

Research Article

A New Iterative Scheme for Countable Families of Weak Relatively Nonexpansive Mappings and System of Generalized Mixed Equilibrium Problems

Yekini Shehu^{1,2}

¹ Department of Mathematics, University of Nigeria, Nsukka, Nigeria

² Mathematics Institute, African University of Science and Technology, Abuja, Nigeria

Correspondence should be addressed to Yekini Shehu, deltanougt2006@yahoo.com

Received 21 September 2010; Revised 9 November 2010; Accepted 22 December 2010

Academic Editor: A. Zafer

Copyright © 2010 Yekini Shehu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We construct a new iterative scheme by hybrid methods and prove strong convergence theorem for approximation of a common fixed point of two countable families of weak relatively nonexpansive mappings which is also a solution to a system of generalized mixed equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth using the properties of generalized f -projection operator. Using this result, we discuss strong convergence theorem concerning general H -monotone mappings and system of generalized mixed equilibrium problems in Banach spaces. Our results extend many known recent results in the literature.

1. Introduction

Let E be a real Banach space with dual E^* , and let C be nonempty, closed and convex subset of E . A mapping $T : C \rightarrow C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.1)$$

A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}. \quad (1.2)$$

The following properties of J are well known (the reader can consult [1–3] for more details).

- (1) If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E .
- (2) $J(x) \neq \emptyset$, $x \in E$.
- (3) If E is reflexive, then J is a mapping from E onto E^* .
- (4) If E is smooth, then J is single valued.

Throughout this paper, we denote by ϕ , the functional on $E \times E$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (1.3)$$

From [4], in uniformly convex and uniformly smooth Banach spaces, we have

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \quad (1.4)$$

Definition 1.1. Let C be a nonempty subset of E and let $\{T_n\}_{n=0}^\infty$ be a countable family of mappings from C into E . A point $p \in C$ is said to be an *asymptotic fixed point* of $\{T_n\}_{n=0}^\infty$ if C contains a sequence $\{x_n\}_{n=0}^\infty$ which converges weakly to p and $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(\{T_n\}_{n=0}^\infty)$. One says that $\{T_n\}_{n=0}^\infty$ is *countable family of relatively nonexpansive mappings* (see, e.g., [5]) if the following conditions are satisfied:

- (R1) $F(\{T_n\}_{n=0}^\infty) \neq \emptyset$;
- (R2) $\phi(p, T_n x) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T_n)$, $n \geq 0$;
- (R3) $\bigcap_{n=0}^\infty F(T_n) = \widehat{F}(\{T_n\}_{n=0}^\infty)$.

Definition 1.2. A point $p \in C$ is said to be a *strong asymptotic fixed point* of $\{T_n\}_{n=0}^\infty$ if C contains a sequence $\{x_n\}_{n=0}^\infty$ which converges strongly to p and $\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(\{T_n\}_{n=0}^\infty)$. One says that a mapping $\{T_n\}_{n=0}^\infty$ is *countable family of weak relatively nonexpansive mappings* (see, e.g., [5]) if the following conditions are satisfied:

- (R1) $F(\{T_n\}_{n=0}^\infty) \neq \emptyset$;
- (R2) $\phi(p, T_n x) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T_n)$, $n \geq 0$;
- (R3) $\bigcap_{n=0}^\infty F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^\infty)$.

Definition 1.3. Let C be a nonempty subset of E and let T be a mapping from C into E . A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^\infty$ which converges weakly to p and $\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. We say that a mapping T is *relatively nonexpansive* (see, e.g., [6–11]) if the following conditions are satisfied:

- (R1) $F(T) \neq \emptyset$;
- (R2) $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T)$;
- (R3) $F(T) = \widehat{F}(T)$.

Definition 1.4. A point $p \in C$ is said to be a *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^\infty$ which converges strongly to p and $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$. We say that a mapping T is *weak relatively nonexpansive* (see, e.g., [12, 13]) if the following conditions are satisfied:

- (R1) $F(T) \neq \emptyset$;
- (R2) $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T)$;
- (R3) $F(T) = \widetilde{F}(T)$.

Definition 1.3 (Definition 1.4) is a special form of Definition 1.1 (Definition 1.2) as $T_n \equiv T$, for all $n \geq 0$. Furthermore, Su et al. [5] gave an example which is a countable family of weak relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings. It is obvious that relatively nonexpansive mapping is weak relatively nonexpansive mapping. In fact, for any mapping $T : C \rightarrow C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$. Kang et al. [12] gave an example of a weak relatively nonexpansive mapping which is not relatively nonexpansive.

Let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction, $A : C \rightarrow E^*$ a mapping and $\varphi : C \rightarrow \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is to find $x \in C$ (see, e.g., [14–19]) such that

$$F(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \tag{1.5}$$

for all $y \in C$. We will denote the solutions set of (1.5) by $\text{GMEP}(F, \varphi)$. Thus

$$\text{GMEP}(F, A, \varphi) := \{x^* \in C : F(x^*, y) + \langle Ax^*, y - x^* \rangle + \varphi(y) - \varphi(x^*) \geq 0, \forall y \in C\}. \tag{1.6}$$

If $\varphi = 0$ and $A = 0$, then problem (1.5) reduces to an equilibrium problem studied by many authors (see, e.g., [20–28]), which is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0 \tag{1.7}$$

for all $y \in C$. We shall denote the solutions set of (1.7) by $\text{EP}(F)$.

If $\varphi = 0$ and $E = H$ (a real Hilbert space), then problem (1.5) reduces to a generalized equilibrium problem studied by many authors (see, e.g., [29–31]), which is to find $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0 \tag{1.8}$$

for all $y \in C$.

If $A = 0$ and $E = H$, then problem (1.5) reduces to mixed equilibrium problem considered by many authors (see, e.g., [32–34]), which is to find $x^* \in C$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \quad (1.9)$$

for all $y \in C$.

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases (see, e.g., [35]). Some methods have been proposed to solve the mixed equilibrium problem (see, e.g., [33, 34, 36]). Numerous problems in physics, optimization and economics reduce to find a solution of problem (1.8).

In [9], Matsushita and Takahashi introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ H_n &= \{w \in C : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ W_n &= \{w \in C : \langle x_n - w, Jx_0 - Jx_n \rangle\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n \geq 0 \end{aligned} \quad (1.10)$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)} x_0$, where $F(T) \neq \emptyset$.

In [37], Plubtieng and Ungchittarakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings: $x_0 \in C$,

$$\begin{aligned} z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT x_n + \beta_n^{(3)} JSx_n), \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1 - \alpha_n)Jz_n), \\ C_n &= \left\{z \in C : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n \left(\|x_0\|^2 + 2\langle w, Jx_n - Jx_0 \rangle \right)\right\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \quad (1.11)$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, and $\{\beta_n^{(3)}\}$ are sequences in $(0, 1)$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and T and S are relatively nonexpansive mappings and J is the single-valued duality mapping on E . They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.11) converges strongly to a common fixed point of T and S .

