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Research Article

Lagrangian Stability of a Class of Second-Order Periodic Systems

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We study the following second-order periodic system: $x'' + V'(x) + p(t) = 0$ where $V(x)$ has a singularity and $p(t) = p(t+1)$. Under some assumptions on the $V(x)$ and $p(t)$, by Moser's twist theorem we obtain the existence of quasiperiodic solutions and boundedness of all the solutions.

1. Introduction and Main Result

In the early 1960s, Littlewood [1] asked whether or not the solutions of the Duffing-type equations

$$x'' + g(x) = e(t), \quad \text{where } e(t+1) = e(t) \tag{1.1}$$

are bounded for all time, that is, whether there are resonances that might cause the amplitude of the oscillations to increase without bound. Littlewood suggested studying the following two cases:

(i) superlinear case: $g(x)/x \to +\infty$ as $x \to \pm\infty$,

(ii) sublinear case: $\operatorname{sign}(x) \cdot x \to +\infty$ and $g(x)/x \to 0$ as $x \to \pm\infty$.

The first positive result of boundedness of solutions in the superlinear case (i) was due to Morris [2]. By means of KAM theorem, Morris proved that every solution of the second-order system (1.1) is bounded if $g(x) = 2x^3$ and $e(t)$ is piecewise continuous and periodic. This result relies on the fact that the nonlinearity $2x^3$ can guarantee the twist condition of KAM theorem. Later, several authors (see [3–5]) improved the Morris's result and obtained similar results for a large class of superlinear function $g(x)$.

In 1999, the first result in the sublinear case was proved by Küpper and You [6] in the study of

$$x'' + |x|^{\alpha-1}x = e(t), \tag{1.2}$$

where $0 < \alpha < 1$ and $e(t) \in C^\infty(T)$. The authors transform (1.2) into a perturbation of an integrable Hamiltonian system and then prove that the Poincaré map of the transformed system is close to a so-called twist map. So, the Moser's twist theorem guarantees the boundedness of all solutions of (1.2). The general sublinear case was considered by Liu [7] under certain reasonable conditions.

The Littlewood problem for singular potentials is known to be challenging, and there are only very a few results. Recently, Capietto et al. [8] studied

$$x'' + V'(x) = p(t), \tag{1.3}$$

with $p(t)$ is a π-periodic function and $V = (1/2)x_+^2 + (1/(1-x_-^2)^\nu) - 1$, where $x_+ = \max\{x, 0\}$, $x_- = \max\{-x, 0\}$ and $\nu > 2$ is a positive integer. Under the Lazer-Leach assumption that

$$1 + \frac{1}{2}\int_0^\pi p(t_0 + \theta)\sin\theta\, d\theta > 0, \qquad \forall t_0 \in R, \tag{1.4}$$

they prove the boundedness of solutions and the existence of quasiperiodic solution by Moser' twist theorem. It is the first time that the equation of the boundedness of all solution is treated in case of a singular potential.

In this paper, We consider the following sublinearly growing potential:

$$x'' + V'(x) = p(t), \tag{1.5}$$

where $V(x) = x_+^{\alpha+1} + (1/1 - x_-^2) - 1,\ 0 < \alpha < 1$.

Our main result is the following theorem.

Theorem 1.1. *If $p(t) \in C^6$ is 1-periodic continuous, then all the solutions of* (1.5) *are bounded.*

The idea for proving the boundedness of solutions of (1.5) is as follows. By means of transformation theory, (1.5) is, outside of a large disc $\mathfrak{D} = \{(x, \dot{x}) \in \mathbb{R}^2 : x^2 + \dot{x}^2 \le r^2\}$ in the $(x, \dot{x})$-plane, transformed into a perturbation of an integrable Hamiltonian system. Then, Poincaré map of the transformed system is close to a so-called twist map in $\mathbb{R}\setminus\mathfrak{D}$. The Moser's twist theorem [9] guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the $(x, \dot{x})$-plane. Every such curves is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, \dot{x}, t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in the interior and which leads to a bound of these solutions.

The paper is organized as follows. In Sections 2.1 and 2.2, we give action-angle variables and some estimates which is useful for our proof. In Section 2.3, we will give an asymptotic expression of the Poincaré map and prove the main result by Moser's twist theorem [9].

2. Proof of Theorem

2.1. Action-Angle Variables and Some Estimates

Without loss of generality and for brevity of arguments, we assume that the average value of $p(t)$ vanishes; that is, $\int_0^1 p(t)dt = 0$. Hence the function $P(t) = \int_0^s p(s)ds$ is also 1-periodic in t and is in C^6.

System (1.5) is equivalent to the planar Hamiltonian system

$$x' = y + P(t), \qquad y' = -V'(x), \tag{2.1}$$

where Hamiltonian is $\mathscr{H}(x, y, t) = (1/2)y^2 + V(x) + yP(t)$.

In order to introduce action and angle variables, we first consider the auxiliary autonomous system

$$x' = y, \qquad y' = -V'(x), \tag{2.2}$$

which is integrable with the Hamiltonian

$$H_0(x, y) = \frac{1}{2}y^2 + V(x). \tag{2.3}$$

The closed curves $H_0(x, y) = h > 0$ are just the integral curves of (2.2). Denote by $T_0(h)$ the time period of the integral curve Γ_h: $H_0(x, y) = h$ and by I_0 the area enclosed by the closed curve Γ_h. Let

$$\alpha_h = \sqrt{\frac{h}{h+1}}, \qquad \beta_h = h^{1/(1+\alpha)}. \tag{2.4}$$

Then, $V(-\alpha_h) = V(\beta_h) = h$.

It is easy to see that

$$\begin{aligned} I_0(h) &= 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h - V(s))}ds, \quad \forall h > 0, \\ T_0(h) = I_0'(h) &= 2\int_{-\alpha_h}^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}}ds, \quad \forall h > 0. \end{aligned} \tag{2.5}$$

Denote

$$\begin{aligned} I_+(h) &= 2\int_0^{\beta_h} \sqrt{2(h - V(s))}ds, \qquad I_-(h) = 2\int_0^{\alpha_h} \sqrt{2(h - V(-s))}ds, \\ T_+(h) &= 2\int_0^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}}ds, \qquad T_-(h) = 2\int_0^{\alpha_h} \frac{1}{\sqrt{2(h - V(-s))}}ds. \end{aligned} \tag{2.6}$$

Then,

$$I_0(h) = I_-(h) + I_+(h), \qquad T_0(h) = T_-(h) + T_+(h). \tag{2.7}$$

The following estimates on the functions I_+, I_-, and I_0 and T_+, T_-, and T_0 are crucial for this paper. We first estimate I_+ and T_+. Since I_+ is the area enclosed by the closed curve Γ_h and y-axis when $x \geq 0$, we can easily prove that

$$I_+(h) = 2\int_0^{\beta_h} \sqrt{2(h - x^{\alpha+1})}\,dx = 2\int_0^{h^{1/(1+\alpha)}} \sqrt{2(h - x^{\alpha+1})}\,dx. \tag{2.8}$$

Let $x = th^{1/(1+\alpha)}$, then we get

$$I_+(h) = 2\int_0^1 \sqrt{2h(1 - t^{1+\alpha})}\,h^{1/(1+\alpha)}dt = 2\sqrt{2}h^{(1/2)+1/(1+\alpha)} \int_0^1 \sqrt{1 - t^{1+\alpha}}\,dt. \tag{2.9}$$

Since $T_+(h) = I_+'(h)$, we have

$$T_+(h) = 2\sqrt{2}\left(\frac{1}{2} + \frac{1}{\alpha+1}\right)h^{-(1/2)+1/(1+\alpha)} \int_0^1 \sqrt{1 - t^{1+\alpha}}\,dt. \tag{2.10}$$

We now give the estimates on the function I_- and T_-.

