infty,\end{aligned} \tag{3.27}$$

which implies that

$$\sum_{n\in\Omega_k(\alpha_\varepsilon,\beta_\varepsilon)}\frac{|\nabla W(n,u_k(n))|}{|u_k(n)|^{p-1}}|z_k(n)|^p \le d_\varepsilon\sum_{n\in\Omega_k(\alpha_\varepsilon,\beta_\varepsilon)}|z_k(n)|^p\longrightarrow 0 \quad \text{as } k\longrightarrow\infty. \tag{3.28}$$

Therefore, there exists $k_0>0$ such that

$$\sum_{n\in\Omega_k(\alpha_\varepsilon,\beta_\varepsilon)}\frac{|\nabla W(n,u_k(n))|}{|u_k(n)|^{p-1}}|z_k(n)|^p \le \varepsilon, \quad \forall k\ge k_0. \tag{3.29}$$

It follows from (3.20), (3.24), and (3.29) that

$$\sum_{n\in\mathbb{Z}}\frac{(\nabla W(n,u_k(n)),u_k(n))}{|u_k(n)|^p}|z_k(n)|^p \le \sum_{n\in\mathbb{Z}}\frac{|\nabla W(t,u_k(n))|}{|u_k(n)|^{p-1}}|z_k(n)|^p < 3\varepsilon < C_5, \quad \forall k\ge k_0, \tag{3.30}$$

which implies that

$$\limsup_{n\to\infty}\sum_{n\in\mathbb{Z}}\frac{(\nabla W(n,u_k(n)),u_k(n))}{|u_k(n)|^p}|z_k(n)|^p < C_5, \tag{3.31}$$

but this contradicts to (3.18). Hence, $\|u_k\|$ is bounded in $E$.

Going to a subsequence if necessary, we may assume that there exists $u\in E$ such that $u_k\rightharpoonup u$ as $k\to\infty$. In order to prove our theorem, it is sufficient to show that $\varphi'(u)=0$. For any $a\in\mathbb{Z}$ with $a>0$, let $\chi_a(t)=1$ for $t\in\mathbb{Z}[-a,a]$ and let $\chi_a(t)=0$ for $t\in\mathbb{Z}(-\infty,-a)\cup\mathbb{Z}(a,\infty)$. Then from (2.16), we have

$$\begin{aligned}&\langle\varphi'(u_k)-\varphi'(u),\chi_a(u_k-u)\rangle \\ &\quad= \sum_{n\in\mathbb{Z}[-a,a]}|\Delta u_k(n-1)|^{p-2}(\Delta u_k(n-1),\Delta u_k(n-1)-\Delta u(n-1)) \\ &\qquad- \sum_{n\in\mathbb{Z}[-a,a]}|\Delta u(n-1)|^{p-2}(\Delta u(n-1),\Delta u_k(n-1)-\Delta u(n-1)) \\ &\qquad+ \sum_{n\in\mathbb{Z}[-a,a]}(\nabla K(n,u_k(n))-\nabla K(n,u(n)),u_k(n)-u(n))\end{aligned}$$