In [10], Takahashi and Zembayashi introduced the following hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mapping which is also a solution to an equilibrium problem in a uniformly convex real Banach space which is also uniformly

smooth: $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}x_0$,

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ F(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle &\geq 0, \quad \forall y \in C, \\ C_{n+1} &= \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}x_0, \quad n \geq 1, \end{aligned} \tag{1.12}$$

where J is the duality mapping on E . Then, they proved that $\{x_n\}_{n=0}^\infty$ converges strongly to $\Pi_\Omega x_0$, where $\Omega = EP(F) \cap F(T) \neq \emptyset$.

Recently, Li et al. [38] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized f -projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ C_{n+1} &= \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}^f x_0, \quad n \geq 1. \end{aligned} \tag{1.13}$$

They proved a strong convergence theorem for finding an element in the fixed points set of T . We remark here that the results of Li et al. [38] extended and improved on the results of Matsushita and Takahashi [9].

Quite recently, motivated by the results of Matsushita and Takahashi [9] and Plubtieng and Ungchittrakool [37], Su et al. [5] proved the following strong convergence theorem by hybrid iterative scheme for approximation of common fixed point of two countable families of weak relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach space.

Theorem 1.5. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E . Suppose $\{T_n\}_{n=1}^\infty$ and $\{S_n\}_{n=1}^\infty$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := (\bigcap_{n=1}^\infty F(T_n)) \cap (\bigcap_{n=1}^\infty F(S_n)) \neq \emptyset$. Suppose that $\{x_n\}_{n=0}^\infty$ is iteratively generated by $x_0 \in C$,*

$$\begin{aligned} z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ C_n &= \{w \in C_{n-1} \cap Q_{n-1} : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ C_0 &= \{w \in C : \phi(w, y_0) \leq \phi(w, x_0)\}, \\ Q_n &= \{w \in C_{n-1} \cap Q_{n-1} : \langle x_n - w, Jx_0 - Jx_n \rangle \geq 0\}, \\ Q_0 &= C, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad n \geq 1, \end{aligned} \tag{1.14}$$

with the conditions

- (i) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iii) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{\Omega}x_0$.

Motivated by the above-mentioned results and the ongoing research, it is our purpose in this paper to prove a strong convergence theorem for two countable families of weak relatively nonexpansive mappings in a uniformly convex real Banach space which is also uniformly smooth using the properties of generalized f -projection operator. Our results extend the results of Li et al. [38], Su et al. [5] and many other recent known results in the literature.

2. Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\rho_E(t) := \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t \right\}. \quad (2.1)$$

E is uniformly smooth if and only if

$$\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0. \quad (2.2)$$

Let $\dim E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0, 2] \rightarrow [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}. \quad (2.3)$$

E is *uniformly convex* if for any $\epsilon \in (0, 2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x - y\| \geq \epsilon$, then $\|(1/2)(x + y)\| \leq 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. A normed space E is called *strictly convex* if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1 - \lambda)y\| < 1$, for all $\lambda \in (0, 1)$.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E . Following Alber [39], the generalized projection Π_C from E onto C is defined by

$$\Pi_C x := \arg \min_{y \in C} \phi(y, x), \quad \forall x \in E. \quad (2.4)$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J (see, e.g., [3, 4, 39–41]). If E is a Hilbert space, then Π_C is the metric projection of H onto C .

Next, we recall the concept of generalized f -projector operator, together with its properties. Let $G : C \times E^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \psi) = \|\xi\|^2 - 2\langle \xi, \psi \rangle + \|\psi\|^2 + 2\rho f(\xi), \quad (2.5)$$

where $\xi \in C$, $\psi \in E^*$, ρ is a positive number and $f : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f , it is easy to see the following properties:

- (i) $G(\xi, \psi)$ is convex and continuous with respect to ψ when ξ is fixed;
- (ii) $G(\xi, \psi)$ is convex and lower semi-continuous with respect to ξ when ψ is fixed.

Definition 2.1 (Wu and Huang [42]). Let E be a real Banach space with its dual E^* . Let C be a nonempty, closed and convex subset of E . One says that $\Pi_C^f : E^* \rightarrow 2^C$ is a generalized f -projection operator if

$$\Pi_C^f \psi = \left\{ u \in C : G(u, \psi) = \inf_{\xi \in C} G(\xi, \psi) \right\}, \quad \forall \psi \in E^*. \quad (2.6)$$

For the generalized f -projection operator, Wu and Huang [42] proved the following theorem basic properties.

Lemma 2.2 (Wu and Huang [42]). *Let E be a real reflexive Banach space with its dual E^* . Let C be a nonempty, closed, and convex subset of E . Then the following statements hold:*

- (i) Π_C^f is a nonempty closed convex subset of C for all $\psi \in E^*$;
- (ii) if E is smooth, then for all $\psi \in E^*$, $x \in \Pi_C^f$ if and only if

$$\langle x - y, \psi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \quad (2.7)$$

- (iii) if E is strictly convex and $f : C \rightarrow \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then Π_C^f is a single valued mapping.

Fan et al. [43] showed that the condition f is positive homogeneous which appeared in Lemma 2.2 and can be removed.

Lemma 2.3 (Fan et al. [43]). *Let E be a real reflexive Banach space with its dual E^* and C a nonempty, closed and convex subset of E . Then if E is strictly convex, then Π_C^f is a single-valued mapping.*

Recall that J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\psi \in E^*$ such that $\psi = Jx$ for each $x \in E$. This substitution in (4.3) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx \rangle + \|x\|^2 + 2\rho f(\xi). \quad (2.8)$$

Now, we consider the second generalized f -projection operator in a Banach space.

Definition 2.4. Let E be a real Banach space and C a nonempty, closed, and convex subset of E . One says that $\Pi_C^f : E \rightarrow 2^C$ is a generalized f -projection operator if

$$\Pi_C^f x = \left\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx) \right\}, \quad \forall x \in E. \quad (2.9)$$

Obviously, the definition of $\{T_n\}_{n=0}^\infty$ is a countably family of weak relatively nonexpansive mappings is equivalent to

$$(R'1) F(\{T_n\}_{n=0}^\infty) \neq \emptyset;$$

$$(R'2) G(p, JT_n x) \leq G(p, Jx), \text{ for all } x \in C, p \in F(T_n), n \geq 0;$$

$$(R'3) \bigcap_{n=0}^\infty F(T_n) = \tilde{F}(\{T_n\}_{n=0}^\infty).$$

Lemma 2.5 (Li et al. [38]). *Let E be a Banach space and $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous convex functional. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E. \quad (2.10)$$

We know that the following lemmas hold for operator Π_C^f .

Lemma 2.6 (Li et al. [38]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E . Then the following statements hold:*

(i) $\Pi_C^f x$ is a nonempty closed and convex subset of C for all $x \in E$;

(ii) for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if

$$\langle \hat{x} - y, Jx - J\hat{x} + \rho f(y) - \rho f(x) \rangle \geq 0, \quad \forall y \in C; \quad (2.11)$$

(iii) if E is strictly convex, then $\Pi_C^f x$ is a single-valued mapping.

Lemma 2.7 (Li et al. [38]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E . Let $x \in E$ and $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \quad (2.12)$$

Lemma 2.8 (Su et al. [5]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex Banach space E . Let T be a weak relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Also, this following lemma will be used in the sequel.