Lemma 2.1. *One has*

$$h^n\left|\frac{d^n T_-(h)}{dh^n}\right| \leq Ch^{-1/2},$$
$$h^n\left|\frac{d^n I_-(h)}{dh^n}\right| \leq Ch^{1/2}, \tag{2.11}$$

where $n = 0, 1, \ldots, 6$, $h \to +\infty$. *Note that here and below, one always uses* C, C_0, *or* C_0' *to indicate some constants.*

Proof. Now, we estimate the first inequality. We choose $V(s)/h = \eta$ as the new variable of integration, then we have

$$T_-(h) = \int_{-\alpha_h}^0 \frac{1}{\sqrt{2(h - V(s))}}\,ds = \int_0^1 \frac{\sqrt{h}}{V'(s(\eta, h))} \frac{1}{\sqrt{2(1-\eta)}}\,d\eta. \tag{2.12}$$

Since $V(s) = (1/(1 - s^2)) - 1$ and $V(s)/h = \eta$, we have $s = \sqrt{\eta h/(1 + \eta h)}$. By direct computation, we have

$$V'(s) = \frac{2s}{(1 - s^2)^2} = \frac{2\sqrt{\eta h}(1 + \eta h)^2}{\sqrt{1 + \eta h}}, \tag{2.13}$$

then we get

$$T_-^{(n)}(h) = \frac{(-3/2)!}{(-(3/2)-n)!}\int_0^1 \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta, \quad n = 0, 1, \ldots, 6. \tag{2.14}$$

When $0 \le \eta \le h^{-1}$ and h sufficient large, there exits C_0 such that $1-\eta > C_0$, so we have

$$\begin{aligned}\int_0^{h^{-1}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta &\le C\int_0^{h^{-1}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}}d\eta \\ &\le \frac{C}{C_0}\int_0^{h^{-1}} \eta^{n-(1/2)}d\eta \le Ch^{-(1/2)-n}.\end{aligned} \tag{2.15}$$

Since $h^{-2/3} \le \eta \le 1$, we have

$$h^{1/3} < 1 + h^{1/3} \le 1 + \eta h \le 1 + h, \tag{2.16}$$

then

$$\begin{aligned}\int_{h^{-2/3}}^1 \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta &\le C\int_{h^{-2/3}}^1 \frac{\eta^n h^n}{\sqrt{2\eta(1-\eta)}h^n(1+\eta h)^n(1+\eta h)^{3/2}}d\eta \\ &\le C\int_{h^{-2/3}}^1 \frac{1}{\sqrt{2\eta(1-\eta)}h^n(1+\eta h)^{3/2}}d\eta \\ &\le C\int_{h^{-2/3}}^1 \frac{1}{\sqrt{2\eta(1-\eta)}h^n h^{1/2}}d\eta \\ &\le Ch^{-(1/2)-n}\int_0^1 \frac{1}{\sqrt{2\eta(1-\eta)}}d\eta \le Ch^{-(1/2)-n}.\end{aligned} \tag{2.17}$$

Observing that there is $C_0 > 0$ such that $\sqrt{1-\eta} \ge C_0$ when $h^{-1} \le \eta \le h^{-2/3}$ and $h \to +\infty$, we have

$$\begin{aligned}\int_{h^{-1}}^{h^{-2/3}} \frac{\eta^n}{\sqrt{2\eta(1-\eta)}(1+\eta h)^{(3/2)+n}}d\eta &\le C_1 h^{-(3/2)-n}\int_{h^{-1}}^{h^{-2/3}} \frac{1}{\sqrt{2\eta(1-\eta)}\eta^{3/2}}d\eta \\ &\le \frac{C_1}{C_0}h^{-(3/2)-n}\int_{h^{-1}}^{h^{-2/3}} \frac{1}{\eta^2}d\eta = \frac{C_1}{C_0}h^{-(3/2)-n}\frac{1}{\eta}\Big|_{h_{-1}}^{h^{-2/3}} \\ &= \frac{C_1}{C_0}h^{-(3/2)-n}\left(h - h^{2/3}\right) \le Ch^{-(1/2)-n}.\end{aligned} \tag{2.18}$$

By (2.15)–(2.18), we have $T_-^{(n)}(h) \le Ch^{-(1/2)-n}$, $n = 0, 1, \ldots, 6$.

The proof of the second inequality is similar to the first one, so we only give the brief proof.

We choose $V(s)/h = \eta$ as the new variable of integration, so we have

$$\frac{\partial s}{\partial h} = \frac{\eta}{V'}, \quad s = \sqrt{\frac{\eta h}{1+\eta h}},$$
$$V'(s) = \frac{2s}{(1-s^2)^2} = \frac{2\sqrt{\eta h}(1+\eta h)^2}{\sqrt{1+\eta h}}. \tag{2.19}$$

By direct computation, we have

$$I_-(h) = 2\int_{-\alpha_h}^{0} \sqrt{2(h-V(s))}\,ds = h\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}(1+\eta h)^{3/2}}\,d\eta. \tag{2.20}$$

By (2.20), we can easily get

$$I_-^{(n)}(h) = I_{-1}^{(n)}(h) + I_{-2}^{(n)}(h) = n\frac{(-3/2)!}{(-(3/2)-n+1)!}\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}}\frac{\eta^{n-1}}{(1+\eta h)^{(3/2)+n-1}}\,d\eta$$
$$+ \frac{(-3/2)!}{(-(3/2)-n)!}h\int_0^1 \frac{\sqrt{2(1-\eta)}}{\sqrt{\eta}}\frac{\eta^{n}}{(1+\eta h)^{(3/2)+n}}\,d\eta, \tag{2.21}$$

where $n = 0, 1, \ldots, 6$.

By the similar way in estimating $T_-^{(n)}(h)$, we get

$$I_{-1}^{(n)}(h) \le Ch^{(1/2)-n}, \qquad I_{-2}^{(n)}(h) \le Ch^{(1/2)-n}, \tag{2.22}$$

which means that

$$I_-^{(n)}(h) \le Ch^{(1/2)-n}, \quad n = 0, 1, \ldots, 6. \tag{2.23}$$

Thus, we complete the proof of Lemma 2.1. □

Remark 2.2. It follows from (2.10) and Lemma 2.1 that

$$\lim_{h\to+\infty} T_-(h) = 0, \qquad \lim_{h\to+\infty} T_+(h) = +\infty. \tag{2.24}$$

Thus, the time period $T_0(h)$ is dominated by $T_+(h)$ when h is sufficiently large. By $T_0(h) = I_0'(h)$, we know $I_0(h)$ is dominated by $I_+(h)$ when h is sufficiently large.

Remark 2.3. It also follows from the definition of $I_0(h)$, $I_-(h)$, $I_+(h)$ and Remark 2.2 that

$$\left|h^n \frac{d^n I_0(h)}{dh^n}\right| \leq C_0 I_0(h), \quad \text{for } n \geq 1. \tag{2.25}$$

In particular,

$$ch^{(1/2)+1/(1+\alpha)} \leq I_0(h) \leq Ch^{(1/2)+1/(1+\alpha)}. \tag{2.26}$$

Remark 2.4. Note that $h = h_0(I_0)$ is the inverse function of I_0. By Remark 2.3, we have

$$\left|I^n \frac{d^n h(I)}{dI^n}\right| \leq C_0 h(I), \quad \text{for } n \geq 1. \tag{2.27}$$

We now carry out the standard reduction to the action-angle variables. For this purpose, we define the generating function $S(x, I) = \int_\Gamma \sqrt{2(h - V(s))}ds$, where Γ is the part of the the closed curve Γ_h connecting the point on the y-axis and point (x, y).

We define the well-known map $(\theta, I) \to (x, y)$ by

$$y = \frac{\partial S}{\partial x}(x, I), \qquad \theta = \frac{\partial S}{\partial I}(x, I). \tag{2.28}$$

It is well-known that the map is symplectic, since

$$\begin{aligned} dx \wedge dy &= dx \wedge (S_{xx}dx + S_{xI}dI) = S_{xI}dx \wedge dI, \\ d\theta \wedge dI &= (S_{Ix}dx + S_{II}dI) \wedge dI = S_{Ix}d \wedge dI. \end{aligned} \tag{2.29}$$

From the above discussion, we can easily get

$$\theta = \begin{cases} \dfrac{1}{T_0(h(x,y))}\left(\displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}}ds\right), & \text{if } y > 0, \\ 1 - \dfrac{1}{T_0(h(x,y))}\left(\displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}}ds\right), & \text{if } y < 0. \end{cases} \tag{2.30}$$

$$I(x, y) = I_0(h(x, y)) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h(x, y) - V(s))}ds. \tag{2.31}$$

In the new variables (θ, I), the system (2.1) becomes

$$\theta' = \frac{\partial H}{\partial I}, \qquad I' = -\frac{\partial H}{\partial \theta}, \tag{2.32}$$

where

$$H(\theta, I, t) = h_0(I) + H_1(I, \theta, t), \tag{2.33}$$

where $h_0(I)$ is the inverse function of $I_0(H)$ and $H_1(I,\theta,t) = y(I,\theta)P(t)$.