$$
\begin{aligned}
&- \sum_{n\in\mathbb{Z}[-a,a]} (\nabla W(n,u_k(n)) - \nabla W(n,u(n)), u_k(n) - u(n)) \\
&\geq \|\Delta u_k\|^p_{l^p\mathbb{Z}[-a,a]} + \|\Delta u\|^p_{l^p\mathbb{Z}[-a,a]} - \sum_{n\in\mathbb{Z}[-a,a]} |\Delta u_k(n-1)|^{p-1}|\Delta u(n-1)| \\
&\quad - \sum_{n\in\mathbb{Z}[-a,a]} |\Delta u(n-1)|^{p-1}|\Delta u_k(n-1)| \\
&\quad + \sum_{n\in\mathbb{Z}[-a,a]} (\nabla K(n,u_k(n)) - \nabla K(n,u(n)), u_k(n) - u(n)) \\
&\quad - \sum_{n\in\mathbb{Z}[-a,a]} (\nabla W(n,u_k(n)) - \nabla W(n,u(n)), u_k(n) - u(n)) \\
&\geq \|\Delta u_k\|^p_{l^p\mathbb{Z}[-a,a]} + \|\Delta u\|^p_{l^p\mathbb{Z}[-a,a]} - \|\Delta u\|_{l^p\mathbb{Z}[-a,a]}\|\Delta u_k\|^{p-1}_{l^p\mathbb{Z}[-a,a]} - \|\Delta u_k\|_{l^p\mathbb{Z}[-a,a]}\|\Delta u\|^{p-1}_{l^p\mathbb{Z}[-a,a]} \\
&\quad + \sum_{n\in\mathbb{Z}[-a,a]} (\nabla K(n,u_k(n)) - \nabla K(n,u(n)), u_k(n) - u(n)) \\
&\quad - \sum_{n\in\mathbb{Z}[-a,a]} (\nabla W(n,u_k(n)) - \nabla W(n,u(n)), u_k(n) - u(n)) \\
&= \left(\|\Delta u_k\|^{p-1}_{l^p\mathbb{Z}[-a,a]} - \|\Delta u\|^{p-1}_{l^p\mathbb{Z}[-a,a]}\right)\left(\|\Delta u_k\|_{l^p\mathbb{Z}[-a,a]} - \|\Delta u\|_{l^p\mathbb{Z}[-a,a]}\right) \\
&\quad + \sum_{n\in\mathbb{Z}[-a,a]} (\nabla K(n,u_k(n)) - \nabla K(n,u(n)), u_k(n) - u(n)) \\
&\quad - \sum_{n\in\mathbb{Z}[-a,a]} (\nabla W(n,u_k(n)) - \nabla W(n,u(n)), u_k(n) - u(n)).
\end{aligned}
\tag{3.32}
$$

Since $\varphi'(u_k) \to 0$ as $k \to +\infty$ and $u_k \rightharpoonup u$ in $E$, it follows from (3.14) that

$$
\begin{gathered}
\left\langle \varphi'(u_k) - \varphi'(u), \chi_a(u_k - u)\right\rangle \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \\
\sum_{n\in\mathbb{Z}[-a,a]} (\nabla K(n,u_k(n)) - \nabla K(n,u(n)), u_k(n) - u(n)) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \\
\sum_{n\in\mathbb{Z}[-a,a]} (\nabla W(n,u_k(n)) - \nabla W(n,u(n)), u_k(n) - u(n)) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.
\end{gathered}
\tag{3.33}
$$

It follows from (3.32) and (3.33) that $\|\Delta u_k\|_{l^p\mathbb{Z}[-a,a]} \to \|\Delta u\|_{l^p\mathbb{Z}[-a,a]}$ as $k \to +\infty$.

For any $w \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)$, and assume that for some $A \in \mathbb{Z}$ with $A > 0$, $\operatorname{supp}(w) \subset \mathbb{Z}[-A, A]$. Since

$$
\begin{aligned}
&\lim_{k\to\infty} \Delta u_k(n-1) = \Delta u(n-1), \quad \forall \text{a.e. } n \in \mathbb{Z}, \\
&\left|\left(|\Delta u_k(n-1)|^{p-2}\Delta u_k(n-1), \Delta w(n-1)\right)\right|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{p-1}{p}|\Delta u_k(n-1)|^p + \frac{1}{p}|\Delta w(n-1)|^p, \quad \forall n \in \mathbb{Z},\ k = 1,2,\ldots, \\
\lim_{k\to\infty} \sum_{n\in\mathbb{Z}[-A,A]} &\left[\frac{p-1}{p}|\Delta u_k(n-1)|^p + \frac{1}{p}|\Delta w(n-1)|^p\right] \\
&= \frac{p-1}{p}\lim_{k\to\infty}\|\Delta u_k\|^p_{l^p\mathbb{Z}[-A,A]} + \frac{1}{p}\|\Delta w\|^p_{l^p\mathbb{Z}[-A,A]} \\
&= \frac{p-1}{p}\|\Delta u\|^p_{l^p\mathbb{Z}[-A,A]} + \frac{1}{p}\|\Delta w\|^p_{l^p\mathbb{Z}[-A,A]} \\
&= \sum_{n\in\mathbb{Z}[-A,A]}\left[\frac{p-1}{p}|\Delta u(n-1)|^p + \frac{1}{p}|\Delta w(n-1)|^p\right] < +\infty,
\end{aligned} \tag{3.34}$$

then, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}[-A,A]}\Big(|\Delta u_k(n-1)|^{p-2}\Delta u_k(n-1), \Delta w(n-1)\Big) \\
&\quad\longrightarrow \sum_{n\in\mathbb{Z}[-A,A]}\Big(|\Delta u(n-1)|^{p-2}\Delta u(n-1), \Delta w(n-1)\Big)
\end{aligned} \tag{3.35}$$

as $k \to \infty$. Noting that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}[-A,A]}(\nabla K(n,u_k(n)), w(n)) &\longrightarrow \sum_{n\in\mathbb{Z}[-A,A]}(\nabla K(n,u(n)), w(n)) \quad \text{as } k \longrightarrow \infty, \\
\sum_{n\in\mathbb{Z}[-A,A]}(\nabla W(n,u_k(n)), w(n)) &\longrightarrow \sum_{n\in\mathbb{Z}[-A,A]}(\nabla W(n,u(n)), w(n)) \quad \text{as } k \longrightarrow \infty.
\end{aligned} \tag{3.36}$$

Hence, we have

$$\langle \varphi'(u), w\rangle = \lim_{k\to\infty}\langle \varphi'(u_k), w\rangle = 0, \tag{3.37}$$

which implies that $\varphi'(u) = 0$; that is, $u$ is a critical point of $\varphi$. From (K1) and (W1), we know that $u \neq 0$. In fact, if $u = 0$, we have from (2.5), (K1), and (W1) that $\varphi(u) = 0$. On the other hand, from Step 1, Step 2, and Lemma 2.5, we know that $\varphi(u) = c > 0$. This is a contradiction. The proof of Theorem 1.1 is complete. □

## 4. An Example

*Example 4.1.* In problem (1.1), let $p = 3/2$, and

$$K(n,x) = \left(1 + \frac{1}{|x|^{3/2}+1}\right)|x|^{3/2}, \qquad W(n,x) = a(n)|x|^{3/2}\left(1 - \frac{1}{(\ln(e+|x|))^{1/2}}\right), \tag{4.1}$$

where $a \in l^{\infty}(\mathbb{Z}, \mathbb{R}^{+})$ with $\inf_{\mathbb{Z}} a(n) > 3$. One can easily check that $K$ satisfies conditions (K1) and (K2) with $b_1 = 1$, $b_2 = 2$, and $b_3 = 3/2$. An easy computation shows that

$$\nabla W(n,x) = \frac{3}{2}a(n)|x|^{-1/2}x\left(1 - \frac{1}{(\ln(e+|x|))^{1/2}}\right) + \frac{a(n)|x|^{1/2}x}{2(e+|x|)(\ln(e+|x|))^{3/2}},$$
$$(\nabla W(n,x), x) - \frac{3}{2}W(n,x) = \frac{a(n)|x|^{5/2}}{2(e+|x|)(\ln(e+|x|))^{3/2}}. \tag{4.2}$$

Then it is easy to check that $W$ satisfies (W1)–(W4). Hence, $K(n,x)$ and $W(n,x)$ satisfy all the conditions of Theorem 1.1 and then problem (1.1) has at least one nontrivial homoclinic solution.