Lemma 2.9 (Kamimura and Takahashi [4]). *Let C be a nonempty, closed, and convex subset of a smooth, uniformly convex Banach space E . Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be sequences in E such that either $\{x_n\}_{n=1}^\infty$ or $\{y_n\}_{n=1}^\infty$ is bounded. If $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.*

Lemma 2.10 (Cho et al. [44]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$ and $\lambda, \mu, \gamma \in [0, 1]$ such that $\lambda + \mu + \gamma = 1$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow \mathbb{R}, \quad g(0) = 0 \tag{2.13}$$

such that for every $x, y, z \in B_r(0)$, the following inequality holds:

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 - \lambda \mu g(\|x - y\|). \tag{2.14}$$

For solving the equilibrium problem for a bifunction $F : C \times C \rightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.11 (Liu et al. [14] and Zhang [19]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex and reflexive Banach space E . Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1)–(A4), $A : C \rightarrow E^*$ a continuous and monotone mapping, and $\varphi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex functional. For $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$Q(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C, \tag{2.15}$$

where $Q(z, y) = F(z, y) + \langle Az, y - z \rangle + \varphi(y) - \varphi(z)$, $z, y \in C$. Furthermore, define a mapping $T_r : E \rightarrow C$ as follows:

$$T_r(x) = \left\{ z \in C : Q(z, y) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}. \tag{2.16}$$

Then, the following hold:

- (i) T_r is single-valued;
- (ii) for any $x, y \in E$,

$$\langle T_r x - T_r y, J T_r x - J T_r y \rangle \leq \langle T_r x - T_r y, J x - J y \rangle; \tag{2.17}$$

- (iii) $F(T_r) = \text{GMEP}(F, A, \varphi)$;
- (iv) $\text{GMEP}(F, A, \varphi)$ is closed and convex.

Lemma 2.12 (Zhang [19]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E . Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1)–(A4), and let $r > 0$. Then for each $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \quad (2.18)$$

For the rest of this paper, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to p shall be denoted by $x_n \rightarrow p$ as $n \rightarrow \infty$, and we shall assume that $\beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(3)} \in [0, 1]$ such that $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$, for all $n \geq 0$.

3. Main Results

Theorem 3.1. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed, and convex subset of E . For each $k = 1, 2, \dots, m$, let F_k be a bifunction from $C \times C$ satisfying (A1)–(A4), $A_k : C \rightarrow E^*$ a continuous and monotone mapping and $\varphi_k : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex functional. Suppose that $\{T_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := \bigcap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap (\bigcap_{n=0}^{\infty} F(T_n)) \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Let $f : E \rightarrow \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \text{int}(D(f))$, and suppose that $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_0 = C$,*

$$\begin{aligned} z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \dots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} &= \{w \in C_n : G(w, Ju_n) \leq G(w, Jx_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}^f x_0, \quad n \geq 0, \end{aligned} \quad (3.1)$$

with the conditions

- (i) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iii) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$;
- (iv) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, \dots, m)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$, $(k = 1, 2, \dots, m)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{\Omega}^f x_0$.

Proof. By Lemma 2.8, we know that $(\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$ is closed and convex. We also know from Lemma 2.11(iv) that $\cap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k)$ is closed and convex. Hence, $\Omega = \cap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$ is a nonempty, closed and convex subset of C . Consequently, $\Pi_{\Omega}^f x_0$ is well defined. We first show that C_n , for all $n \geq 0$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Thus, we only need to show that C_n is closed and convex for each $n \geq 1$. Since $G(z, Ju_n) \leq G(z, Jx_n)$ is equivalent to

$$2(\langle z, Jx_n \rangle - \langle z, Ju_n \rangle) \leq \|x_n\|^2 - \|u_n\|^2, \quad (3.2)$$

this implies that C_{n+1} is closed and convex for all $n \geq 0$. This shows that $\Pi_{C_{n+1}}^f x_0$ is well defined for all $n \geq 0$. By taking $\theta_n^k = T_{r_k, n}^{Q_k} T_{r_{k-1}, n}^{Q_{k-1}} \cdots T_{r_2, n}^{Q_2} T_{r_1, n}^{Q_1}$, $k = 1, 2, \dots, m$ and $\theta_n^0 = I$ for all $n \geq 1$, we obtain $u_n = \theta_n^m y_n$. We next show that $\Omega \subset C_n$, for all $n \geq 0$. For $n = 0$, we have $\Omega \subset C = C_0$. Then for each $x^* \in \Omega$, we obtain

$$\begin{aligned} G(x^*, Ju_n) &\leq G(x^*, J\theta_n^m y_n) \leq G(x^*, Jy_n) \\ &= G(x^*, (\alpha_n Jx_n + (1 - \alpha_n) Jz_n)) \\ &= \|x^*\|^2 - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n) \langle x^*, Jz_n \rangle \\ &\quad + \|\alpha_n Jx_n + (1 - \alpha_n) Jz_n\|^2 + 2\rho f(x^*) \\ &\leq \|x^*\|^2 - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n) \langle x^*, Jz_n \rangle \\ &\quad + \alpha_n \|Jx_n\|^2 + (1 - \alpha_n) \|Jz_n\|^2 + 2\rho f(x^*) \\ &= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) G(x^*, Jz_n) \\ &= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) G(x^*, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n) \\ &\leq \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) \left(\|x^*\|^2 - 2\beta_n^{(1)} \langle x^*, Jx_n \rangle - 2\beta_n^{(2)} \langle x^*, JT_n x_n \rangle \right. \\ &\quad \left. - 2\beta_n^{(3)} \langle x^*, JS_n x_n \rangle \right. \\ &\quad \left. + \beta_n^{(1)} \|x_n\|^2 + \beta_n^{(2)} \|T_n x_n\|^2 + \beta_n^{(3)} \|S_n x_n\|^2 + 2\rho f(x^*) \right) \\ &= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) \left(\beta_n^{(1)} G(x^*, Jx_n) + \beta_n^{(2)} G(x^*, JT_n x_n) + \beta_n^{(3)} G(x^*, JS_n x_n) \right) \\ &\leq G(x^*, Jx_n). \end{aligned} \quad (3.3)$$

So, $x^* \in C_n$. This implies that $\emptyset \neq \Omega \subset C_n$, for all $n \geq 0$. It follows that $\{x_n\}_{n=0}^{\infty}$ is well defined for all $n \geq 0$.