In order to estimate $H_1(I,\theta,t)$, we need the estimate on the functions $x(I,\theta)$ and $y(I,\theta)$. For this purpose, we first give some definitions which are very similar to those in [4].

Define a function L in terms of h and V by

$$L(x, I) = -\frac{h_{II}}{h_I} + \frac{h_I}{h}\left(W' - \frac{1}{2}\right), \tag{2.34}$$

and a linear differential operator acting on functions of x, I, according to

$$\mathcal{L}(f) = \frac{h_I}{h}\left\{\left(f\frac{V}{V_x}\right)_x - \frac{1}{2}f\right\} + f_I, \tag{2.35}$$

where $f(x, I)$ is a smooth function, and we denote $\mathcal{L}^n = \underbrace{\mathcal{L} \circ \cdots \circ \mathcal{L}}_{n}$.

The following equality (its proof can be found in [4]) is crucial for the proof of the following lemmas:

$$\begin{aligned}\frac{d}{dI}\int_{-\alpha_h}^{x} f(s, I)\frac{1}{\sqrt{h - V(s)}}ds &= \int_{-\alpha_h}^{x} \mathcal{L}(f)\frac{1}{\sqrt{h - V(s)}}ds \\ &\quad + f(x, I)\int_{-\alpha_h}^{x} L(s, I)\frac{1}{\sqrt{h - V(s)}}ds.\end{aligned} \tag{2.36}$$

Before giving the estimates on $x(I,\theta)$ and $y(I,\theta)$, we now prove some lemmas which will be used frequently in the following proof.

Lemma 2.5. *Suppose that there is a constant C_0 such that $|g(x, I)| \le C_0 I^{-k}$, then one can find a constant C_0' and C_1' such that, for $-\alpha_h \le x \le 0$,*

$$\left|V'(x)\int_{-\alpha_h}^{x} \frac{g(s, I)}{\sqrt{h - V(s)}}ds\right| \le C_0' I^{-k}\sqrt{h - V(x)}, \tag{2.37}$$

$$\left|\sqrt{h - V(x)}\int_{-\alpha_h}^{x} \frac{g(s, I)}{\sqrt{h - V(x)}}\right| \le C_1' I^{-k}(\alpha_h + x). \tag{2.38}$$

Proof. We now prove (2.37). Let

$$G(x, I) = I^{-k}\frac{\sqrt{h - V(x)}}{V'(x)}, \qquad F(x, I) = \int_{-\alpha_h}^{x} g(s, I)\frac{1}{\sqrt{h - V(s)}}ds, \tag{2.39}$$

then

$$F(-\alpha_h, I) = \lim_{x \to -\alpha_h} G(x, I) = 0. \tag{2.40}$$

By direct calculation,

$$\begin{gathered}
\left|\frac{\partial F(x,I)}{\partial x}\right| = \left| g(x,I)\frac{1}{\sqrt{h-V(x)}}\right| \le C_0 I^{-k}\frac{1}{\sqrt{h-V(x)}}, \\
\frac{\partial G(x,I)}{\partial x} = I^{-k}\left(\frac{(V'(x))^2 + (h-V(x))V''(x)}{(V'(x))^2}\right)\frac{1}{\sqrt{h-V(x)}}.
\end{gathered} \tag{2.41}$$

Since

$$V''(x) = \frac{(1-x^2)^2 + 4x^2(1-x^2)}{(1-x^2)^4} > 0, \tag{2.42}$$

we have

$$\frac{(V'(x))^2 + (h-V(x))V''(x)}{(V'(x))^2} > 0. \tag{2.43}$$

By $|g(x,I)| \le C_0 I^{-k}$, there is $C_0' = 2C_0$ such that

$$\begin{aligned}
-C'_0 I^{-k}&\left(\frac{(V'(x))^2 + (h-V(x))V''(x)}{(V'(x))^2}\right)\frac{1}{\sqrt{(h-V(x))}} \\
&\le \frac{\partial F(x,I)/\partial x}{\sqrt{2h-V(x)}} \\
&\le C_0' I^{-k}\left(\frac{(V'(x))^2 + (h-V(x))V''(x)}{(V'(x))^2}\right)\frac{1}{\sqrt{h-V(x)}},
\end{aligned} \tag{2.44}$$

that is

$$-C_0'\frac{\partial G(x,I)}{\partial x} \le \frac{\partial F(x,I)}{\partial x} \le C_0'\frac{\partial G(x,I)}{\partial x}, \tag{2.45}$$

which means that

$$-C_0' G(x,I) \le F(x,I) \le C_0' G(x,I). \tag{2.46}$$

That is,

$$\left| V'(x) \int_{-\alpha_h}^{x} g(s,I) \frac{1}{\sqrt{h - V(x)}} \right| \leq C I^{-k} \sqrt{h - V(x)}. \tag{2.47}$$

Thus, we complete the proof (2.37).

Now, we prove (2.38). Let

$$G(x,I) = I^{-k} \frac{(\alpha_h + x)}{\sqrt{h - V(x)}}, \qquad F(x,I) = \int_{-\alpha_h}^{x} g(s,I) \frac{1}{\sqrt{h - V(s)}} ds. \tag{2.48}$$

Then, we have

$$F(-\alpha_h, I) = \lim_{x \to -\alpha_h} G(x,I) = 0. \tag{2.49}$$

By direct computation, we have

$$\begin{aligned} \left| \frac{\partial F(x,I)}{\partial x} \right| &= \left| g(x,I) \frac{1}{\sqrt{(h - V(x))}} \right| \leq C_0 I^{-k} \frac{1}{\sqrt{h - V(x)}}, \\ \frac{\partial G(x,I)}{\partial x} &= I^{-k} \left(1 + \frac{V'(x)(\alpha_h + x)}{2(h - V(x))} \right) \frac{1}{\sqrt{h - V(x)}}. \end{aligned} \tag{2.50}$$

Since $V''(x) > 0$ and $h = V(-\alpha_h + x)$, it follows that for

$$\left| \frac{V'(x)(\alpha_h)}{2(h - V(x))} \right| \leq \frac{1}{2}, \tag{2.51}$$

so for $C_1' > 2C_0 + 1$, we have

$$-C_1' \frac{\partial G(x,I)}{\partial x} \leq \frac{\partial F(x,I)}{\partial x} \leq C_1' \frac{\partial G(x,I)}{\partial x}, \tag{2.52}$$

which means that

$$-C_1' G(x,I) \leq F(x,I) \leq C_1' G(x,I). \tag{2.53}$$

By the definition of $G(x,I)$, we have

$$\left| \sqrt{h - V(x)} F(x,I) \right| \leq C_1' I^{-k} (\alpha_h + x). \tag{2.54}$$

Thus, we complete the proof of (2.38) and Lemma 2.5. □

By Lemma 2.5, we have the following Lemma which is important to our estimation.