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## References

[1] H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, Gauthier-Villars, Pairs, France, 1899.
[2] M. Izydorek and J. Janczewska, "Homoclinic solutions for a class of the second order Hamiltonian systems," *Journal of Differential Equations*, vol. 219, no. 2, pp. 375–389, 2005.
[3] X. H. Tang and L. Xiao, "Homoclinic solutions for a class of second-order Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 1140–1152, 2009.
[4] L. L. Wan and C. L. Tang, "Homoclinic orbits for a class of the second order Hamiltonian systems," *Acta Mathematica Scientia Series B*, vol. 30, no. 1, pp. 312–318, 2010.
[5] Q. F. Zhang and X. H. Tang, "Existence of homoclinic solutions for a class of second-order non-autonomous Hamiltonian systems ," *Mathematica Slovaca*. In press.
[6] R. P. Agarwal, *Difference Equations and Inequalities*, vol. 228 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.
[7] C. D. Ahlbrandt and A. C. Peterson, *Discrete Hamiltonian Systems*, vol. 16 of *Kluwer Texts in the Mathematical Sciences*, Kluwer Academic Publishers Group, Dordrecht, 1996, Difference equations, continued fractions, and Riccati equation.
[8] S. N. Elaydi, *An Introduction to Difference Equations*, Undergraduate Texts in Mathematics, Springer, New York, NY, USA, 2nd edition, 1999.
[9] R. P. Agarwal and J. Popenda, "Periodic solutions of first order linear difference equations," *Mathematical and Computer Modelling*, vol. 22, no. 1, pp. 11–19, 1995.
[10] R. P. Agarwal, K. Perera, and D. O'Regan, "Multiple positive solutions of singular discrete $p$-Laplacian problems via variational methods," *Advances in Difference Equations*, no. 2, pp. 93–99, 2005.
[11] Z. M. Guo and J. S. Yu, "Existence of periodic and subharmonic solutions for second-order superlinear difference equations," *Science in China Series A*, vol. 46, no. 4, pp. 506–515, 2003.
[12] Z. M. Guo and J. S. Yu, "Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 7-8, pp. 969–983, 2003.
[13] Z. M. Guo and J. S. Yu, "The existence of periodic and subharmonic solutions of subquadratic second order difference equations," *Journal of the London Mathematical Society*, vol. 68, no. 2, pp. 419–430, 2003.
[14] H. H. Liang and P. X. Weng, "Existence and multiple solutions for a second-order difference boundary value problem via critical point theory," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 511–520, 2007.
[15] J. Rodriguez and D. L. Etheridge, "Periodic solutions of nonlinear second-order difference equations," *Advances in Difference Equations*, no. 2, pp. 173–192, 2005.

[16] Y.-F. Xue and C.-L. Tang, "Existence of a periodic solution for subquadratic second-order discrete Hamiltonian system," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 7, pp. 2072–2080, 2007.

[17] J. S. Yu, Z. M. Guo, and X. Zou, "Periodic solutions of second order self-adjoint difference equations," *Journal of the London Mathematical Society*, vol. 71, no. 2, pp. 146–160, 2005.

[18] J. S. Yu, Y. H. Long, and Z. M. Guo, "Subharmonic solutions with prescribed minimal period of a discrete forced pendulum equation," *Journal of Dynamics and Differential Equations*, vol. 16, no. 2, pp. 575–586, 2004.

[19] J. S. Yu, X. Q. Deng, and Z. M. Guo, "Periodic solutions of a discrete Hamiltonian system with a change of sign in the potential," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 2, pp. 1140–1151, 2006.

[20] Z. M. Zhou, J. S. Yu, and Z. M. Guo, "Periodic solutions of higher-dimensional discrete systems," *Proceedings of the Royal Society of Edinburgh*, vol. 134, no. 5, pp. 1013–1022, 2004.

[21] P. Chen and X. H. Tang, "Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation," *Applied Mathematics and Computation*, vol. 217, no. 9, pp. 4408–4415, 2011.

[22] P. Chen and X. H. Tang, "Existence of homoclinic solutions for a class of nonlinear difference equations," *Advances in Difference Equations*, Article ID 470375, 19 pages, 2010.

[23] P. Chen and X. Tang, "Existence of homoclinic orbits for 2nth-order nonlinear difference equations containing both many advances and retardations," *Journal of Mathematical Analysis and Applications*, vol. 381, no. 2, pp. 485–505, 2011.

[24] H. Fang and D. P. Zhao, "Existence of nontrivial homoclinic orbits for fourth-order difference equations," *Applied Mathematics and Computation*, vol. 214, no. 1, pp. 163–170, 2009.

[25] X. Lin and X. H. Tang, "Existence of infinitely many homoclinic orbits in discrete Hamiltonian systems," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 1, pp. 59–72, 2011.

[26] M. Ma and Z. M. Guo, "Homoclinic orbits and subharmonics for nonlinear second order difference equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 6, pp. 1737–1745, 2007.

[27] M. Ma and Z. M. Guo, "Homoclinic orbits for second order self-adjoint difference equations," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 513–521, 2006.

[28] J. S. Yu, H. P. Shi, and Z. M. Guo, "Homoclinic orbits for nonlinear difference equations containing both advance and retardation," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 799–806, 2009.

[29] H. L. Royden, *Real Analysis*, The Macmillan, New York, NY, USA, 2nd edition, 1968.

[30] G. Cerami, "An existence criterion for the critical points on unbounded manifolds," *Istituto Lombardo Accademia di Scienze e Lettere*, vol. 112, no. 2, pp. 332–336, 1978.