We now show that $\lim_{n \rightarrow \infty} G(x_n, Jx_0)$ exists. Since $f : E \rightarrow \mathbb{R}$ is a convex and lower semi-continuous, applying Lemma 2.5, we see that there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, u^* \rangle + \alpha, \quad \forall y \in E. \quad (3.4)$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 + 2\rho \langle x_n, u^* \rangle + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho u^* \rangle + \|x_0\|^2 + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|x_n\| \|Jx_0 - \rho u^*\| + \|x_0\|^2 + 2\rho\alpha \\ &= (\|x_n\| - \|Jx_0 - \rho u^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho u^*\|^2 + 2\rho\alpha. \end{aligned} \quad (3.5)$$

Since $x_n = \Pi_{C_n}^f x_0$, it follows from (3.5) that

$$G(x^*, Jx_0) \geq G(x_n, Jx_0) \geq (\|x_n\| - \|Jx_0 - \rho u^*\|)^2 + \|x_0\|^2 - \|Jx_0 - \rho u^*\|^2 + 2\rho\alpha \quad (3.6)$$

for each $x^* \in \Omega$. This implies that $\{x_n\}_{n=0}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$. By the construction of C_n , we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m}^f x_0 \in C_n$ for any positive integer $m \geq n$. It then follows from Lemma 2.7 that

$$\phi(x_m, x_n) + G(x_n, Jx_0) \leq G(x_m, Jx_0). \quad (3.7)$$

It is obvious that

$$\phi(x_m, x_n) \geq (\|x_m\| - \|x_n\|)^2 \geq 0. \quad (3.8)$$

In particular,

$$\begin{aligned} \phi(x_{n+1}, x_n) + G(x_n, Jx_0) &\leq G(x_{n+1}, Jx_0), \\ \phi(x_{n+1}, x_n) &\geq (\|x_{n+1}\| - \|x_n\|)^2 \geq 0, \end{aligned} \quad (3.9)$$

and so $\{G(x_n, Jx_0)\}_{n=0}^\infty$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^\infty$ exists. By the fact that $C_m \subset C_n$ and $x_m = \Pi_{C_m}^f x_0 \in C_n$ for any positive integer $m \geq n$, we obtain

$$\phi(x_m, u_n) \leq \phi(x_m, x_n). \quad (3.10)$$

Now, (3.7) implies that

$$\phi(x_m, u_n) \leq \phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0). \quad (3.11)$$

Taking the limit as $m, n \rightarrow \infty$ in (3.11), we obtain

$$\lim_{n \rightarrow \infty} \phi(x_m, x_n) = 0. \quad (3.12)$$

It then follows from Lemma 2.9 that $\|x_m - x_n\| \rightarrow 0$ as $m, n \rightarrow \infty$. Hence, $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since E is a Banach space and C is closed and convex, then there exists $p \in C$ such that $x_n \rightarrow p$ as $n \rightarrow \infty$.

Now since $\phi(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$ we have in particular that $\phi(x_{n+1}, x_n) \rightarrow 0$ as $n \rightarrow \infty$ and this further implies that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 0. \quad (3.13)$$

Then, we obtain

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = 0. \quad (3.14)$$

Since E is uniformly convex and smooth, we have from Lemma 2.9 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 = \lim_{n \rightarrow \infty} \|x_{n+1} - u_n\|. \quad (3.15)$$

So,

$$\|x_n - u_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - u_n\|. \quad (3.16)$$

Hence,

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \quad (3.17)$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$, we obtain

$$\lim_{n \rightarrow \infty} \|Jx_n - u_n\| = 0. \quad (3.18)$$

Since $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{JT_n x_n\}$, and $\{JS_n x_n\}$. Let $r := \sup_{n \geq 0} \{\|x_n\|, \|T_n x_n\|, \|S_n x_n\|\}$. Then from Lemma 2.10, we have

$$\begin{aligned}
G(x^*, Ju_n) &\leq G(x^*, J\theta_n^m y_n) \leq G(x^*, Jy_n) \\
&= G(x^*, (\alpha_n Jx_n + (1 - \alpha_n)Jz_n)) \\
&= \|x^*\|^2 - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n) \langle x^*, Jz_n \rangle \\
&\quad + \|\alpha_n Jx_n + (1 - \alpha_n)Jz_n\|^2 + 2\rho f(x^*) \\
&\leq \|x^*\|^2 - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1 - \alpha_n) \langle x^*, Jz_n \rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + (1 - \alpha_n) \|Jz_n\|^2 + 2\rho f(x^*) \\
&= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) G(x^*, Jz_n) \\
&= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) G\left(x^*, \left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\right)\right) \\
&\leq \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) \left(\|x^*\|^2 - 2\beta_n^{(1)} \langle x^*, Jx_n \rangle - 2\beta_n^{(2)} \langle x^*, JT_n x_n \rangle \right. \\
&\quad \left. - 2\beta_n^{(3)} \langle x^*, JS_n x_n \rangle + \beta_n^{(1)} \|x_n\|^2 + \beta_n^{(2)} \|T_n x_n\|^2 \right. \\
&\quad \left. + \beta_n^{(3)} \|S_n x_n\|^2 - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) + 2\rho f(x^*) \right) \\
&= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) \left(\beta_n^{(1)} G(x^*, Jx_n) + \beta_n^{(2)} G(x^*, JT_n x_n) \right. \\
&\quad \left. + \beta_n^{(3)} G(x^*, JS_n x_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \right) \\
&\leq \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) \left(\beta_n^{(1)} G(x^*, Jx_n) + \beta_n^{(2)} G(x^*, Jx_n) \right. \\
&\quad \left. + \beta_n^{(3)} G(x^*, Jx_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \right) \\
&= \alpha_n G(x^*, Jx_n) + (1 - \alpha_n) \left(G(x^*, Jx_n) - \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \right) \\
&\leq G(x^*, Jx_n) - (1 - \alpha_n) \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|).
\end{aligned} \tag{3.19}$$

It then follows that

$$(1 - \alpha) \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \leq (1 - \alpha_n) \beta_n^{(1)} \beta_n^{(2)} g(\|Jx_n - JT_n x_n\|) \leq G(x^*, Jx_n) - G(x^*, Ju_n). \tag{3.20}$$

But

$$\begin{aligned}
 G(x^*, Jx_n) - G(x^*, Ju_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x^*, Jx_n - Ju_n \rangle \\
 &\leq \left| \|x_n\|^2 - \|u_n\|^2 \right| + 2|\langle x^*, Jx_n - Ju_n \rangle| \\
 &\leq \| \|x_n\| - \|u_n\| \| (\|x_n\| + \|u_n\|) + 2\|x^*\| \|Jx_n - Ju_n\| \\
 &\leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + 2\|x^*\| \|Jx_n - Ju_n\|.
 \end{aligned} \tag{3.21}$$

From $\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0$ and $\lim_{n \rightarrow \infty} \|Jx_n - Ju_n\| = 0$, we obtain

$$G(x^*, Jx_n) - G(x^*, Ju_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.22}$$

Using the condition $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$, we have

$$\lim_{n \rightarrow \infty} g(\|Jx_n - JT_n x_n\|) = 0. \tag{3.23}$$

By property of g , we have $\lim_{n \rightarrow \infty} \|Jx_n - JT_n x_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_n - T_n x_n\| = 0. \tag{3.24}$$

Similarly, we can show that

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{3.25}$$

Since $x_n \rightarrow p$ and $\{T_n\}, \{S_n\}$ are uniformly closed, we have $p \in (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$.