Lemma 2.6. *One can find a constant C such that, for* $-\alpha_h \leq x \leq 0$*,*

$$\left|\partial_I^k\left(\int_{-\alpha_h}^{x}\frac{L(x,I)}{\sqrt{h-V(s)}}ds\right)\right| \leq \frac{C}{V'(x)}I^{-(k+1)}\sqrt{h-V(x)}, \tag{2.55}$$

$$\left|\partial_I^k\left(\int_{-\alpha_h}^{x}\frac{L(x,I)}{\sqrt{h-V(s)}}ds\right)\right| \leq \frac{C}{\sqrt{h-V(x)}}I^{-(k+1)}(\alpha_h+x), \tag{2.56}$$

where $0 \leq k \leq 6$*.*

Proof. When $k = 1$, we have

$$\partial_I\left(\int_{-\alpha_h}^{x}L\frac{1}{\sqrt{h-V(s)}}ds\right) = \int_{-\alpha_h}^{x}\mathcal{L}(L)\frac{1}{\sqrt{h-V(s)}}ds + L\int_{-\alpha_h}^{x}L\frac{1}{\sqrt{h-V(s)}}ds. \tag{2.57}$$

By the definitions of L and $\mathcal{L}$, we have $|L| \leq CI^{-1}$ and $|\mathcal{L}(L)| \leq CI^{-2}$. By (2.37), we obtain

$$\int_{-\alpha_h}^{x}L\frac{1}{\sqrt{h-V(s)}}ds \leq \frac{C}{V'}I^{-2}\sqrt{h-V}. \tag{2.58}$$

Suppose that $k = l$, we have

$$\partial_I^l\left(\int_{-\alpha_h}^{x}L\frac{1}{\sqrt{h-V(s)}}ds\right) \leq \frac{C}{V'}I^{-(l+1)}\sqrt{h-V}. \tag{2.59}$$

We now proven that when $k = l+1$,

$$\partial_I^{l+1}\left(\int_{-\alpha_h}^{x}L\frac{1}{\sqrt{h-V(s)}}ds\right) \leq \frac{C}{V'}I^{-(l+2)}\sqrt{h-V}. \tag{2.60}$$

By direct computation, we have

$$\partial_I^{l+1}\left(\int_{-\alpha_h}^{x}\frac{L}{\sqrt{h-V(s)}}ds\right) = \left\{\partial_I^n(\mathcal{L}^m(f))\partial_I^{l-m-n}\left(\int_{-\alpha_h}^{x}\frac{L}{\sqrt{h-V(s)}}ds\right), \int_{-\alpha_h}^{x}\frac{\mathcal{L}^{l+1}(L)}{\sqrt{h-V(s)}}ds\right\}, \tag{2.61}$$

where $\{f_1, f_2, \ldots, f_n\}$ denotes linear combination of functions $f_1, \ldots, f_n$ with integer coefficients and $0 \leq m+n \leq l$.

Since $|\mathcal{L}(L)| \leq CI^{-2}$, we have $\mathcal{L}^{l+1}(L) \leq CI^{-(l+2)}$. By (2.37), we obtain

$$\left|\int_{-\alpha_h}^{x} \frac{\mathcal{L}^{l+1}(L)}{\sqrt{h-V(s)}} ds\right| \leq \frac{C}{V'} I^{-(l+2)} \sqrt{h-V}. \tag{2.62}$$

By direct computation, we get

$$\partial^n(\mathcal{L}^m(L)) \leq CI^{-(m+n+1)}. \tag{2.63}$$

By assumption (2.59), we have

$$\left|\partial_I^{l-m-n}\left(\int_{-\alpha_h}^{x} \frac{L}{\sqrt{h-V(s)}} ds\right)\right| \leq \frac{C}{V'} I^{-(l-m-n+1)} \sqrt{h-V}. \tag{2.64}$$

So, we have

$$\left|\partial_I^{n}(\mathcal{L}^m(f))\partial_I^{l-m-n}\left(\int_{-\alpha_h}^{x} \frac{L}{\sqrt{h-V(s)}} ds\right)\right| \leq \frac{C}{V'} I^{-(l+2)} \sqrt{h-V}. \tag{2.65}$$

By (2.62) and (2.65), we have

$$\left|\partial_I^{l+1}\left(\int_{-\alpha_h}^{x} \frac{L}{\sqrt{h-V(s)}} ds\right)\right| \leq \frac{C}{V'} I^{-(l+2)} \sqrt{h-V}. \tag{2.66}$$

Thus, we have proved (2.55).

The inequality (2.56) can be proved by (2.38), and the process of proof is similar to that of (2.55), so we omit it.

Thus, we have proved Lemma 2.6. □

Now, we give the estimates of $x(I,\theta)$ and $y(I,\theta)$.

Lemma 2.7. *For I sufficient large and $-\alpha_h \leq x < 0$, the following estimates hold:*

$$\left|I^n \frac{\partial^n x(I,\theta)}{\partial I^n}\right| \leq C|x(I,\theta)+1|, \qquad \left|I^n \frac{\partial^n y(I,\theta)}{\partial I^n}\right| \leq C\left|y(I,\theta)\right|, \quad \textit{for } 0 \leq n \leq 6. \tag{2.67}$$

Proof. We now prove the first inequality. It is sufficient to prove that

$$I^n \frac{\partial^n x(I,\theta)}{\partial I^n}\Bigg| \leq C(1+x). \tag{2.68}$$

Case $k = 1$. Differentiating (2.30) by I and noting $1/T_0 = h_I$, we have

$$0 = h_{II} \int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}} ds + h_I \partial_I \left(\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}} ds \right). \tag{2.69}$$

Now, we choose $V(s)/h = \eta$ as the new variable of integration, so

$$\begin{aligned}
&\partial_I \left(\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}} ds \right) \\
&\quad = \partial_I \left(\int_{0}^{V(x)/h} \frac{\sqrt{h}}{V'(s(\eta,t,h))} \frac{1}{\sqrt{2(1-\eta)}} d\eta \right) \\
&\quad = \frac{V' x_I h - V h_I}{h^2} \frac{h^{1/2}}{V'} \frac{1}{\sqrt{2(1-\eta)}} + \int_{0}^{V(x)/h} \frac{(1/2)h^{-1/2} V' h_I - h^{1/2} V''(\partial s/\partial I)}{(V')^2} \frac{1}{\sqrt{2(1-\eta)}} d\eta \\
&\quad = \frac{x_I}{\sqrt{2(h-V)}} - \frac{h_I}{h\sqrt{2(h-V)}} \frac{V}{V'} + \frac{h_I}{h} \\
&\qquad + \int_{0}^{V(x)/h} \frac{(1/2)h^{-1/2} V' h_I - h^{1/2} V''(\partial s/\partial I)}{(V')^2} \frac{\sqrt{h}}{\sqrt{2(h-V)}} \frac{V'}{h} ds.
\end{aligned} \tag{2.70}$$

Observing that

$$\frac{\partial s}{\partial I} = \frac{h_I \eta}{V'}, \qquad W' = 1 - \frac{V V''}{(V')^2}, \tag{2.71}$$

and simplifying, we have

$$(2.70) = \frac{x_I}{\sqrt{2(h-V)}} - \frac{h_I}{h\sqrt{2(h-V)}} \frac{V}{V'} + \frac{h_I}{h} \int_{0}^{x} \left(W' - \frac{1}{2} \right) \frac{ds}{\sqrt{2(h-V)}}. \tag{2.72}$$

By (2.69)–(2.72), we have

$$x_I(\theta, I) = \sqrt{2(h-V)} \int_{-\alpha_h}^{x} L(s,I) \frac{1}{\sqrt{2(h - V(s))}} ds + \frac{h_I}{h} W(x). \tag{2.73}$$

Since $|L| \le C I^{-1}$, by Lemma 2.6, we have

$$\left| \sqrt{(2(h-V))} \int_{-\alpha_h}^{x} L(s,I) \frac{1}{\sqrt{2(h-V(s))}} ds \right| \le C I^{-1} (\alpha_h + x). \tag{2.74}$$

We observe that

$$W(x) = \frac{1}{2}x(1-x)(1+x), \tag{2.75}$$

where $-1 < x < 0$, so $|W(x)| \le C(1+x)$, which means that

$$\left|\frac{h_I}{h}W(x)\right| \le CI^{-1}(1+x). \tag{2.76}$$

By (2.74) and (2.76), we have $|x_I| \le I^{-1}(1+x)$.

We suppose that

$$\left|\partial_I^{k-1}x\right| \le I^{-(k-1)}(1+x), \tag{2.77}$$

where $1 \le k \le 6$. We will prove $|\partial_I^k x| \le I^{-k}(1+x)$.

For this purpose, we firstly estimate $\sqrt{2(h-V(x))}$. We differentiate $\sqrt{2(h-V(x))}$ in (2.73) and using (2.73), then we obtain

$$\frac{d}{dI}\sqrt{2(h-V(x))} = \frac{h_I}{h}\sqrt{2(h-V(x))} - \frac{V'(x)}{2}\int_{-\alpha_h}^{x} L(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds. \tag{2.78}$$

Since $|h_I/h| \le CI^{-1}$, we have

$$\left|\frac{h_I}{h}\sqrt{2(h-V(x))}\right| \le CI^{-1}\sqrt{2(h-V(x))}. \tag{2.79}$$

By (2.37), we have

$$\left|\frac{V'(x)}{2}\int_{-\alpha_h}^{x} L(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds\right| \le CI^{-1}\sqrt{2(h-V(x))}. \tag{2.80}$$

By (2.78)–(2.80), we have

$$\left|\partial_I\left(\sqrt{2(h-V(x))}\right)\right| \le CI^{-1}\sqrt{2(h-V(x))}. \tag{2.81}$$

We suppose that when $n \le k-1$,

$$\left|\partial_I^n\left(\sqrt{2(h-V(x))}\right)\right| \le CI^{-n}\sqrt{2(h-V(x))}. \tag{2.82}$$

We will prove that when $n = k$,

$$\left|\partial_I^n\left(\sqrt{2(h - V(x))}\right)\right| \le CI^{-n}\sqrt{2(h - V(x))}. \tag{2.83}$$

By direct computation, we have

$$\begin{aligned}\partial_I^k\left(\sqrt{2(h - V(x))}\right) = \Bigg\{&V^{(k_1)}\partial_I^{l_1}x\cdots\partial_I^{l_{k_1-1}}x\partial_I^{k_3}\left(\int_{-\alpha_h}^{x} L(s,I)\frac{1}{\sqrt{2(h - V(s))}}ds\right),\\ &\partial_I^m\left(\frac{h_I}{h}\right)\partial_I^{k-m-1}\left(\sqrt{2(h - V(x))}\right)\Bigg\},\end{aligned} \tag{2.84}$$

where $l_1 + \cdots + l_{k_1-1} = k_2,\ k_2 + k_3 = k - 1$ and $k_2 < k_1 < k,\ k_3 < k$.

By the assume (2.77), we have

$$\left|\partial_I^{l_1}x\cdots\partial_I^{l_{k_1-1}}x\right| \le (1 + x)^{k_1-1}I^{-k_2}. \tag{2.85}$$

By (2.55), we get

$$\left|\partial_I^{k_3}\left(\int_{-\alpha_h}^{x} L(s,I)\frac{1}{\sqrt{2(h - V(s))}}ds\right)\right| \le \frac{C}{V'}I^{-k_3}\sqrt{2(h - V(x))} \tag{2.86}$$

By (2.85), (2.86) and noting the fact that

$$(1 + x)^{k-1}\partial_I^k(V) \le C\left(|V'| + (1 + x)^{k_1+1}\right), \quad -1 < x < 0, \tag{2.87}$$

we obtain

$$\begin{aligned}&\left|\sum V^{(k_1)}\partial_I^{l_1}x\cdots\partial_I^{l_{k_1-1}}x\partial_I^{k_3}\left(\int_{-\alpha_h}^{x} L(s,I)\frac{1}{\sqrt{2(h - V(s))}}ds\right)\right|\\ &\quad\le \sum\frac{C}{V'}V^{(k_1)}(1 + x)^{k_1-1}I^{-(k_2+k_3+1)}\sqrt{2(h - V(x))}\\ &\quad\le \sum\frac{\left[|V'| + (1 + x)^{k_1+1}\right]}{V'}I^{-(k)}\sqrt{2(h - V(x))}\\ &\quad\le CI^{-(k)}\sqrt{2(h - V(x))}.\end{aligned} \tag{2.88}$$

By assumption (2.82), we have

$$\left|\partial_I^{k-m-1}\left(\sqrt{2(h - V(x))}\right)\right| \le CI^{k-m-1}\left(\sqrt{2(h - V(x))}\right). \tag{2.89}$$

By (2.89) and the fact that $|\partial_I^m(h_I/h)| \le CI^{-(m+1)}$, we have

$$\left|\sum \partial_I^m\left(\frac{h_I}{h}\right)\partial_I^{k-m-1}\left(\sqrt{2(h-V(x))}\right)\right| \le CI^{-k}\sqrt{2(h-V(x))}. \tag{2.90}$$

So, by (2.88) and (2.90), we have proved (2.83).

By (2.56), we have

$$\left|\partial_I^m\left(\int_{-\alpha_h}^{x}\frac{L(x,I)}{\sqrt{2(h-V(s))}}ds\right)\right| \le \frac{C}{\sqrt{2(h-V(x))}}I^{-(m+1)}(\alpha_h+x). \tag{2.91}$$

By (2.83) and (2.91), we have

$$\begin{aligned}\left|\partial^n\left(\sqrt{2(h-V(x))}\right)\partial^m\left(\int_{-\alpha_h}^{x}L(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds\right)\right| &\le CI^{-(m+n+1)}(\alpha_h+x)\\ &\le CI^{-(k+1)}(\alpha_h+x).\end{aligned} \tag{2.92}$$

By the assumption (2.77), the fact that $\partial_I^{m_3}(h_I/h) \le CI^{-(m_3+1)}$, and noting that

$$\partial_I\left(\frac{h_I}{h}W\right) = \sum^{k} W^{(m_1)}\partial_I^{l_1}x\cdots\partial_I^{l_{m_1}}x\partial_I^{m_3}\left(\frac{h_I}{h}\right), \tag{2.93}$$

where $m_2 < m_1,\ l_1+\cdots+l_{m_1} = m_2,\ m_2+m_3 = k$, we have

$$\begin{aligned}\partial_I\left(\frac{h_I}{h}W\right) &\le \sum^{k} W^{(m_1)}I^{-(l_1+\cdots+l_{m_1})}(1+x)^{m_1}I^{-(m_3+1)}\\ &\le C\sum^{k} W^{(m_1)}(1+x)^{m_1}I^{-(m_2+m_3+1)}\\ &\le C(1+x)I^{-(k+1)}.\end{aligned} \tag{2.94}$$

By (2.92) and (2.94), we have

$$\begin{aligned}\partial_I^k x_I &= \sum^{k}\partial^n\left(\sqrt{2(h-V(x))}\right)\partial^m\left(\int_{-\alpha_h}^{x}L(s,I)\frac{1}{\sqrt{2(h-V(s))}}ds\right)+\partial^k\left(\frac{h_I}{h}W(x)\right)\\ &\le CI^{-(k+1)}(1+x) \le CI^{-(k+1)}(1+x),\end{aligned} \tag{2.95}$$

which means

$$\partial_I^k x \le CI^{-k}(1+x). \tag{2.96}$$

We now prove

$$\left| I^n \frac{\partial^n y(I,\theta)}{\partial I^n} \right| \le c\sqrt{I}. \tag{2.97}$$

Since

$$h(I) = \frac{1}{2}y^2 + V(x), \tag{2.98}$$

we have

$$y = \pm\sqrt{2(h(I) - V(x))}. \tag{2.99}$$

we have proved (2.83), so we have

$$\left| I^n \frac{\partial^n y(I,\theta)}{\partial I^n} \right| \le C \cdot I^{-n} y, \tag{2.100}$$

which means that

$$\left| I^n \frac{\partial^n y(I,\theta)}{\partial I^n} \right| \le C \cdot h_0(I). \tag{2.101}$$

The proof of Lemma 2.7 is complete. □

Remark 2.8. Lemma 2.7 also holds when $x \ge 0$. Since the idea and the process of the proof is more easily than that of Lemma 2.7, we omit the details.

Now, we give the estimate of $H_1(I,\theta,t)$.

Lemma 2.9.

$$I^k \left| \frac{\partial_I^k \partial_t^l \partial_\theta^i H_1(I,\theta,t)}{\partial I^k \partial t^l \partial \theta^i} \right| \le C \cdot \sqrt{h_0(I)}, \quad k+i \le 7, \ i = 0,1. \tag{2.102}$$

Proof. This lemma can be proved easily form the definition of H_1 and $|y(\theta,I)| \le \sqrt{2h_0(I)}$. □

2.2. New Action and Angle Variables

Now, we are concerned with the Hamiltonian system (2.32) with Hamiltonian function $H(\theta,I,t)$ given by (2.33). Note that

$$Id\theta - Hdt = -(Hdt - Id\theta). \tag{2.103}$$

This means that if one can solve I form (2.32) as a function of H (θ and t as parameters), then

$$\frac{dH}{d\theta} = -\frac{\partial I}{\partial t}(t,H,\theta), \qquad \frac{dt}{d\theta} = -\frac{\partial I}{\partial H}(t,H,\theta) \tag{2.104}$$

is also a Hamiltonian system with Hamiltonian function I, and now, the action, angle, and time variables are H, t, and θ.