Next, we show that $p \in \cap_{k=1}^m \text{GMPEP}(F_k, A_k, \varphi_k)$. Now, by Lemma 2.12, we obtain

$$\begin{aligned}
 \phi(u_n, y_n) &= \phi(\theta_n^m y_n, y_n) \\
 &\leq \phi(x^*, y_n) - \phi(x^*, \theta_n^m y_n) \\
 &\leq \phi(x^*, x_n) - \phi(x^*, u_n) \longrightarrow 0, \quad n \longrightarrow \infty.
 \end{aligned} \tag{3.26}$$

Using Lemma 2.9, we have $\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0$. Furthermore,

$$\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.27}$$

Since $x_n \rightarrow p$ as $n \rightarrow \infty$ and $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $y_n \rightarrow p$ as $n \rightarrow \infty$. By the fact that θ_n^k , $k = 1, 2, \dots, m$ is relatively nonexpansive and using Lemma 2.12 again, we have that

$$\begin{aligned} \phi(\theta_n^k y_n, y_n) &\leq \phi(x^*, y_n) - \phi(x^*, \theta_n^k y_n) \\ &\leq \phi(x^*, x_n) - \phi(x^*, \theta_n^k y_n). \end{aligned} \quad (3.28)$$

Observe that

$$\begin{aligned} \phi(x^*, u_n) &= \phi(x^*, \theta_n^m y_n) \\ &= \phi(x^*, T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \dots T_{r_{k,n}}^{Q_k} T_{r_{k-1,n}}^{Q_{k-1}} \dots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n) \\ &= \phi(x^*, T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \dots \theta_n^k y_n) \\ &\leq \phi(x^*, \theta_n^k y_n). \end{aligned} \quad (3.29)$$

Using (3.29) in (3.28), we obtain

$$\phi(\theta_n^k y_n, y_n) \leq \phi(x^*, x_n) - \phi(x^*, u_n) \rightarrow 0, \quad n \rightarrow \infty. \quad (3.30)$$

Then Lemma 2.9 implies that $\lim_{n \rightarrow \infty} \|y_n - \theta_n^k y_n\| = 0$, $k = 1, 2, \dots, m$. Now

$$\|p - \theta_n^k y_n\| \leq \|y_n - \theta_n^k y_n\| + \|y_n - p\| \rightarrow 0, \quad n \rightarrow \infty, \quad k = 1, 2, \dots, m. \quad (3.31)$$

Similarly, $\lim_{n \rightarrow \infty} \|p - \theta_n^{k-1} y_n\| = 0$, $k = 1, 2, \dots, m$. This further implies that

$$\lim_{n \rightarrow \infty} \|\theta_n^k y_n - \theta_n^{k-1} y_n\| = 0. \quad (3.32)$$

Also, since J is uniformly norm-to-norm continuous on bounded sets and using (3.32), we obtain

$$\lim_{n \rightarrow \infty} \|J\theta_n^k y_n - J\theta_n^{k-1} y_n\| = 0. \quad (3.33)$$

Since $\liminf_{n \rightarrow \infty} r_{k,n} > 0$, ($k = 1, 2, \dots, m$),

$$\lim_{n \rightarrow \infty} \frac{\|J\theta_n^k y_n - J\theta_n^{k-1} y_n\|}{r_{k,n}} = 0. \quad (3.34)$$

By Lemma 2.11, we have that for each $k = 1, 2, \dots, m$

$$Q_k(\theta_n^k y_n, y) + \frac{1}{r_{k,n}} \langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n \rangle \geq 0, \quad \forall y \in C. \quad (3.35)$$

Furthermore, using (A2) we obtain

$$\frac{1}{r_{k,n}} \langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n \rangle \geq Q_k(y, \theta_n^k y_n). \quad (3.36)$$

By (A4), (3.34), and $\theta_n^k y_n \rightarrow p$, we have for each $k = 1, 2, \dots, m$

$$Q_k(y, p) \leq 0, \quad \forall y \in C. \quad (3.37)$$

For fixed $y \in C$, let $z_{t,y} := ty + (1-t)p$ for all $t \in (0, 1]$. This implies that $z_t \in C$. This yields that $Q_k(z_t, p) \leq 0$. It follows from (A1) and (A4) that

$$\begin{aligned} 0 &= Q_k(z_t, z_t) \leq tQ_k(z_t, y) + (1-t)Q_k(z_t, p) \\ &\leq tQ_k(z_t, y) \end{aligned} \quad (3.38)$$

and hence

$$0 \leq Q_k(z_t, y). \quad (3.39)$$

From condition (A3), we obtain

$$Q_k(p, y) \geq 0, \quad \forall y \in C. \quad (3.40)$$

This implies that $p \in \text{GMEP}(F_k, A_k, \varphi_k)$, $k = 1, 2, \dots, m$. Thus, $p \in \bigcap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k)$. Hence, we have $p \in \Omega = \bigcap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap (\bigcap_{n=0}^{\infty} F(T_n)) \cap (\bigcap_{n=0}^{\infty} F(S_n))$.

Finally, we show that $p = \Pi_{\Omega}^f x_0$. Since $\Omega = \bigcap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap (\bigcap_{n=0}^{\infty} F(T_n)) \cap (\bigcap_{n=0}^{\infty} F(S_n))$ is a closed and convex set, from Lemma 2.6, we know that $\Pi_{\Omega}^f x_0$ is single valued and denote $w = \Pi_{\Omega}^f x_0$. Since $x_n = \Pi_{C_n}^f x_0$ and $w \in \Omega \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0), \quad \forall n \geq 0. \quad (3.41)$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p, Jx_0) \leq \liminf_{n \rightarrow \infty} G(x_n, Jx_0) \leq \limsup_{n \rightarrow \infty} G(x_n, Jx_0) \leq G(w, Jx_0). \quad (3.42)$$

From the definition of $\Pi_{\Omega}^f x_0$ and $p \in \Omega$, we see that $p = w$. This completes the proof. \square

Corollary 3.2 (Li et al. [38]). *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed, and convex subset of E . Suppose that $T : C \rightarrow C$ is a relatively nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and $f : E \rightarrow \mathbb{R}$ is a convex and lower semicontinuous mapping with $C \subset \text{int}(D(f))$. Suppose that $\{x_n\}_{n=0}^\infty$ is iteratively generated by $x_0 \in C$, $C_0 = C$*

$$\begin{aligned} y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ C_{n+1} &= \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}^f x_0, \quad n \geq 0, \end{aligned} \tag{3.43}$$

where J is the duality mapping on E . Suppose that $\{\alpha_n\}_{n=0}^\infty$ is a sequence in $(0, 1)$ such that $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Then, $\{x_n\}_{n=0}^\infty$ converges strongly to $\Pi_{F(T)}^f x_0$.