Form Remarks 2.3 and 2.4, we have

$$\frac{\partial H}{\partial I} \longrightarrow 1, \quad \text{as } I \longrightarrow +\infty. \tag{2.105}$$

Hence, by the implicit function theorem, there is a function $I(t,H,\theta)$ such that

$$H(\theta, I(t,H,\theta), t) = H. \tag{2.106}$$

By Lemma 2.9, we have

$$\frac{H_1(\theta,I,t)}{H_0(I)} \longrightarrow 0, \quad \text{as } I \rightarrow +\infty. \tag{2.107}$$

So, there is a function $R(t,H,\theta)$ with $|R| \le (1/2)H$ such that

$$I(t,H,\theta) = I_0(H - R(t,H,\theta)), \quad \text{for } H \longrightarrow +\infty. \tag{2.108}$$

Let

$$I_1(t,H,\theta) = I_0(H - R(t,H,\theta)) - I_0(H) = \int_0^{\pi} I_0'(H - sR(t,H,\theta))R(t,H,\theta)d\theta. \tag{2.109}$$

Then,

$$I(t,H,\theta) = I_0(H) + I_1(t,H,\theta). \tag{2.110}$$

From Remark 2.3, we have known the estimate of $I_0(H)$, so we need to give the estimate of $I_1(t,H,\theta)$. For this propose, we need firstly the following Lemma on the estimate of $R(t,H,\theta)$.

Lemma 2.10. *The function $R(t,H,\theta)$ possesses the following estimates:*

$$H^k \left| \frac{\partial^{k+l} R(H,t,\theta)}{\partial^k H \partial^l t} \right| \le H^{1/2}, \tag{2.111}$$

for $k + l \le 6$.

Proof. From (2.108) and (2.110), it follows that

$$R(t,H,\theta) = H_1(\theta, I_0(H-R), t). \tag{2.112}$$

When $k+l=0$, by Lemma 2.9, we have

$$\begin{aligned} |R(t,H,\theta)| &= |H_1(\theta, I_0(H-R), t)| \\ &\le C\cdot\sqrt{h_0(I_0(H-R))} \\ &\le C\cdot\sqrt{H-R} \\ &\le C\cdot\sqrt{H}. \end{aligned} \tag{2.113}$$

When $k+l=1$, we first denote

$$\Delta = \frac{\partial H_1}{\partial I}(\theta, I_0(H-R), t) I_0'(H-R). \tag{2.114}$$

By Remark 2.3, we observe that I_0 is increasing and

$$I_0\left(\frac{1}{2}H\right) \le I_0(H) \le I_0\left(\frac{3}{2}H\right) \le C\cdot I_0\left(\frac{1}{2}H\right). \tag{2.115}$$

By Lemma 2.9 and (2.115), we have

$$\begin{aligned} |\Delta| &= \left|\frac{\partial H_1}{\partial I}(\theta, I_0(H-R), t)\cdot I_0'(H-R)\right| \\ &\le C\cdot\frac{1}{I_0(H-R)}\cdot\sqrt{h_0(I_0(H-R))}\cdot I_0'(H-R) \\ &\le C\cdot\frac{1}{I_0(H/2)}\cdot\sqrt{\frac{3}{2}H}\cdot\frac{1}{(3/2)H}\cdot I_0\left(\frac{3}{2}H\right) \\ &\le C\cdot H^{-1/2} \\ &\le \frac{1}{2}. \end{aligned} \tag{2.116}$$

So,

$$1+\Delta \ge \frac{1}{2}. \tag{2.117}$$

By direct computation, we have

$$\begin{aligned}
\left|\frac{\partial R}{\partial H}\right| &= \left|\frac{(\partial H_1/\partial I)(\theta, I_0(H-R), t)I_0'(H-R)}{1+\Delta}\right| \\
&\leq C\cdot\left|\frac{\partial H_1}{\partial I}(\theta, I_0(H-R), t)I_0'(H-R)\right| \\
&\leq C\cdot H^{-1/2}, \\
\left|\frac{\partial R}{\partial t}\right| &= \left|\frac{(\partial H_1/\partial t)(I_0(H-R))}{1+\Delta}\right| \\
&\leq C\cdot\left|\frac{\partial H_1}{\partial t}(I_0(H-R))\right| \\
&\leq C\cdot H^{1/2}.
\end{aligned} \tag{2.118}$$

By (2.118), we can easily get

$$H^k\left|\frac{\partial^{k+l}R(H,t,\theta)}{\partial^k H\partial^l t}\right| \leq H^{1/2}. \tag{2.119}$$

When $k+l\geq 2$, one may get

$$\frac{\partial^{k+l}R(H,t,\theta)}{\partial^k H\partial^l t} = \sum c_{n,j_1\cdots j_n}\frac{\partial^n H_1}{\partial^n I_0^{j_1}}\frac{\partial^{j_1}I_0(H-R)}{\partial H^{j_1}}\cdots\frac{\partial^{j_n}I_0(H-R)}{\partial H^{j_n}}, \tag{2.120}$$

where $1\leq n\leq k$, $j_1+\cdots+j_n<k$. It is easy to verify

$$\left|\frac{\partial^{k+l}R(H,t,\theta)}{\partial^k H\partial^l t}\right| \leq C\cdot H^{(1/2)-k}, \tag{2.121}$$

for $k+l\geq 2$. This complete the proof. □

Now, we give the estimates of $I_1(t,H,\theta)$.

Lemma 2.11. *The function $I_1(t,H,\theta)$ possesses the following estimates:*

$$H^k\left|\frac{\partial^{k+l}I_1(H,t,\theta)}{\partial^k H\partial^l t}\right| \leq H^{1/(1+\alpha)}, \tag{2.122}$$

for $k+l\leq 6$.

Proof. When $k = l = 0$. By Remark 2.3 and $|R| \leq (1/2)H$, we have

$$I_0'(H - sR(t,H,\theta)) \leq C \cdot \frac{(H - sR)^{1/(1+\alpha)}}{(H - sR)^{1/2}} \leq C \cdot \frac{((3/2)H)^{1/(1+\alpha)}}{((1/2)H)^{1/2}} \leq C \cdot H^{1/(1+\alpha)-(1/2)}. \tag{2.123}$$

By Lemma 2.10, we know

$$|R| \leq C \cdot H^{1/2}. \tag{2.124}$$

Since

$$I_1(t,H,\theta) = \int_0^{\pi} I_0'(H - sR(t,H,\theta))R(t,H,\theta)d\theta, \tag{2.125}$$

it is easy to get that $|I_1| \leq C \cdot H^{1/(1+\alpha)}$.

When $k + l \geq 1$. By direct computation, we have

$$\frac{\partial^{k+l} I_1(t,H,\theta)}{\partial H^k \partial t^l} = \sum \int_0^{\pi} \frac{\partial^{k_1+l_1} I_0'(t,H,\theta)}{\partial H^{k_1} \partial t^{l_1}} \frac{\partial^{k_2+l_2} R(t,H,\theta)}{\partial H^{k_2} \partial t^{l_2}} ds, \tag{2.126}$$

where $k_1 + k_2 = k$, $l_1 + l_2 = l$. Now, we need to estimate the first term of the integrand. The following equality is important:

$$\frac{\partial^{k_1+l_1} I_0'(t,H,\theta)}{\partial H^{k_1} \partial t^{l_1}} = \sum I_0^{(p+q+1)}(H - sR) \frac{\partial^{m_1} u}{\partial H^{m_1}} \cdots \frac{\partial^{m_p} u}{\partial H^{m_p}} \cdot \frac{\partial^{j_1+n_1} u}{\partial H^{j_1} \partial t^{n_1}} \cdots \frac{\partial^{j_q+n_q} u}{\partial H^{j_q} \partial t^{n_q}}, \tag{2.127}$$

where $u = H - sR$, $p \leq m$, $q \leq n$, $n_1, \ldots, n_q > 0$, $m_1, \ldots, m_p > 0$, $n_1 + \cdots + n_q = n$, $m_1 + \cdots + m_p + j_1 + \cdots + j_q = m$. Assume that there are $\beta(\leq p)$ members: $m_1, \ldots, m_\beta$ in $\{m_1, \ldots, m_p\}$ which equal to 1. Noting that

$$\begin{gathered}\left|\frac{\partial u}{\partial H}\right| = \left|1 + s\frac{\partial R}{\partial H}\right| \leq C, \quad \left|\frac{\partial^k u}{\partial H^k}\right| \leq \left|\frac{\partial^k R}{\partial H^k}\right| \leq C \cdot H^{(1/2)-k}, \quad k > 1, \\ \left|\frac{\partial^{k+l} u}{\partial H^k \partial t^l}\right| \leq \left|\frac{\partial^{k+l} R}{\partial H^k \partial t^l}\right| \leq C \cdot H^{(1/2)-k}, \quad l > 0.\end{gathered} \tag{2.128}$$

By the above discussions, we have

$$\begin{aligned}\left|\frac{\partial^{k_1+l_1} I_0'(t,H,\theta)}{\partial H^{k_1}\partial t^{l_1}}\right| &\le C\cdot\frac{(H-sR)^{1/(1+\alpha)+(1/2)}}{(H-sR)^{p+q+1}}\cdot H^{(1/2)(p-\beta)-(m_{\beta+1}+\cdots+m_p)}\cdot H^{(1/2)q-(j_1+\cdots+j_q)}\\ &\le C\frac{H^{(1/2)+1/(1+\alpha)}}{H^{p+q+1}}\cdot H^{(1/2)(p+q-\beta)}H^{-(m-\beta)}\\ &\le CH^{-(1/2)+1/(1+\alpha)-m}\cdot H^{(p+q-\beta)((1/2)-1)}\\ &\le CH^{-(1/2)+1/(1+\alpha)-m}.\end{aligned} \tag{2.129}$$

By Lemma 2.10, we have known

$$H^k\left|\frac{\partial^{k+l}R(H,t,\theta)}{\partial^k H\partial^l t}\right|\le H^{1/2}, \tag{2.130}$$

then we have

$$H^k\left|\frac{\partial^{k+l}I_1(H,t,\theta)}{\partial^k H\partial^l t}\right|\le CH^{1/(1+\alpha)}. \tag{2.131}$$

□

2.3. Proof of the Main Result

Up to now, we have given an equivalent form of (1.5), that is, the system (2.32), which is expressed in the action and angle variables (H,t). In this section, we first introduce some transformations such that in the transformed system, the perturbation terms of (2.32) depending on the new angle variable are very small if the new action variable is sufficiently large and then prove, by Moser's twist theorem, the statement of Theorem 1.1.

Lemma 2.12. *There is a canonical transformation $\Psi:(\lambda,\tau)\to(H,t)$ of the form*

$$\Psi: H=\lambda+U(\tau,\lambda,\theta),\qquad t=\tau+V(\tau,\lambda,\theta), \tag{2.132}$$

where the functions U and V are 1-periodic in θ and satisty

$$\frac{U(\tau,\lambda,\theta)}{\lambda},V(\tau,\lambda,\theta)\longrightarrow 0\quad \text{as } \lambda\longrightarrow\infty, \tag{2.133}$$

uniformly for $(\tau,\theta)\in T^2$ such that under this mapping, the system and the Hamiltonian function I in (2.110) *is changed into the form*

$$\frac{d\lambda}{d\theta}=-\frac{\partial\mathcal{K}}{\partial\tau}(\tau,\lambda,\theta),\qquad \frac{d\tau}{d\theta}=\frac{\partial\mathcal{K}}{\partial H}(\tau,\lambda,\theta), \tag{2.134}$$

where

$$\mathcal{K}(\tau,\lambda,\theta) = I_0(\lambda) + [I_1](\lambda,\theta) + \mathcal{M}(\tau,\lambda,\theta), \tag{2.135}$$

with

$$[I_1](\lambda,\theta) = \int_0^1 I_1(t,\lambda,\theta)dt. \tag{2.136}$$

Moreover, the new perturbation $\mathcal{M}$ *possesses the estimate*

$$\left|\frac{\partial^{k+l}}{\partial\lambda^k\partial\tau^l}\mathcal{M}(\tau,\lambda,\theta)\right| \le C\cdot\lambda^{-k+(1/2)}. \tag{2.137}$$

Proof. We will look for the required transformation Ψ by a generating function $\mathcal{F}(t,\lambda,\theta)$ in the following way:

$$H = \lambda + \frac{\partial\mathcal{F}}{\partial t}(t,\lambda,\theta), \qquad \tau = t + \frac{\partial\mathcal{F}}{\partial\lambda}(t,\lambda,\theta), \tag{2.138}$$

where the function $\mathcal{F}$ will be given later. Under Ψ, the transformed system of (2.104) is of the form

$$\frac{d\lambda}{d\theta} = -\frac{\partial\mathcal{K}}{\partial\tau}(\tau,\lambda,\theta), \qquad \frac{d\tau}{d\theta} = -\frac{\partial\mathcal{K}}{\partial\lambda}(\tau,\lambda,\theta), \tag{2.139}$$

where

$$\mathcal{K}(\tau,\lambda,\theta) = I_0\left(\lambda + \frac{\partial\mathcal{F}}{\partial t}\right) + I_1\left(t,\lambda + \frac{\partial\mathcal{F}}{\partial t},\theta\right) + \frac{\partial\mathcal{F}}{\partial\theta}. \tag{2.140}$$

By Taylor's formula, one can write

$$\mathcal{K}(\tau,\lambda,\theta) = I_0(\lambda) + I_0'(\lambda)\frac{\partial\mathcal{F}}{\partial t} + I_1(t,\lambda,\theta) + \mathcal{M}(\tau,\lambda,\theta), \tag{2.141}$$

where

$$\mathcal{M}(\tau,\lambda,\theta) = \frac{\partial\mathcal{F}}{\partial\theta} + \int_0^1 (1-s)I_0''\left(\lambda + s\frac{\partial\mathcal{F}}{\partial t}\right)\cdot\left(\frac{\partial\mathcal{F}}{\partial\theta}\right)^2 ds + \int_0^1 \frac{\partial I_1}{\partial H}\left(t,\lambda + s\frac{\partial\mathcal{F}}{\partial t},\theta\right)\cdot\frac{\partial\mathcal{F}}{\partial t}ds. \tag{2.142}$$

We choose $\mathcal{F}$

$$\mathcal{F}(t,\lambda,\theta) = -\int_0^t \frac{1}{\overline{I_0'}(\lambda)}\cdot\left(I_1(t,\lambda,\theta) - [I_1](\lambda,\theta)\right). \tag{2.143}$$

Then, $\mathcal{K}$ is of the form (2.135).

We now show that $\mathcal{M}$ satisfies (2.137). From Remark 2.3 and Lemma 2.11, it follows that

$$\left| \frac{\partial^{k+k+i}}{\partial \lambda^k \partial t^l \partial \theta^i} \mathcal{F}(t,\lambda,\theta) \right| \leq C \cdot \lambda^{-k+(1/2)}, \tag{2.144}$$

for $k+i+l \leq 6$ and $i = 0, 1$. In particular

$$\left| \frac{\partial^2}{\partial \lambda \partial t} \mathcal{F}(t,\lambda,\theta) \right| \leq C\lambda^{-1/2} \leq \frac{1}{2}, \tag{2.145}$$

if $\lambda \gg 1$. So we can solve the second equation of (2.138) for t,

$$t = \tau + V(\tau,\lambda,\theta), \tag{2.146}$$

where the function V satisfies

$$V(\tau,\lambda,\theta) = -\frac{\partial \mathcal{F}}{\partial \lambda}(\tau + V, \lambda, \theta). \tag{2.147}$$

Set

$$U(\tau,\lambda,\theta) = \frac{\partial \mathcal{F}}{\partial t}(\tau + V, \lambda, \theta). \tag{2.148}$$