Corollary 3.3. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E . For each $k = 1, 2, \dots, m$, let F_k be a bifunction from $C \times C$ satisfying (A1)–(A4), $A_k : C \rightarrow E^*$ a continuous and monotone mapping and $\varphi_k : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex functional. Suppose $\{T_n\}_{n=0}^\infty$ and $\{S_n\}_{n=0}^\infty$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := \bigcap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap (\bigcap_{n=0}^\infty F(T_n)) \cap (\bigcap_{n=0}^\infty F(S_n)) \neq \emptyset$. Suppose that $\{x_n\}_{n=0}^\infty$ is iteratively generated by $x_0 \in C$, $C_0 = C$*

$$\begin{aligned} z_n &= J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n), \\ u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \dots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} &= \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 0, \end{aligned} \tag{3.44}$$

with the conditions

- (i) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iii) $0 \leq \alpha_n \leq \alpha < 1$ for some $\alpha \in (0, 1)$;
- (iv) $\{r_{k,n}\}_{n=1}^\infty \subset (0, \infty)$, $(k = 1, 2, \dots, m)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$, $(k = 1, 2, \dots, m)$.

Then, $\{x_n\}_{n=0}^\infty$ converges strongly to $\Pi_\Omega x_0$.

Proof. Take $f(x) = 0$ for all $x \in E$ in Theorem 3.1, $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x_0 = \Pi_C x_0$. Then, the desired conclusion follows. \square

Remark 3.4. Corollary 3.3 extends and improves on Theorem 1.5. In fact, the iterative procedure (3.44) is simpler than (1.14) in the following two aspects: (a) the process of computing $Q_n = \{w \in C_{n-1} \cap Q_{n-1} : \langle x_n - w, Jx_0 - Jx_n \rangle \geq 0\}$ is removed; (b) the process of computing $\Pi_{C_n \cap Q_n}$ is replaced by computing Π_{C_n} .

4. Applications

A mapping H from E to E^* is said to be

- (i) monotone if $\langle Hx - Hy, x - y \rangle \geq 0$, for all $x, y \in E$;
- (ii) strictly monotone if H is monotone and $\langle Hx - Hy, x - y \rangle = 0$ if and only if $x = y$;
- (iii) β -Lipschitz continuous if there exists a constant $\beta \geq 0$ such that $\|Hx - Hy\| \leq \beta\|x - y\|$, for all $x, y \in E$.

Let M be a set-valued mapping from E to E^* with domain $D(M) = \{z \in E : Mz \neq \emptyset\}$ and range $R(M) = \cup\{Mz : z \in D(M)\}$. A set-valued mapping M is said to be

- (i) monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ for each $x_i \in D(M)$ and $y_i \in Mx_i, i = 1, 2$;
- (ii) r -strongly monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq r\|x_1 - x_2\|^2$ for each $x_i \in D(M)$ and $y_i \in Mx_i, i = 1, 2$;
- (iii) maximal monotone if M is monotone and its graph $G(M) := \{(x, y) : y \in Mx\}$ is not properly contained in the graph of any other monotone operator;
- (iv) a general H -monotone if M is monotone and $(H + \lambda M)E = E^*$ holds for every $\lambda > 0$, where H is a mapping from E to E^* .

We denote the set $\{x \in E : 0 \in Mx\}$ by $M^{-1}0$. From Li et al. [38], we know that if $H : E \rightarrow E^*$ is strictly monotone and $M : E \rightarrow 2^{E^*}$ is general H -monotone mapping, then $M^{-1}0$ is closed and convex. Furthermore, for every $\lambda > 0$ and $x^* \in E^*$, there exists a unique $x \in D(M)$ such that $x = (H + \lambda M)^{-1}x^*$. Thus, we can define a single-value mapping $T_\lambda : E \rightarrow D(M)$ by $T_\lambda x = (H + \lambda M)^{-1}Hx$. It is obvious that $M^{-1}0 = F(T_\lambda)$ for all $\lambda > 0$.

Lemma 4.1 (Alber, [39]). *If E is a uniformly convex and uniformly smooth Banach space, $\delta_E(\epsilon)$ is the modulus of convexity of E , and $\rho_E(t)$ is the modulus of smoothness of E , then the inequalities*

$$8d^2\delta_E\left(\frac{\|x - \xi\|}{4d}\right) \leq \phi(x, \xi) \leq 4d^2\rho_E\left(\frac{4\|x - \xi\|}{d}\right) \tag{4.1}$$

hold for all x and ξ in E , where $d = \sqrt{(\|x\|^2 + \|\xi\|^2)/2}$.

Lemma 4.2 (Xia and Huang [45]). *Let E be a Banach space with dual space E^* , $H : E \rightarrow E^*$ a strictly monotone mapping, and $M : E \rightarrow 2^{E^*}$ a general H -monotone mapping. Then*

- (i) $(H + \lambda M)^{-1}$ is a single-valued mapping;
- (ii) if E is reflexive and $M : E \rightarrow 2^{E^*}$ is r -strongly monotone, $(H + \lambda M)^{-1}$ is Lipschitz continuous with constant $1/\lambda r$, where $r > 0$.

Theorem 4.3. *Let E be a uniformly convex real Banach space which is also uniformly smooth with $\delta_E(\epsilon) \geq k\epsilon^2$ and $\rho_E(t) \leq ct^2$ for some $k, c > 0$. For each $k = 1, 2, \dots, m$, let F_k be a bifunction from $E \times E$ satisfying (A1)–(A4), $A_k : E \rightarrow E^*$ a continuous and monotone mapping, and $\varphi_k : E \rightarrow \mathbb{R}$ a lower semicontinuous and convex functional. Suppose that $H : E \rightarrow E^*$ is a strictly monotone and β -Lipschitz continuous mapping and $M_i : E \rightarrow 2^{E^*}$ is a general H -monotone mapping and r_i -strongly monotone mapping with $r_i > 0, i = 1, 2$ such that $\Omega := \cap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap M_1^{-1}0 \cap M_2^{-1}0 \neq \emptyset$. Let $T_\lambda^{M_i} = (H + \lambda M_i)^{-1}H, i = 1, 2$, let $f : E \rightarrow \mathbb{R}$ a convex and lower semicontinuous*

mapping with $D(f) = E$ and suppose for each $n \geq 0$ that there exists a $\lambda_n > 0$ such that $64c\beta^2 \leq \min\{(1/2)k\lambda_n^2 r_1^2, (1/2)k\lambda_n^2 r_2^2\}$. Let $\{x_n\}_{n=0}^\infty$ be iteratively generated by $x_0 \in E$, $C_0 = E$,

$$\begin{aligned} z_n &= J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_{\lambda_n}^{M_1} x_n + \beta_n^{(3)} JT_{\lambda_n}^{M_2} x_n\right), \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \dots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} &= \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}^f x_0, \quad n \geq 0, \end{aligned} \tag{4.2}$$

with the conditions

- (i) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iii) $0 \leq \alpha_n \leq \alpha < 1$;
- (iv) $\liminf_{n \rightarrow \infty} \lambda_n > 0$;
- (v) $\{r_{k,n}\}_{n=1}^\infty \subset (0, \infty)$, $(k = 1, 2, \dots, m)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$, $(k = 1, 2, \dots, m)$.

Then, $\{x_n\}_{n=0}^\infty$ converges strongly to $\Pi_{\Omega}^f x_0$.