Then, the canonical transformation Ψ is of the form (2.132). Moreover, similar to the proof of [5, Lemma 2], we can verify that

$$\left| \frac{\partial^{k+l}}{\partial \lambda^k \partial \tau^l} U(\tau,\lambda,\theta) \right| \leq C \cdot \lambda^{-k+(1/2)}, \qquad \left| \frac{\partial^{k+l}}{\partial \lambda^k \partial \tau^l} V(\tau,\lambda,\theta) \right| \leq C \cdot \lambda^{-k-(1/2)}, \tag{2.149}$$

for $k+l \leq 5$ and $U/\lambda, V \to 0$ as $\lambda \to +\infty$. Let

$$\begin{aligned} \phi_1(\tau,\lambda,\theta) &= \frac{\partial \mathcal{F}}{\partial \theta}(\tau + V, \lambda, \theta), \\ \phi_2(\tau,\lambda,\theta) &= \int_0^1 (1-s) I_0''(\lambda + sU) \cdot U^2 ds, \\ \phi_3(\tau,\lambda,\theta) &= \int_0^1 \frac{\partial I_1}{\partial H}(\tau + V, \lambda + sU, \theta) \cdot U\, ds. \end{aligned} \tag{2.150}$$

It is not difficult to prove that

$$\left|\frac{\partial^{k+l}}{\partial\lambda^k\partial\tau^l}\phi_1(\tau,\lambda,\theta)\right| \le C\cdot\lambda^{-k+(1/2)},$$
$$\left|\frac{\partial^{k+l}}{\partial\lambda^k\partial\tau^l}\phi_1(\tau,\lambda,\theta)\right| \le C\cdot\lambda^{-k+(1/(1+\alpha))-(1/2)}, \tag{2.151}$$
$$\left|\frac{\partial^{k+l}}{\partial\lambda^k\partial\tau^l}\phi_1(\tau,\lambda,\theta)\right| \le C\cdot\lambda^{-k+(1/(1+\alpha))-(1/2)},$$

for $k+l\le 5$. Note that $0<\alpha<1$, we have $1/(1+\alpha)-1/2<1/2$.

Hence, we have

$$\left|\frac{\partial^{k+l}}{\partial\lambda^k\partial\tau^l}\mathcal{M}(\tau,\lambda,\theta)\right| \le \lambda^{-k+(1/2)}. \tag{2.152}$$

The proof of Lemma 2.12 is complete.

For $\lambda_0>0$, we denote by A_{λ_0} the domain

$$A_{\lambda_0} = \left\{(\lambda,\tau,\theta) \mid \lambda\ge\lambda_0, (\tau,\theta)\in T^2\right\}. \tag{2.153}$$

□

Lemma 2.13. *The Poincaré mapping P of* (2.134) *has the intersection property on A_{H_0}; that is, if Γ is an embedded circle in A_{H_0} homotopic to a circle $\lambda = const.$ in A_{H_0}, then $P(\Gamma)\bigcap\Gamma\neq\emptyset$.*

Proof. The proof can be found in [5].

Definite a diffeomorphism $\Psi_1 : A_{H_0}\times\mathbb{S}^1 \longrightarrow A_{H_0}\times\mathbb{S}^1$

$$\nu = I_0'(H), \qquad \tau=\tau, \qquad \theta=\theta. \tag{2.154}$$

Then, the system (2.134) under the transformation Ψ_1 becomes

$$\frac{d\nu}{d\theta} = f_1(\lambda,t,\theta), \qquad \frac{dt}{d\theta} = \nu + f_2(\lambda,t,\theta), \tag{2.155}$$

where

$$f_1(\tau,\nu,\theta) = -I_0''(\lambda)\frac{\partial\mathcal{M}}{\partial\tau}(\tau,\lambda,\theta), \qquad f_2(\tau,\nu,\theta) = \frac{\partial[I_1]}{\partial\lambda}(\lambda,\theta) + \frac{\partial\mathcal{M}}{\partial\lambda}(\tau,\lambda,\theta), \tag{2.156}$$

with $\lambda=\lambda(\nu)$ defined through the transformation Ψ_1.

Now, we estimate f_1 and f_2. Since

$$c \cdot \lambda^{1/(1+\alpha)-(1/2)} \le I_0'(\lambda) \le C \cdot \lambda^{1/(1+\alpha)-(1/2)}, \qquad \left|\frac{\partial [I_1]}{\partial \lambda}\right| \le C \cdot \lambda^{1/(1+\alpha)-1}, \tag{2.157}$$

we have

$$c \cdot \lambda^{2(1+\alpha)/(1-\alpha)} \le \lambda(\nu) \le C \cdot \lambda^{2(1+\alpha)/(1-\alpha)}, \tag{2.158}$$

then

$$\lambda \gg 1 \Longleftrightarrow \nu \gg 1. \tag{2.159}$$

Moreover, we have

$$\left|\frac{\partial^k}{\partial \nu^k}\lambda(\nu)\right| \le C \cdot \nu^{-k}\lambda(\nu). \tag{2.160}$$

When $0 < \alpha < 1$, we have

$$\begin{gathered} \left|\frac{\partial^{k+l}}{\partial \lambda^k \partial \tau^l}\mathcal{M}(\tau,\lambda,\theta)\right| \le C \cdot \lambda^{-k+(1/2)}, \\ \frac{1}{2} < \frac{1}{1+\alpha} < 1, \end{gathered} \tag{2.161}$$

so

$$\begin{aligned} \left|\frac{\partial^{k+l}}{\partial \nu^k \partial \tau^l} f_1(\tau,\nu,\theta)\right| &\le C \cdot \nu^{-k} \cdot \lambda^{-2} \cdot \nu^{-k} I_0(\lambda) \cdot \lambda^{1/2} \\ &\le C \cdot \nu^{-k}\lambda^{-\alpha/(1+\alpha)} \\ &\le C \cdot \nu^{-k+(-4\alpha/(1-\alpha))} \\ &\le C \cdot \nu^{-k-\sigma}, \\ \left|\frac{\partial^{k+l}}{\partial \nu^k \partial \tau^l} f_2(\tau,\nu,\theta)\right| &\le C \cdot \nu^{-k} \cdot \left(\lambda^{-1/2} + \lambda^{1/(1+\alpha)-1}\right) \\ &\le C \cdot \nu^{-k} \cdot \lambda^{1/(1+\alpha)-1} \\ &\le C \cdot \nu^{-k-(2\alpha/(1-\alpha))} \\ &\le C \cdot \nu^{-k-\sigma}, \end{aligned} \tag{2.162}$$

for all $k + l \le 4$, where $\sigma = \min\{4\alpha/(1-\alpha), 2\alpha/(1-\alpha)\} = 2\alpha/(1-\alpha) > 0$.

Since f_1 and f_2 are sufficiently small as $\nu \gg 1$, all solution of (2.155) exist for $0 \leq \theta \leq 1$ when the initial values $\nu(0) = \lambda$ are sufficiently large. Hence, the Poincaré map Φ associated to (2.155) is well defined on A_{λ_0} as $\lambda_0 \gg 1$. In fact, by integrating (2.155) from $\theta = 0$ to $\theta = \pi$, we see that Φ has the form

$$\Phi : \tau_1 = \tau_0 + \mu_0 + \Xi_1(t_0, \lambda_0), \qquad \nu_1 = \nu_0 + \Xi_2(t_0, \lambda_0), \tag{2.163}$$

where Ξ_1 and Ξ_2 satisfy the same estimates as those of f_1 and f_2; that is,

$$\partial_\nu^k \partial_\tau^l \Xi_i \leq \nu^{-\sigma}, \tag{2.164}$$

where $i = 1, 2,\ 0 \leq k + l \leq 4$.

Since Φ satisfies all the assumptions of Moser's twist theorem [9], from which we conclude that for any $\omega \gg 1$ satisfying

$$\left|\omega - \frac{p}{q}\right| \geq c_0 |q|^{-5/2}, \quad \frac{p}{q} \in \mathbb{Q}. \tag{2.165}$$

There is an invariant curve Γ_ω of Φ which is conjugated to pure rotation of the circle with rotation number ω. Tracing back to the system (2.32), Γ_ω gives rise to an invariant closed curve of the Poincaré map Φ of (2.32) with rotation number $1/\omega$ which surrounds and is arbitrarily far away form the origin. Hence, all solutions of (1.5) are bounded. This completes the proof of the Theorem. □

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