Proof. We only need to prove that $\{T_{\lambda_n}^{M_1}\}$ and $\{T_{\lambda_n}^{M_2}\}$ are countable families of weak relatively nonexpansive mappings with common fixed points sets $\bigcap_{n=0}^\infty F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$ and $\bigcap_{n=0}^\infty F(T_{\lambda_n}^{M_2}) = M_2^{-1}0$, respectively. Firstly, we have $\bigcap_{n=0}^\infty F(T_{\lambda_n}^{M_1}) = M_1^{-1}0 \neq \emptyset$. Secondly, we show that $\phi(p, T_{\lambda_n}^{M_1} w) \leq \phi(p, w)$, for all $w \in E$, $p \in F(T_{\lambda_n}^{M_1})$, $n \geq 0$. Now, by Lemma 4.2 and the Lipschitz continuity of H , we have

$$\begin{aligned} \left\| T_{\lambda_n}^{M_1} p - T_{\lambda_n}^{M_1} w \right\| &= \left\| (H + \lambda_n M_1)^{-1} H p - (H + \lambda_n M_1)^{-1} H w \right\| \\ &\leq \frac{1}{\lambda_n r_1} \|H p - H w\| \\ &\leq \frac{\beta}{\lambda_n r_1} \|p - w\|. \end{aligned} \tag{4.3}$$

By (4.3) and Lemma 4.1,

$$\begin{aligned}
 \phi(p, T_{\lambda_n}^{M_1} w) &= \phi(T_{\lambda_n}^{M_1} p, T_{\lambda_n}^{M_1} w) \\
 &\leq 4d^2 \rho_E \left(\frac{4 \|T_{\lambda_n}^{M_1} p - T_{\lambda_n}^{M_1} w\|}{d} \right) \\
 &\leq 64c \|T_{\lambda_n}^{M_1} p - T_{\lambda_n}^{M_1} w\|^2 \\
 &\leq \frac{64c\beta^2}{\lambda_n^2 r_1^2} \|p - w\|^2, \\
 \phi(p, w) &\geq 8d^2 \delta_E \left(\frac{\|p - w\|}{4d} \right) \geq \frac{1}{2} k \|p - w\|^2.
 \end{aligned} \tag{4.4}$$

Since $64c\beta^2 \leq (1/2)k\lambda_n^2 r_1^2$, it follows from (4.4) that $\phi(p, T_{\lambda_n}^{M_1} w) \leq \phi(p, w)$, for all $w \in E$, $p \in F(T_{\lambda_n}^{M_1})$, $n \geq 0$. Thirdly, we show that $\tilde{F}(\{T_{\lambda_n}^{M_1}\}) = \cap_{n=0}^\infty F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$. We first show that $\tilde{F}(\{T_{\lambda_n}^{M_1}\}) \subset M_1^{-1}0$. Let $p \in \tilde{F}(\{T_{\lambda_n}^{M_1}\})$, then there exists $\{x_n\} \subset E$ such that $x_n \rightarrow p$ and $\lim_{n \rightarrow \infty} \|x_n - T_{\lambda_n}^{M_1} x_n\| = 0$. Since H is β -Lipschitz continuous,

$$\|Hx_n - HT_{\lambda_n}^{M_1} x_n\| \leq \beta \|x_n - T_{\lambda_n}^{M_1} x_n\|. \tag{4.5}$$

Letting $n \rightarrow \infty$, we obtain

$$\frac{1}{\lambda_n} (Hx_n - HT_{\lambda_n}^{M_1} x_n) \rightarrow 0. \tag{4.6}$$

It follows from $(1/\lambda_n)(Hx_n - HT_{\lambda_n}^{M_1} x_n) \in M_1 T_{\lambda_n}^{M_1}$ and the monotonicity of M_1 that

$$\left\langle x - T_{\lambda_n}^{M_1} x, x^* - \frac{1}{\lambda_n} (Hx_n - HT_{\lambda_n}^{M_1} x_n) \right\rangle \geq 0 \tag{4.7}$$

for all $x \in D(M_1)$ and $x^* \in M_1 x$. Taking the limit as $n \rightarrow \infty$, we obtain

$$\langle x - p, x^* \rangle \geq 0 \tag{4.8}$$

for all $x \in D(M_1)$ and $x^* \in M_1 x$. By the maximality of M_1 , we know that $p \in M_1^{-1}0$. On the other hand, we know that $F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$, $F(T_{\lambda_n}^{M_1}) \subset \tilde{F}(T_{\lambda_n}^{M_1})$ for all $n \geq 0$, therefore, $M_1^{-1}0 = \cap_{n=0}^\infty F(T_{\lambda_n}^{M_1}) = \tilde{F}(\cap_{n=0}^\infty T_{\lambda_n}^{M_1})$. Thus, we have proved that $\{T_{\lambda_n}^{M_1}\}$ is a countable family of weak relatively nonexpansive mappings with common fixed points sets $\cap_{n=0}^\infty F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$. By following the same arguments, we can show that $\{T_{\lambda_n}^{M_2}\}$ is a countable family of weak relatively nonexpansive mappings with common fixed points sets $\cap_{n=0}^\infty F(T_{\lambda_n}^{M_2}) = M_2^{-1}0$. \square

Let E be a uniformly convex and uniformly smooth Banach space, $H = J$ and M a maximal monotone mapping. Then, we can define $J_\lambda = (J + \lambda M)^{-1}J$ for all $\lambda > 0$. We know that J_λ is relatively nonexpansive and therefore weak relatively nonexpansive and $M^{-1}0 = F(J_\lambda)$ for all $\lambda > 0$ (see, e.g., [2]), where $F(J_\lambda)$ denotes the fixed points set of J_λ . By Corollary 3.3, we obtain the following theorem.

Theorem 4.4. *Let E be a uniformly convex real Banach space which is also uniformly smooth. For each $k = 1, 2, \dots, m$, let F_k be a bifunction from $E \times E$ satisfying (A1)–(A4), $A_k : E \rightarrow E^*$ a continuous and monotone mapping, and $\varphi_k : E \rightarrow \mathbb{R}$ a lower semicontinuous and convex functional. For each $i = 1, 2$, let $M_i \subset E \times E^*$ a maximal monotone operator, and let $J_\lambda^{M_i} = (J + \lambda M_i)^{-1}J$ for all $\lambda > 0$, and suppose C is a nonempty closed and convex subset of E such that $D(M_i) \subset C \subset J^{-1}(\cap_{\lambda>0} R(J + \lambda M_i))$, $i = 1, 2$. Assume that $\Omega := \cap_{k=1}^m \text{GMEP}(F_k, A_k, \varphi_k) \cap M_1^{-1}0 \cap M_2^{-1}0 \neq \emptyset$ and let $\{x_n\}_{n=0}^\infty$ be iteratively generated by $x_0 \in E$, $C_0 = E$,*

$$\begin{aligned} z_n &= J^{-1}\left(\beta_n^{(1)} Jx_n + \beta_n^{(2)} J J T_{\lambda_n}^{M_1} x_n + \beta_n^{(3)} J J T_{\lambda_n}^{M_2} x_n\right), \\ y_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) Jz_n), \\ u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \dots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} &= \{w \in C_n : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \geq 0, \end{aligned} \tag{4.9}$$

with the conditions

- (i) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
- (ii) $\liminf_{n \rightarrow \infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
- (iii) $0 \leq \alpha_n \leq \alpha < 1$;
- (iv) $\liminf_{n \rightarrow \infty} \lambda_n > 0$;
- (v) $\{r_{k,n}\}_{n=1}^\infty \subset (0, \infty)$, $(k = 1, 2, \dots, m)$ satisfying $\liminf_{n \rightarrow \infty} r_{k,n} > 0$, $(k = 1, 2, \dots, m)$.

Then, $\{x_n\}_{n=0}^\infty$ converges strongly to $\Pi_\Omega x_0$.

Acknowledgments

The author would like to express his thanks to the referees for their valuable suggestions. This research work is dedicated to Professor Isaac U. Asuzu of the University of Nigeria, Nsukka.

References

- [1] C. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, vol. 1965 of *Lecture Notes in Mathematics*, Springer, London, UK, 2009.
- [2] W. Takahashi, *Nonlinear Functional Analysis-Fixed Point Theory and Applications*, Yokohama Publishers, Yokohama, Japan, 2000.
- [3] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.
- [4] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

- [5] Y. Su, H.-K. Xu, and X. Zhang, "Strong convergence theorems for two countable families of weak relatively nonexpansive mappings and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 12, pp. 3890–3906, 2010.
- [6] D. Butnariu, S. Reich, and A. J. Zaslavski, "Asymptotic behavior of relatively nonexpansive operators in Banach spaces," *Journal of Applied Analysis*, vol. 7, no. 2, pp. 151–174, 2001.
- [7] D. Butnariu, S. Reich, and A. J. Zaslavski, "Weak convergence of orbits of nonlinear operators in reflexive Banach spaces," *Numerical Functional Analysis and Optimization*, vol. 24, no. 5-6, pp. 489–508, 2003.
- [8] Y. Censor and S. Reich, "Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization," *Optimization*, vol. 37, no. 4, pp. 323–339, 1996.
- [9] S. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.
- [10] W. Takahashi and K. Zembayashi, "Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 528476, 11 pages, 2008.
- [11] X. Qin and Y. Su, "Strong convergence theorems for relatively nonexpansive mappings in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 6, pp. 1958–1965, 2007.
- [12] J. Kang, Y. Su, and X. Zhang, "Hybrid algorithm for fixed points of weak relatively nonexpansive mappings and applications," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 4, pp. 755–765, 2010.
- [13] H. Zegeye and N. Shahzad, "Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2707–2716, 2009.
- [14] M. Liu, S. Chang, and P. Zuo, "On a hybrid method for generalized mixed equilibrium problem and fixed point problem of a family of quasi- ϕ -asymptotically nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 157278, 18 pages, 2010.
- [15] N. Petrot, K. Wattanawitton, and P. Kumam, "A hybrid projection method for generalized mixed equilibrium problems and fixed point problems in Banach spaces," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 4, pp. 631–643, 2010.
- [16] S. Saewan, P. Kumam, and K. Wattanawitton, "Convergence theorem based on a new hybrid projection method for finding a common solution of generalized equilibrium and variational inequality problems in Banach spaces," *Abstract and Applied Analysis*, vol. 2010, Article ID 734126, 25 pages, 2010.
- [17] S. Saewan and P. Kumam, "Modified hybrid block iterative algorithm for convex feasibility problems and generalized equilibrium problems for uniformly quasi- ϕ -asymptotically nonexpansive mappings," *Abstract and Applied Analysis*, vol. 2010, Article ID 357120, 22 pages, 2010.
- [18] P. Kumam and S. Saewan, "A hybrid iterative scheme for a maximal monotone operator and two countable families of relatively quasi-nonexpansive mappings for generalized mixed equilibrium and variational inequality problems," *Abstract and Applied Analysis*, vol. 2010, Article ID 123027, 31 pages, 2010.
- [19] S.-S. Zhang, "Generalized mixed equilibrium problem in Banach spaces," *Applied Mathematics and Mechanics. English Edition*, vol. 30, no. 9, pp. 1105–1112, 2009.
- [20] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 6, no. 1, pp. 117–136, 2005.
- [21] Y. Liu, "A general iterative method for equilibrium problems and strict pseudo-contractions in Hilbert spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 10, pp. 4852–4861, 2009.
- [22] A. Moudafi, "Weak convergence theorems for nonexpansive mappings and equilibrium problems," *Journal of Nonlinear and Convex Analysis*, vol. 9, no. 1, pp. 37–43, 2008.
- [23] S. Plubtieng and R. Punpaeng, "A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 548–558, 2008.
- [24] X. Qin, M. Shang, and Y. Su, "Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems," *Mathematical and Computer Modelling*, vol. 48, no. 7-8, pp. 1033–1046, 2008.
- [25] Y. Su, M. Shang, and X. Qin, "An iterative method of solution for equilibrium and optimization problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2709–2719, 2008.
- [26] W. Takahashi and K. Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 1, pp. 45–57, 2009.

- [27] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.
- [28] R. Wangkeeree, "An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 134148, 17 pages, 2008.
- [29] X. Qin, Y. J. Cho, and S. M. Kang, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 99–112, 2010.
- [30] Y. Shehu, "Fixed point solutions of generalized equilibrium problems for nonexpansive mappings," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 892–898, 2010.
- [31] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.
- [32] J.-W. Peng and J.-C. Yao, "Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1816–1828, 2009.
- [33] S. Plubtieng and K. Sombut, "Weak convergence theorems for a system of mixed equilibrium problems and nonspreading mappings in a Hilbert space," *Journal of Inequalities and Applications*, vol. 2010, Article ID 246237, 12 pages, 2010.
- [34] Y. Yao, Y.-C. Liou, and J.-C. Yao, "A new hybrid iterative algorithm for fixed-point problems, variational inequality problems, and mixed equilibrium problems," *Fixed Point Theory and Applications*, vol. 2008, Article ID 417089, 15 pages, 2008.
- [35] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.
- [36] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.
- [37] S. Plubtieng and K. Ungchittrakool, "Strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 149, no. 2, pp. 103–115, 2007.
- [38] X. Li, N.-J. Huang, and D. O'Regan, "Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1322–1331, 2010.
- [39] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Dekker, New York, NY, USA, 1996.
- [40] Y. I. Alber and S. Reich, "An iterative method for solving a class of nonlinear operator equations in Banach spaces," *Panamerican Mathematical Journal*, vol. 4, no. 2, pp. 39–54, 1994.
- [41] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, vol. 62 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.
- [42] K. Wu and N. Huang, "The generalised f -projection operator with an application," *Bulletin of the Australian Mathematical Society*, vol. 73, no. 2, pp. 307–317, 2006.
- [43] J. Fan, X. Liu, and J. Li, "Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 11, pp. 3997–4007, 2009.
- [44] Y. J. Cho, H. Zhou, and G. Guo, "Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings," *Computers & Mathematics with Applications*, vol. 47, no. 4-5, pp. 707–717, 2004.
- [45] F.-Q. Xia and N.-J. Huang, "Variational inclusions with a general H -monotone operator in Banach spaces," *Computers & Mathematics with Applications*, vol. 54, no. 1, pp. 24–30, 2007.